COURANT ELEMENTS FOR THE INVERSE PROBLEM IN THE FILTRATION COEFFICIENT IDENTIFICATION

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Abstract

We consider the inverse problem for ground water filtration. Courant elements are used for discretization together with Tikhonov regularization method for ill-posed problems. A special model problem is constructed for finding out the best regularization parameter for the case when the input data must be interpolated to get a regular mesh. Real-life problem is solved on MasPar MP-2 computer.

Keywords. Ill-posed problems, parameter estimation, Courant elements, Tikhonov regularization, identifiability.

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1 Introduction

We are dealing with the following inverse problem: find the coefficient $a \in L^2(\Omega)$ such that

$$-div(a(x)\nabla u(x)) = f(x), \qquad x \in \Omega \tag{1}$$

$$[a(x)\nabla u(x)] \cdot \nu(x) = g(x), \qquad x \in \Gamma \tag{2}$$

where $\Omega \subset \mathcal{R}^n$ is an open bounded region with a piecewise smooth boundary $\partial\Omega$ and outer unit normal ν , $\Gamma \subset \partial\Omega$ is a relatively open set where the boundary condition is defined; $u \in W^{1,\infty}(\Omega)$, $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$ are given functions. Physically, u can be interpreted as the piezometrical head of ground water in Ω; function f characterizes the sources and sinks in Ω andfunction g characterizes the inflow and outflow through $\Gamma \subset \partial\Omega$. The

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filtration (transmissivity) coefficient a is, physically, positive and piecewise smooth with possible discontinuities of the first kind on some surfaces in Ω.

Our aim is to find good ways for solving real-life problems of filtration coefficient identification. For this purpose we introduce the weak formulation of the problem which will be the basis for our discretization scheme.

Let us provisionally assume that $a \in H^1(\Omega)$, $u \in W^{2,\infty}(\Omega)$. Multiplying the equation (1) by $w \in H^1(\Omega)$, integrating by parts and using the equation (2) we obtain

$$\int_\Omega a\nabla u \cdot \nabla w \, dx = \int_\Omega f w \, dx + \int_\Gamma g w \, dS + \int_{\partial\Omega\setminus\Gamma} (a\nabla u)\cdot \nu w \, dS.$$

After introducing the subspace $H^1(\Omega,\Gamma) = \{w \in H^1(\Omega) : w(x) = 0$ for $x \in \partial\Omega\setminus\Gamma\} \subseteq H^1(\Omega)$ we obtain the following weak formulation of inverse problem (1),(2): find $a \in L^2(\Omega)$ such that

$$\int_\Omega a\nabla u \cdot \nabla w \, dx = \int_\Omega f w \, dx + \int_\Gamma g w \, dS, \ \forall w \in H^1(\Omega,\Gamma). \tag{3}$$

The same formulation can be obtained in case of piecewise smooth a and u. Problem (3) has sense for $u \in W^{1,\infty}(\Omega)$, $a \in L^2(\Omega)$.

Introduce the space of gradients of functions $w \in H^1(\Omega,\Gamma)$

$$G = G(\Omega,\Gamma) = \{\nabla w : w \in H^1(\Omega,\Gamma)\} \subset (L^2(\Omega))^n,$$

Let Q_G be the orthoprojector in $(L^2(\Omega))^n$ corresponding to G. The weak formulation (3) is equivalent to the equation

$$Ta = \nabla\psi, \tag{4}$$

where $T = T_u \in \mathcal{L}(L^2(\Omega), G)$ is defined by the formula

$$Ta = Q_G(a\nabla u), \ \ a \in L^2(\Omega),$$

and $\psi = \psi_{f,g}$ is the solution to boundary value problem

$$\begin{array}{rcl} -\nabla\psi(x) & = & f(x), \ x \in \Omega, \\ \nabla\psi \cdot \nu(x) & = & g(x), \ x \in \Gamma, \ \ \psi(x) = 0, \ x \in \partial\Omega\setminus\Gamma. \end{array} \tag{5}$$

The adjoint operator $T^* \in \mathcal{L}(G, L^2(\Omega))$ has a very simple form:

$$T^*\nabla w = \nabla u(x) \cdot \nabla w(x).$$

It can be seen from properties of T^*, that the problem under consideration is ill-posed problem with a non-compact operator. This yields that together with discretization some kind of regularization should be used. We were using Tikhonov regularization method, but it seems that the problem can also be successively regularized using some other method.

Let S_h be a finite dimensional subsace of $H^1(\Omega,\Gamma)$, $w_j = w_{j,h}$, $(j=1,\dots,l_h)$ its basis, and Q_h the orthoprojector in $(L^2(\Omega))^n$ corresponding to $\nabla S_h \subset G \subset (L^2(\Omega))^n$. To discretize (4) we use the method of minimal error which is a special projection method and has the operator form

$$Q_h T a_h = Q_h \nabla\psi,\ a_h \in T^*\nabla S_h.$$

Regularizing the last equation by Tikhonov method we obtain

$$(\alpha I + T^* Q_h T)a_h = T^* Q_h \nabla\psi,$$

which has the matrix form

$$(\alpha B + A)c = d,\quad a_h = \sum_{j=1}^{l_h} c_j \nabla u \cdot \nabla w; \tag{6}$$

where $c=(c_j)$, $A=(a_{ij})$, $B=(b_{ij})$, $(i,j=1,\dots,l_h)$,

$$\begin{aligned} a_{ij} &= \int_\Omega (\nabla u\cdot\nabla w_i)(\nabla u\cdot\nabla w_j)\,dx, \\ b_{ij} &= \int_\Omega \nabla w_i\cdot\nabla w_j\,dx, \\ d_i &= \int_\Omega f w_i\,dx + \int_\Gamma g w_i\,dS. \end{aligned}$$

For more details see [6].

2 Courant elements

Let us look at first the case $\Omega = \{(x_1,x_2)\in R^2 : 0 < x_1, x_2 < 1\}$. We introduce a standard uniform triangulation of Ω corresponding to a mesh size $h = 1/N$, where N is an integer (see Figure 1 a)). Denote $x_{i_1 i_2} = (i_1 h, i_2 h)$ and introduce Courant element $w_{i_1 i_2}$ which is a linear function on every triangle, continuos on Ω, equals to 1 at $x_{i_1 i_2}$ and 0 at other knots. Figures 1 b) and c) describe the support of $w_{i_1 i_2}$ for inner knots and boundary knots, respectively. The linear span of $w_{i_1 i_2}$ with appropriate values of indexes $i_1 i_2$ constitutes a finite subspace S_h. We maintain the double indexing of the basis funcions $w_{i_1 i_2}$.

2.1 Case $\Gamma = \emptyset$.

In this case $H^1(\Omega,\Gamma) = \{w\in H^1(\Omega) : w(x) = 0 \text{ for } x\in\partial\Omega\} = H_0^1(\Omega)$ and the subspace $S_h \subset H^1(\Omega,\Gamma)$ where we are looking for the solution, consists of the piecewise linear functions

$$w_h = \sum_{i_1=1}^{N-1}\sum_{i_2=1}^{N-1} c_{i_1 i_2} w_{i_1 i_2}$$

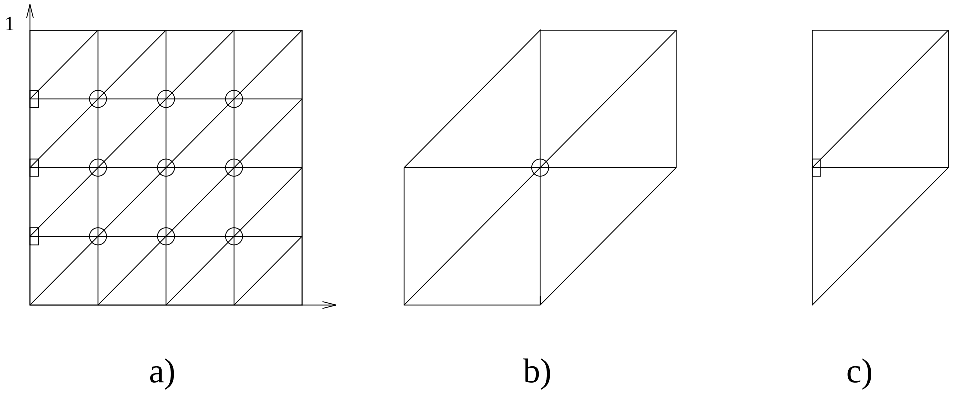

Figure 1: a) Triangulation of unit square; Courant element b) for inner points of Ω, c) on boundary $\Gamma = \{(x_1, x_2) : x_1 = 0,\ 0 < x_2 < 1\}$

vanishing on $\partial\Omega$. We approximate

$$d_{i_1 i_2} = \int_\Omega f w_{i_1 i_2}\, dx \approx h^2 f(x_{i_1 i_2}), \quad 1 \le i_1, i_2 \le N-1.$$

The elements of the matrix B can be found exactly: for $1 \le i_1, i_2, j_1, j_2 \le N-1$,

$$b_{i_1 i_2 j_1 j_2} = \int_\Omega \nabla w_{j_1 j_2} \cdot \nabla w_{i_1 i_2}\, dx = \begin{cases} 4 & \text{if } (i_1, i_2) = (j_1, j_2) \\ \text{-1} & \text{if } i_1 = j_1, i_2 = j_2 \pm 1 \\ & \text{or } i_2 = j_2, i_1 = j_1 \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

To approximate the elements of the matrix A we first replace u by its piecewise linear interpolant

$$u_h = \sum_{k_1=1}^{N-1} \sum_{k_2=1}^{N-1} u(x_{k_1 k_2}) w_{k_1 k_2}$$

and then it is possible to get exact expressions for the matrix $A = \{a_{i_1 i_2 j_1 j_2}\}$ for $1 \le i_1, i_2, j_1, j_2 \le N-1$:

$$\begin{aligned} a_{i_1 i_2 j_1 j_2} &= \int_\Omega (\nabla u \cdot \nabla w_{j_1 j_2})(\nabla u \cdot \nabla w_{i_1 i_2})\, dx \\ &\approx \int_\Omega (\nabla u_h \cdot \nabla w_{j_1 j_2})(\nabla u_h \cdot \nabla w_{i_1 i_2})\, dx. \end{aligned}$$

After solving the system (6), we specify the formula for the filtration coefficient $a_{\alpha,h}$ as follows: $a_{\alpha,h}$ is constant on any triangle of our triangulation; in the triangle with $x_{i_1} < x_1 < x_{i_1+1},\ x_{i_2-1} < x_2 < x_{i_2}$,

$$a_{\alpha,h} = \left[(c_{i_1+1 i_2} - c_{i_1 i_2})(u_{i_1+1 i_2} - u_{i_1 i_2}) + (c_{i_1 i_2} - c_{i_1 i_2-1})(u_{i_1 i_2} - u_{i_1 i_2-1})\right]/h^2,$$

in the triangle with $x_{i_1-1} < x_1 < x_{i_1},\ x_{i_2} < x_2 < x_{i_2+1}$,

$$a_{\alpha,h} = \left[(c_{i_1 i_2+1} - c_{i_1 i_2})(u_{i_1 i_2+1} - u_{i_1 i_2}) + (c_{i_1 i_2} - c_{i_1-1 i_2})(u_{i_1 i_2} - u_{i_1-1 i_2})\right]/h^2.$$

2.2 Case $\Gamma = \{(0, x_2) : 0 < x_2 < 1\}$

In this case the subspace $S_h \subset H^1(\Omega, \Gamma)$ consists of the piecewise linear functions of the form

$$w_h = \sum_{i_1=0}^{N-1} \sum_{i_2=1}^{N-1} c_{i_1 i_2} w_{i_1 i_2}.$$

The formulas corresponding to the inner knots are the same as in Section 2.1. In addition, we obtain the formulas corresponding to the boundary knots:

$$d_{0i_2} = \int_\Omega f w_{0i_2}\, dx + \int_\Gamma g w_{0i_2}\, dS \approx \frac{1}{2} h^2 f(x_{0i_2}) + h g(x_{0i_2}), \quad 1 \le i_2 \le N-1,$$

$$b_{0i_2 j_1 j_2} = b_{i_1 i_2 0 j_2} = \begin{cases} 2 & \text{if } j_1 = 0, j_2 = i_2 \\ -1/2 & \text{if } j_1 = 0, j_2 = i_2 \pm 1 \\ -1 & \text{if } j_1 = 1, j_2 = i_2 \\ 0 & \text{otherwise,} \end{cases}$$

also, corresponding additional elements will appear in the matrix A. We refer to [7] for more general results of discretization, regularization (including iterative regularization) and choices of parameter α that guarantee convergence of the problem; and [3, 4, 5, 10] for the methods (without regularization) to solve an inverse problem type (1),(2).

3 Numerical results

3.1 Model problem

Here we consider a model problem defining

$$a(x) = e^{x_1+x_2} + \sin 2(x_1 - x_2),$$
$$u(x) = 10\left[(x_1 + 1/4)^2 + (x_2 - 1\tfrac{1}{3})^2\right],$$

where the functions f and g are calculated in correspondance to (1),(2). Let us, for the purpose of modelling the real-life situation, assume that the real measurements of the ground water levels u are available in the region $\Omega = \{(x_1, x_2) : 0 \le x_1, x_2 \le 1\}$ at the points (x_{i_1}, x_{i_2}) that are situated quite randomly in Ω, but still, cover the whole region uniformly enough. To solve the problem on the regular mesh we used a simple interpolation method based on the weighted sums in the circle of radius $r = 0.1$. After that we solved the problem with different values of α choosen by the method of golden section for minimizing the error $\|a_{\alpha,h} - a\|_{L^2(\Omega)}$, and so, found out the best regularization parameter α_{opt}. In Table 1 we present some results of solving the model problem on MasPar MP-2 computer using the conjugate

Table 1: Optimal α for model problem together with corresponding error and the number of iterations l_{CG} with the method of Conjugate Gradients.

N	α_{opt}	Error	l_{CG}
8	1.62	0.798	85
16	1.33	0.618	219
32	1.15	0.507	509
64	2.07	0.467	1147
120	2.55	0.488	2522

gradient method (see, e.g. [2]). Here the boundary condition was given on two lines: $\Gamma = \{(0, x_2) : 0 \le x_2 \le 1\} \cup \{(x_1, 0) : 0 \le x_1 \le 1\}$. We mention here that the solution of the problem with exact values of u (i.e. without interpolation) gives us $\alpha_{opt} = 0.0026$ and that with the value of $\alpha = 2.07$ the average of the elements of matrix αB is only about 0.4% of that of matrix A. For solving a problem with inexact data it is reasonable to choose more bigger value of α because we need to add regularization due to the errors in input data. Numerical results solving a similar model problem with block SOR iterations combined with the method of golden section for finding out the best regularization parameter α in the case of inexact data can be found in [6].

From Table 1 one can see that in case of $N = 120$ there is no improvement of error compared to $N = 64$. It can be concluded from here that the value $N = 64$ is already critical for the linear system to be solved and there is no improvement in results if choosing greater N. Still, during the computations with real data, described in the following subsection, we could not mention any dramatic differences in the graphical plots of the solution with different mesh sizes and the plots on the bigger meshes are presented.

3.2 Computations with real-life data

We made our computations for estimation of the filtration coefficient in the region of 292.5×317.5 kilometers covering a big part of the map of Estonia. The measurements of ground water levels in the innermost layer are taken from 1155 different places. The first task was to interpolate the data to regular grids as the measurement points were quite scattered around the region. Although, as the data is from above the deepest watertight layer, some kind of seasonal variation in levels may be present. For interpolation we used the method described in the previous subsection; we tried two different radiuses: $r = 30000m$ and $r = 15000m$. Interpolated ground water levels u for $r = 30000$ are shown on the Figure 2. We used all the measurements available; note that there are regions in the corners of our mesh (except the northeasten one) where there are no measurements.

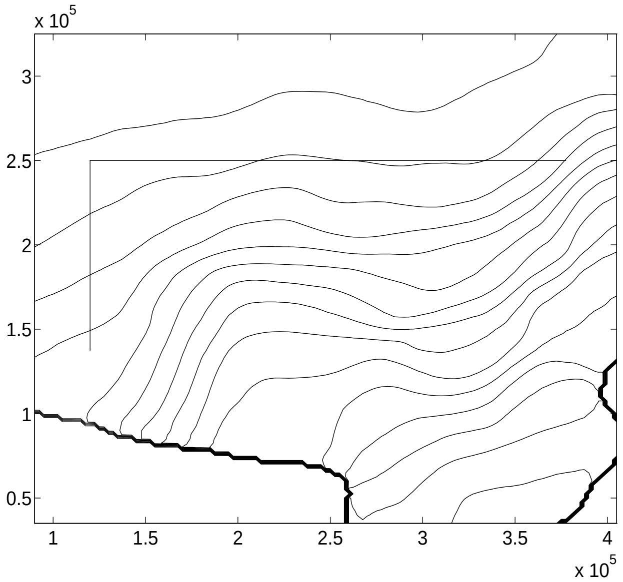

Figure 2: Levels of ground water: contour lines at every 5 meters

The levels of the ground water u are the easiest part of the problem. As it is seen from the Figure 2, the flow curves pass through the region during finite time period, this is the case when the boundary conditions must be present to get identifiability. (We refer here to [8] for detailed treatment of identifiability, see also [1, 5], where the C^1-identifiability is examined.) In practice, it is hard to get values of high accuracy to g in boundary condition (2). Only guesses to these values were available on two straight lines (see Figure 2) which devide our region into different independent subproblems. The beauty of using massively parallel computer is that these subproblems can be solved at the same time! The other "painful" point of the model is: how to find the values of function f in (1). Our computations are based on the condition $f = 0$.

We used the computational scheme where each processor element of MasPar holds the values of one knot characteristics – including the corresponding matrix elements that are attached to this knot. In such a way all the data is "smeared" on to the processor array. As the basic operations of conjugate gradient method are matrix–vector application and dot products, MasPar computer architecture (see, e.g. [9]) is really suitable for solving similar problems as the communication here is needed between the nearest neighbour processors and there are also fast procedures for summing up vectors on MasPar processor array.

On Figures 3 and 4 there is represented the computed filtration coefficient in the case when the regularization parameter $\alpha = \alpha_{10}$ is choosen such a way that the values of regularization matrix B elements to be about 10%

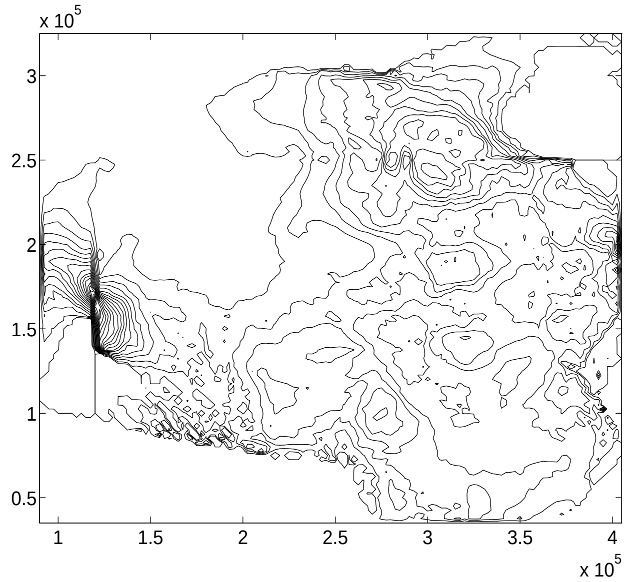

Figure 3: Filtration coefficient (mesh size $117{\times}127, h = 2500m$): contour lines at every 0.5 units

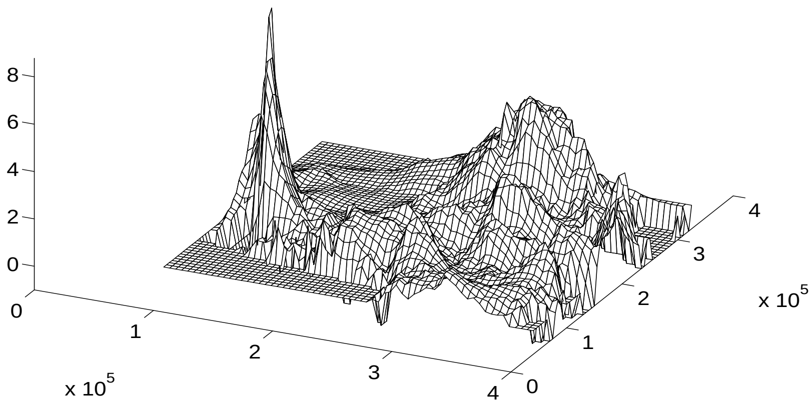

Figure 4: 3-D view of the filtration coefficient (mesh size $117{\times}127, h = 2500m$)

of the average values of matrix A elements. In this case the convergence of the method took ca 3000 iteration steps. If to use some preconditioning technique, this, surely, can be reduced. There occured some regions where the filtration coefficient became negative. Two of such regions can be found on the figures where the boundary condition lines end up — nonidentifiability is present there and no reasonable results should be expected. If to take α smaller (e.g. $\alpha = \alpha_{10}/2$ which took about 4000 iterations to converge) we detect more and more such "negativism" also in the region where the filtration coefficient should be well identifiable. We conclude from here that some kind of regularization must be always present to get true results.

For solving the two subproblems at the same time we moved all the knot values below and right from the boundary condition lines one step southeast. Such way we got two different problems with two different boundary conditions (as the outer unit normals for the neighbour subregions are in the opposite directions) on the same processor array. Afterwards, the solution was shifted back to its natural position; as it is seen from the figure 3, many of the contour lines do not even notice, that they are passing to the solution region of the different subproblem.

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References

[1] C. Chicone and J.Gerlach, *Identifiability of distributed parameters,* in: Inverse and Ill-Posed Problems, H. W. Engl and C. W. Groetsch (Ed.). Academic Press, Boston etc., (1987), pp 513 - 521.

[2] G. H. Golub and C. F. Van Loan, *Matrix Computations,* The John Hopkins Univ. Press, 1989.

[3] R. V. Kohn and B. C. Lowe, *A variational method for parameter estimation,* RAIRO Math. Mod. and Num. Anal. 22, (1988), pp 119 - 158.

[4] K. Kunisch, *A review of some recent results on the output least squares formulation of parameter estimation problem,* Automatica 24, (1988), pp 531 - 539.

[5] G. R. Richter, *An inverse problem for the steady state diffusion equation,* SIAM J. Appl. Math. 41, (1981), pp 210 - 221.

[6] E. Vainikko and G. Vainikko, *Some numerical schemes for the identification of the filtration coefficient,* Acta et comment. univ. Tartuensis 937, (1992), pp 90 - 102.

[7] G. Vainikko, *On the discretization and regularization of ill-posed problems with noncomact operators,* Numer. Funct. Anal. and Optimiz. 13, (1992), pp 381 - 396.

[8] G. Vainikko and K. Kunisch, *Identifiability of the transmissivity coefficient in an elliptic boundary value problem,* Z. für Anal. und Anwend. 12, (1992), pp 327 - 342.

[9] J. Nickolls, *The design of the MasPar MP-1, a cost effective massively parallel computer,* in: Proc. IEEE Compcon, Spring 1990.

[10] W. W.-G. Yeh, *Review of parameter identification procedures in groundwater hydrology: the inverse problem,* Water Resources Res. 22, (1986), pp 95 -108.