Expander Codes: Constructions and Bounds

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Turbo Codes

Presented by Berrou, Glavieux and Thitimajshima in 1993.

- Very efficient in practice.
- Perform at rates very close to Shannon capacity.
- Hard to analyze.

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Low-density parity-check codes.

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- [Richardson Urbanke '01] Good *average* behavior over binary memoryless channels.
- [Richardson Shokrollahi Urbanke '01] Codes, which are extremely close to the capacity, found by the exhaustive search.

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- [Barg Zémor '01–'04] Capacity-achieving codes for BSC with linear-time decoding, exponentially small decoding error.
- [Guruswami Indyk '02] Linear-time encodable and decodable codes that attain the Zyablov bound, used concatenation with nearly-MDS code. Based on construction in [Zémor '01] as a building block.

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- *The minimum distance* of a code $\mathcal{C}$ is

$$d = \min_{\boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}, \boldsymbol{x} \neq \boldsymbol{y}} \mathsf{d}(\boldsymbol{x}, \boldsymbol{y}).$$

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- *The relative minimum distance* of $\mathcal{C}$ is defined as $\delta = d/n$.

Linear Code

Definition

- A code $\mathcal{C}$ over field Φ is said to be a *linear* $[n, k, d]$ *code* if there exists a matrix H with n columns and rank $n - k$ such that
$$H\boldsymbol{x}^t = \bar{\mathbf{0}} \Leftrightarrow \boldsymbol{x} \in \mathcal{C}.$$
- The matrix H is called a *parity-check matrix*.
- The value k is called the *dimension* of the code $\mathcal{C}$.
- The ratio $r = k/n$ is called the *rate* of the code $\mathcal{C}$.
- The words of $\mathcal{C}$ can be obtained as linear combinations of rows of a *generating* $k \times n$ *matrix* G.

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- A linear $[n, k{=}rn, \delta_{in}n]$ code $\mathcal{C}_{in}$ over $F = \mathrm{GF}(q)$ (inner code).

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Concatenated code $\mathbb{C}_{cont}$ of length $n \cdot N$ over F is defined as

$$\mathbb{C}_{cont} = \Big\{ (\boldsymbol{c}_1|\boldsymbol{c}_2|\cdots|\boldsymbol{c}_n) \in F^{n\cdot N} : \boldsymbol{c}_i = \mathcal{E}_0(\Xi_i) \ , \\ \text{for } i \in 1, 2, \cdots, n, \quad \text{and } (\Xi_1\Xi_2\cdots\Xi_n) \in \mathcal{C}_{out} \Big\} \ .$$

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The rate $R_{cont} = r \cdot R$. The relative minimum distance $\delta \geq \delta_{in} \cdot \delta_{out}$.

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$$\mathsf{Prob}_e(\mathbb{C}_{cont}) \leq \max_{\mathcal{R} \leq r \leq \mathsf{C}_q(p)} \mathsf{e}^{-NE(r)(1-\frac{\mathcal{R}}{r})},$$

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- [Justesen '72] For a wide range of rates, concatenated codes attain the Zyablov bound

$$\delta \geq \max_{\mathcal{R} \leq r \leq 1} \left(1 - \frac{\mathcal{R}}{r}\right) \mathsf{H}_q^{-1}(1-r).$$

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Alternative Description

Bipartite undirected graph $\mathcal{G} = (V, E)$.

- Vertex set $V = V_m \cup V_c$, $|V_m| = n$, $|V_c| = n - k$.
- Edge set E. There is an edge between the message vertex i and the check vertex j if and only if $(H)_{i,j} \neq 0$.

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$\mathbb{C} = (\mathcal{G}, \mathcal{C})$ is the following linear $[N, K, D]$ code over $\mathrm{GF}(q)$:

$$\mathbb{C} = \left\{ \boldsymbol{c} \in (\mathrm{GF}(q))^N \ : \ (\boldsymbol{c})_{E(v)} \in \mathcal{C} \text{ for every } v \in V \right\} \ ,$$

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- The code $\mathbb{C} = (\mathcal{G}, \mathcal{C})$ is a *low-complexity* code.

Low-Complexity Codes – Example

Take $\Delta = 3$, $k = 2$, $|V| = 4$.
Let G be a generating matrix of $\mathcal{C}$ over $F = \mathrm{GF}(2^2) = \{0, 1, \alpha, \alpha^2\}$:

$$G = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \alpha \end{pmatrix} .$$

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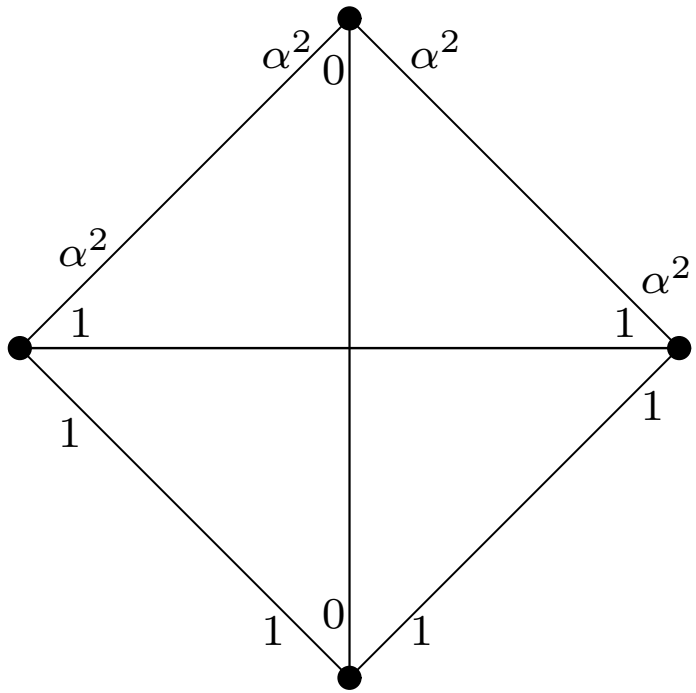

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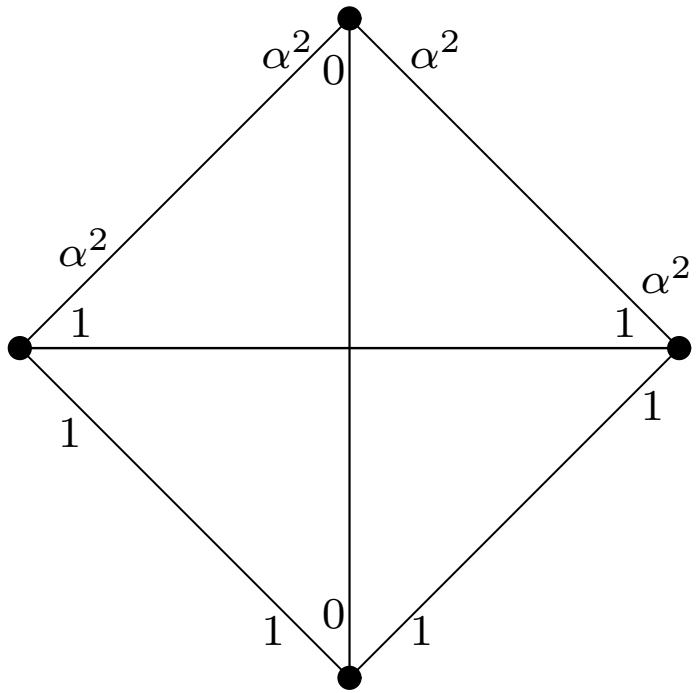

The resulting code $\mathbb{C}$ is of length $N = 6$. For instance,

$$(1\ 1\ \alpha^2\ 0\ \alpha^2\ 1) \in \mathbb{C}.$$

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- The graph $\mathcal{G}$ is an (α, ζ)-*expander* if every subset of at most $\alpha|V|$ vertices expands by a factor of ζ.

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- Lower ratios of $\gamma_{\mathcal{G}} = \frac{\lambda_{\mathcal{G}}}{\Delta}$ imply greater values ζ of expansion. [Alon '86]
- Expander graph with

$$\lambda_{\mathcal{G}} \leq 2\sqrt{\Delta - 1}$$

is called a *Ramanujan graph*. Constructions are due to [Lubotsky Philips Sarnak '88], [Margulis '88].

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- A linear $[\Delta, k{=}r\Delta, d{=}\delta\Delta]$ code $\mathcal{C}$ over $F = \mathrm{GF}(q)$.
- Let $\mathbb{C} = (\mathcal{G}, \mathcal{C})$ be the low-complexity linear $[N, \mathcal{R}N, D]$ code.

Example

Take the graph $\mathcal{G}$ with $\Delta = 3$ and $n = 4$. ($\mathcal{G}$ as on the slide is both $(1/8, 1)$-expander and $(1/4, 2/3)$-expander.)

Let $k = 2$, and pick $F = \mathrm{GF}(2)$.

Take $\mathcal{C}$ parity code over F:

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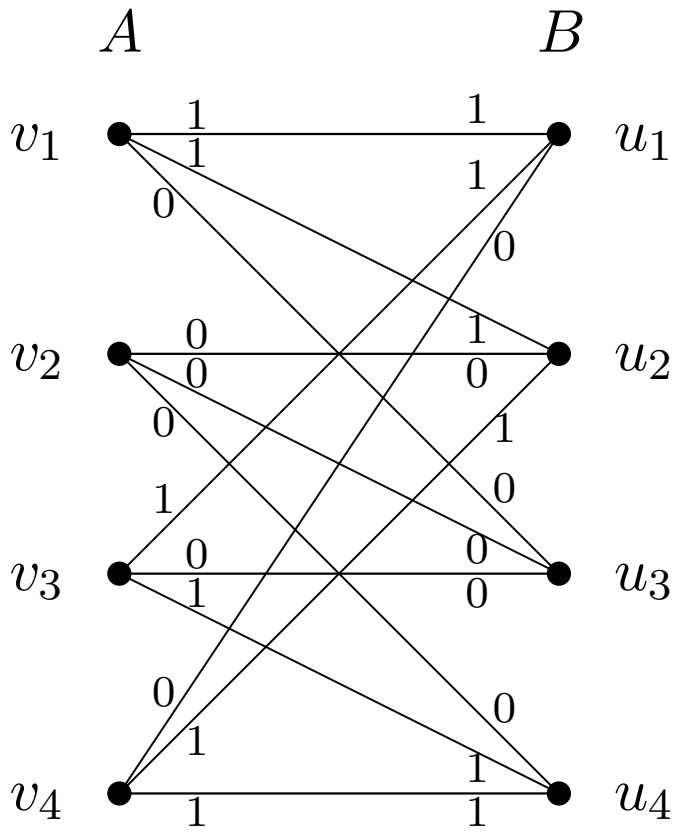

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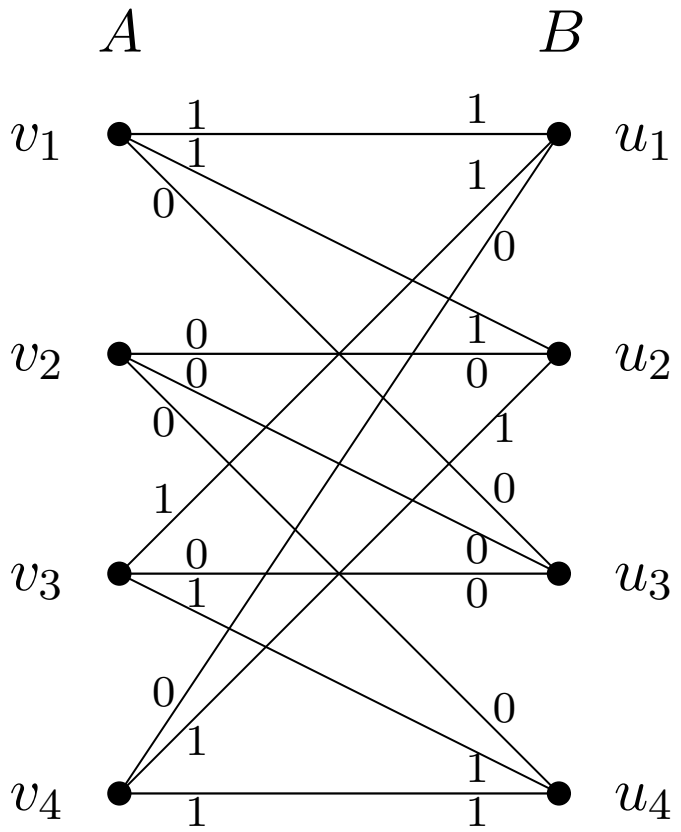
Thus,

$$(\,1\,1\,0\;\;0\,0\,0\;\;1\,0\,1\;\;0\,1\,1\,) \in \mathbb{C}.$$



Parameters of Expander Codes

The Code Rate

Each sub-code vertex contributes $\Delta - k$ parity-check equations. Thus,

$$N(1-\mathcal{R}) \leq (\Delta - k) \cdot 2n,$$

$$\Rightarrow \quad \mathcal{R} \geq 1 - (\Delta - k)\frac{2n}{N} = 1 - 2\frac{\Delta - k}{\Delta} = 2r - 1.$$

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Relative Minimum Distance

[Sipser Spielman '96]

$$D \geq N\left(\frac{\delta - \gamma_{\mathcal{G}}}{1 - \gamma_{\mathcal{G}}}\right)^2 .$$

Linear-time Decoder of Zémor

Input: Received word $\boldsymbol{y} = (y_e)_{e \in E}$.

Let $\boldsymbol{z} \leftarrow \boldsymbol{y}$.

For $t \leftarrow 1$ **to** m **do** {

Let X stand for A if t is odd, and for B otherwise.

Iteration t: For every $v \in X$ let $(\boldsymbol{z})_{E(v)} \leftarrow \mathcal{D}\left((\boldsymbol{z})_{E(v)}\right)$.

`/* Decoder for` $\mathcal{C}$ `*/`

}

Output: $\boldsymbol{z}$.

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- Using combination of Zémor and GMD decoding [Forney '66] the number of correctable errors becomes [Skachek Roth '03]:

$$J_{SR} \approx \frac{1}{2} \cdot N \left(\delta^2 - O(\gamma_{\mathcal{G}}) \right) \ .$$

Decoder Analysis

Let S_i be the set of corrupted vertices in A (in B) in i-th iteration for odd i (even i).

- Using expansion property, Zémor shows:

$$|S_{i+1}| \leq \rho |S_i|,$$

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- Using expansion property, Zémor shows:

$$|S_{i+1}| \leq \rho |S_i|,$$

  for some constant $\rho < 1$.
- There are only $O(\log n)$ iterations needed to correct all errors.

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- On next iteration, only vertices that have neighbors, which value was changed during the last iteration, could be unsatisfied.
- The amount of work to be done is:

$$\begin{aligned} \Delta|S_0|(1+\rho+\rho^2+\rho^3+\cdots+\rho^{m-1}) &= \Delta|S_0|\frac{1-\rho^m}{1-\rho} \\ &< \Delta|S_0|\frac{1}{1-\rho} = O(N)\,. \end{aligned}$$

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  - Edge set E of size $n\Delta$ such that every edge in E has one endpoint in A and one endpoint in B.
- Linear $[\Delta, k{=}r_A\Delta, \delta_A\Delta]$ and $[\Delta, r_B\Delta, \delta_B\Delta]$ codes $\mathcal{C}_A$ and $\mathcal{C}_B$, respectively, over $F = \mathrm{GF}(q)$.

Barg-Zémor's Construction

$\mathbb{C}$ is a linear code of length $|E|$ over F:

$$\mathbb{C} = \left\{ \boldsymbol{c} \in F^{|E|} \ : \ \begin{array}{c} (\boldsymbol{c})_{E(u)} \in \mathcal{C}_A \text{ for every } u \in A \text{ and} \\ (\boldsymbol{c})_{E(u)} \in \mathcal{C}_B \text{ for every } u \in B \end{array} \right\},$$

where $(\boldsymbol{c})_{E(u)}$ = the sub-word of $\boldsymbol{c}$ that is indexed by the set of edges incident with u.

Example

Take $k = 2$, $\Delta = 3$, $n = 4$.
Let G_A and G_B be generating matrices of $\mathcal{C}_A$ and $\mathcal{C}_B$ (respectively) over $F = \mathrm{GF}(2^2) = \{0, 1, \alpha, \alpha^2\}$:

$$G_A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & 0 \end{pmatrix},$$

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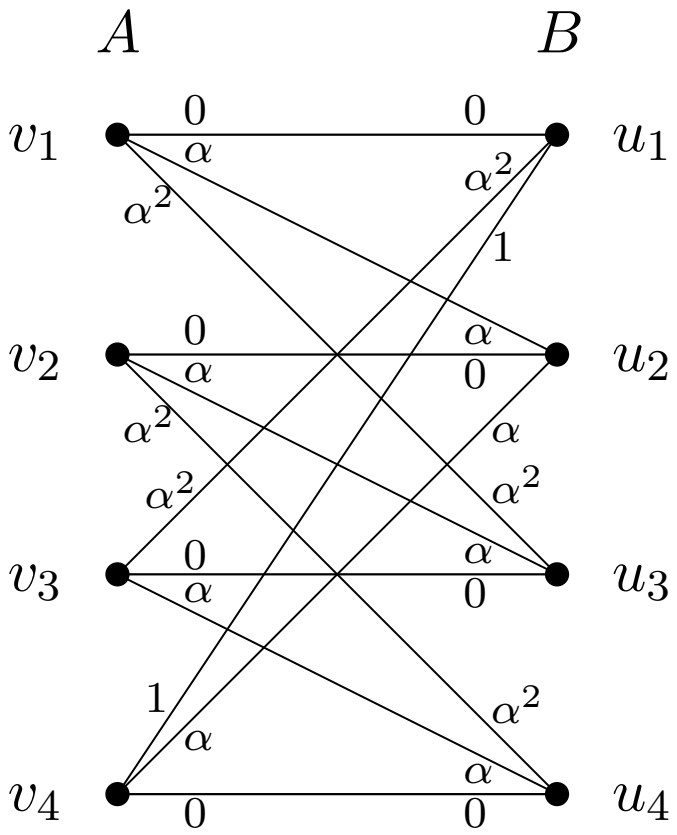

Results in [Barg Zémor '03]

In the presented construction:

- Codes $\mathcal{C}_A$ and $\mathcal{C}_B$ are random codes;
- Code $\mathbb{C}$ achieves the capacity of BSC under the linear-time expander iterative decoding. The decoding error probability decreases exponentially with the overall length.

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- Code $\mathbb{C}$ achieves the capacity of BSC under the linear-time expander iterative decoding. The decoding error probability decreases exponentially with the overall length.

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Further modified construction [Barg Zémor '03]

- The error exponent similar to the error exponent of concatenated codes [Forney '66].
- The trade-offs between the code rate and the minimum distance attain the Zyablov bound.

Analysis in [Barg Zémor '04]

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Lower bounds on the relative minimum distance

(i)

$$\delta(\mathcal{R}) \geq \frac{1}{4}(1-\mathcal{R})^2 \cdot \min_{\delta_{GV}((1+\mathcal{R})/2)<\mathsf{B}<\frac{1}{2}} \frac{g(\mathsf{B})}{\mathsf{H}_2(\mathsf{B})} \ ,$$

where the function $g(\mathsf{B})$ is defined in the next slides.

(ii)

$$\delta(\mathcal{R}) \geq \max_{\mathcal{R}\leq r\leq 1} \left\{ \min_{\delta_{GV}(r)<\mathsf{B}<\frac{1}{2}} \left(\delta_0(\mathsf{B},r) \cdot \frac{1-\mathcal{R}/r}{\mathsf{H}_2(\mathsf{B})} \right) \right\} \ ,$$

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Definition of the Function $g(\mathsf{B})$

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Let $\delta_{GV}(\mathcal{R}) = \mathsf{H}_2^{-1}(1-\mathcal{R})$, and let B_1 be the largest root of the equation

$$\mathsf{H}_2(\mathsf{B}) = \mathsf{H}_2(\mathsf{B}) \left(\mathsf{B} - \mathsf{H}_2(\mathsf{B}) \cdot \frac{\delta_{GV}(\mathcal{R})}{1-\mathcal{R}} \right) = -\left(\mathsf{B} - \delta_{GV}(\mathcal{R})\right) \cdot \log_2(1-\mathsf{B}) \ .$$

Moreover, let

$$a_1 = \frac{\mathsf{B}_1}{\mathsf{H}_2(\mathsf{B}_1)} - \frac{\delta_{GV}(\mathcal{R})}{\mathsf{H}_2(\delta_{GV}(\mathcal{R}))} \ ,$$

and

$$b_1 = \frac{\delta_{GV}(\mathcal{R})}{\mathsf{H}_2(\delta_{GV}(\mathcal{R}))} \cdot \mathsf{B}_1 - \frac{\mathsf{B}_1}{\mathsf{H}_2(\mathsf{B}_1)} \cdot \delta_{GV}(\mathcal{R})) \ .$$

Definition of the Function $g(\mathsf{B})$ (Cont.)

The function $g(\mathsf{B})$ is defined as

$$g(\mathsf{B}) = \begin{cases} \dfrac{\delta_{GV}(\mathcal{R})}{1-\mathcal{R}} & \text{if } \mathsf{B} \leq \delta_{GV}(\mathcal{R}) \\ \dfrac{\mathsf{B}}{\mathsf{H}_2(\mathsf{B})} & \text{if } \delta_{GV}(\mathcal{R}) \leq \mathsf{B} \text{ and } \mathcal{R} \leq 0.284 \\ \dfrac{a_1\mathsf{B}+b_1}{\mathsf{B}_1-\delta_{GV}(\mathcal{R})} & \text{if } \delta_{GV}(\mathcal{R}) \leq \mathsf{B} \leq \mathsf{B}_1 \text{ and } 0.284 < \mathcal{R} \leq 1 \\ \dfrac{\mathsf{B}}{\mathsf{H}_2(\mathsf{B})} & \text{if } \mathsf{B}_1 < \mathsf{B}_1 \leq 1 \text{ and } 0.284 < \mathcal{R} \leq 1 \end{cases}.$$

Definition of the Function $\delta_0(\mathsf{B}, r)$

The function $\delta_0(\mathsf{B}, r)$ is defined to be $\omega^{\star\star}(\mathsf{B})$ for $\delta_{GV}(r) \leq \mathsf{B} \leq \mathsf{B}_1$, where

$$\omega^{\star\star}(\mathsf{B}) = r\mathsf{B} + (1-r)\mathsf{H}_2^{-1}\left(1 - \frac{r}{1-r}\mathsf{H}_2(\mathsf{B})\right) ,$$

and B_1 is the only root of the equation

$$\delta_{GV}(r) = w^{\star}(\mathsf{B}) ,$$

where

$$w^{\star}(\mathsf{B}) = (1-r)\left((2^{\mathsf{H}_2(\mathsf{B})/\mathsf{B}}+1)^{-1} + \frac{\mathsf{B}}{\mathsf{H}_2(\mathsf{B})}\left(1 - \mathsf{H}_2\left((2^{\mathsf{H}_2(\mathsf{B})/\mathsf{B}}+1)^{-1}\right)\right)\right) .$$

For $\mathsf{B}_1 \leq \mathsf{B} \leq \frac{1}{2}$, the function $\delta_0(\mathsf{B}, r)$ is defined to be a tangent to the function $\omega^{\star\star}(\mathsf{B})$ drawn from the point $\left(\frac{1}{2}, \omega^{\star}(\frac{1}{2})\right)$.

Nearly-MDS Codes of Guruswami and Indyk

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- The alphabet size is

$$\exp\left\{O\left((\log(1/\epsilon))/\mathcal{R}\epsilon^4\right)\right\} \ .$$

List of Results

- Nearly-MDS linear-time encodable and decodable expander codes.
  - Improvement on the minimum distance in [Barg Zémor '03].
  - Improvement on the number of correctable errors over [Barg Zémor '03].
  - Nearly-MDS codes that improve on the alphabet size in [Guruswami Indyk '02].
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- For capacity-approaching codes: reduction of the decoding error probability (polynomial $\rightarrow$ exponential), while preserving linear-time (in the length) and polynomial (in the gap to capacity) decoding.
- Bounds on the minimum distance of expander codes with *weak* constituent codes.

Our Construction

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  $\mathbb{C}_\Phi$ is linear over F.
- The Barg-Zémor construction can be represented as a concatenated code with $\mathbb{C}_\Phi$ as the outer code and the inner code taken over a sub-field of F.

Example

Let $k = 2$, $\Delta = 3$, $n = 4$.
Pick $F = \mathrm{GF}(2)$ and $\Phi = F^2$.
Take $\mathcal{C}_A = \mathcal{C}_B =$ parity code over F.
Let $\mathcal{E}_A(\boldsymbol{x}) = \boldsymbol{x}G_A$,

$$G_A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

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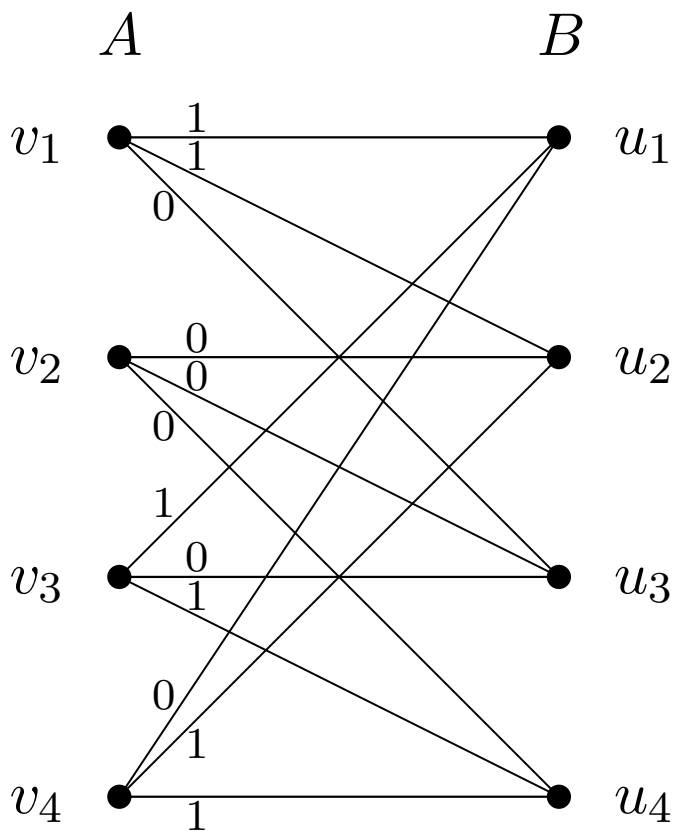

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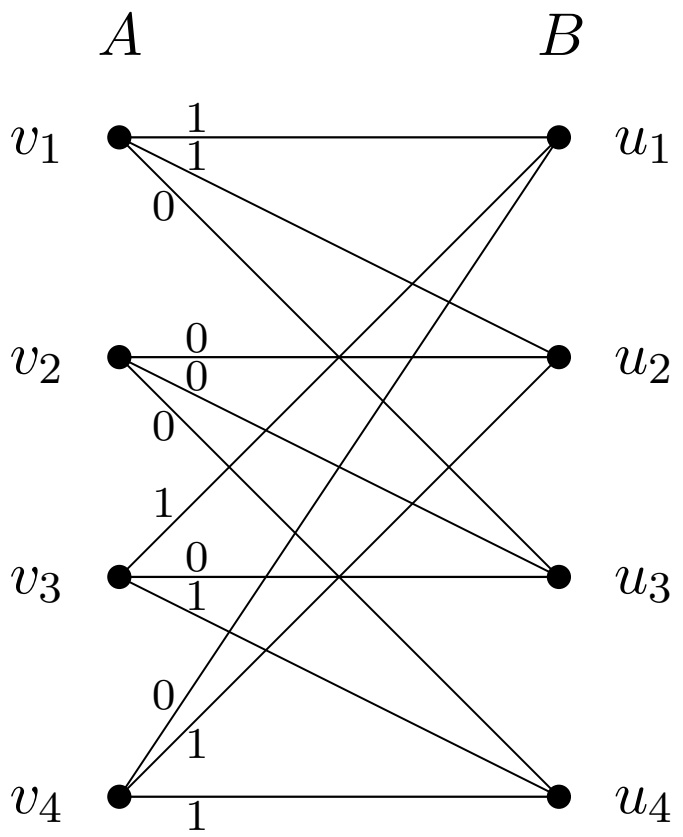
Then,

$$\begin{aligned} \boldsymbol{c} &= (\,1\,1\,0\,|\,0\,0\,0\,|\,1\,0\,1\,|\,0\,1\,1\,)\ , \\ \psi(\boldsymbol{c}) &= (\,\mathcal{E}_A^{-1}(110)\quad \mathcal{E}_A^{-1}(000) \\ &\qquad\quad \mathcal{E}_A^{-1}(101)\quad \mathcal{E}_A^{-1}(011)\,) \\ &= (\,11\ \ 00\ \ 10\ \ 01\,). \end{aligned}$$



Minimum Distance of $\mathbb{C}_\Phi$

- Let $\lambda_{\mathcal{G}}$ be the second largest eigenvalue of the adjacency matrix of $\mathcal{G}$, and let $\gamma_{\mathcal{G}} = \lambda_{\mathcal{G}}/\Delta$.

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- Let $\lambda_\mathcal{G}$ be the second largest eigenvalue of the adjacency matrix of $\mathcal{G}$, and let $\gamma_\mathcal{G} = \lambda_\mathcal{G}/\Delta$.
- Relative minimum distance of $\mathbb{C}_\Phi$:

$$\delta_\Phi \geq \frac{\delta_B - \gamma_\mathcal{G}\sqrt{\delta_B/\delta_A}}{1-\gamma_\mathcal{G}} \; ;$$

in particular, $\delta_\Phi \to \delta_B$ whenever $\gamma_\mathcal{G} \to 0$.

Alphabet size

- For any design rate $\mathcal{R} < 1$ and $\epsilon > 0$ we obtain arbitrarily long codes $\mathbb{C}_\Phi$ such that $R_\Phi > \mathcal{R}$ and $\delta_\Phi \geq 1 - \mathcal{R} - \epsilon$ (thus approaching the Singleton bound when $\epsilon \to 0$).

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- The alphabet size of $\mathbb{C}_\Phi$ is

$$\exp\left\{O\left((\log(1/\epsilon))/\epsilon^3\right)\right\} ,$$

compared with

$$\exp\left\{O\left((\log(1/\epsilon))/R\epsilon^4\right)\right\}$$

in [Guruswami Indyk '02].

Error-Erasure Decoder for $\mathbb{C}_\Phi$

Input: Received word $\boldsymbol{y} = (\boldsymbol{y}_u)_{u \in A}$ in $(\Phi \cup \{?\})^n$.

For $u \in A$ **do** $\quad (\boldsymbol{z})_{E(u)} \leftarrow \begin{cases} \mathcal{E}_A(\boldsymbol{y}_u) & \text{if } \boldsymbol{y}_u \in \Phi \\ ?? \cdots ? & \text{if } \boldsymbol{y}_u = ? \end{cases}$.

For $i \leftarrow 1, 2, \ldots, m$ **do** {

If i is even **then** $X \equiv A$, $\mathcal{D} \equiv \mathcal{D}_A$, **else** $X \equiv B$, $\mathcal{D} \equiv \mathcal{D}_B$.

For $u \in X$ **do** $(\boldsymbol{z})_{E(u)} \leftarrow \mathcal{D}((\boldsymbol{z})_{E(u)})$.

}

Output: $\psi(\boldsymbol{z})$ if $\boldsymbol{z} \in \mathbb{C}$ (and declare 'error' otherwise).

Error-Erasure Decoder for $\mathbb{C}_\Phi$ (Cont.)

- The algorithm makes use of a word $\boldsymbol{z} = (z_e)_{e \in E}$ over $F \cup \{?\}$, initialized by the contents of the received word $\boldsymbol{y}$ over $\Phi \cup \{?\}$.

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- Iterations $i = 1, 3, 5, \ldots$ use a decoder $\mathcal{D}_B : (F \cup \{?\})^\Delta \to \mathcal{C}_B$ that recovers correctly any pattern of θ errors and ν erasures, provided that $2\theta + \nu < \delta_B \Delta$.

Error-Correcting Capabilities

- The decoder corrects any pattern of μ errors and ρ erasures, provided that $\mu + \frac{1}{2}\rho < \alpha n$, where

$$\alpha = \frac{(\delta_B/2) - \gamma_{\mathcal{G}}\sqrt{\delta_B/\delta_A}}{1 - \gamma_{\mathcal{G}}} \; ;$$

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- The value of m is logarithmic in n.
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Applications

Pick $\mathbb{C}_\Phi$ to replace an MDS outer code in asymptotic concatenated code constructions. This leads to codes attaining:

- the Zyablov bound — our bound on the minimum distance and the number of correctable errors improves on the analysis of Barg-Zemor;

Applications

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- the Zyablov bound — our bound on the minimum distance and the number of correctable errors improves on the analysis of Barg-Zemor;
- the capacity of the memoryless symmetric channel with linear-time decoding and exponentially decaying error probability.

Linear Encoding

Using $\mathbb{C}_\Phi$ as a building block, we were able to construct a nearly-MDS code family that is *linear-time encodable and decodable*. The alphabet size of the new codes is again

$$\exp\left\{O\left((\log(1/\epsilon))/\epsilon^3\right)\right\} ,$$

compared with

$$\exp\left\{O\left((\log(1/\epsilon))/\mathcal{R}\epsilon^4\right)\right\}$$

in the construction of linear-time encodable and decodable code of [Guruswami Indyk '02].

Decoding over Non-bipartite Graph

- Recall that the relative minimum distance of the expander code based on a constituent code $\mathcal{C}$ of minimum distance δ is $\delta^2 + o_\Delta(1)$.

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Problem

Could the fraction of correctable errors become close to a half of the minimum distance when using a non-bipartite graph?

Reduction

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  - For each edge $e = a—b$ in $\mathcal{G}$, we let $\widehat{\mathcal{G}}$ contain two edges:

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- Define the code $\widehat{\mathbb{C}}$ of length $n\Delta$ over F using the graph $\widehat{\mathcal{G}}$:

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- Define mapping $\widehat{\varphi}$, such that for $\boldsymbol{y} \in F^{|E|}$,

$$\left(\widehat{\varphi}(\boldsymbol{y})\right)_{e_1} = \left(\widehat{\varphi}(\boldsymbol{y})\right)_{e_2} = y_e \ .$$

Decoding

Input: Received word $\boldsymbol{y} = (y_e)_{e \in E}$ in $F^{|E|}$.

Let $\boldsymbol{z} \leftarrow \widehat{\varphi}(\boldsymbol{y})$.

Let $\boldsymbol{z} \leftarrow \mathcal{D}(\boldsymbol{z})$.

Output: $\widehat{\varphi}^{-1}(\boldsymbol{z})$ if there exists $\boldsymbol{c} \in \mathbb{C}$ such that $\boldsymbol{z} = \widehat{\varphi}(\boldsymbol{c})$ (and declare 'error' otherwise).

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We define the linear code of length $|E|$ over F:

$$\mathbb{C} = \Big\{ \boldsymbol{c} \in F^{|E|} \ : \ (\boldsymbol{c})_{E(u)} \in \mathcal{C}_A \text{ for every } u \in A,$$

$$(\boldsymbol{c})_{E(u)} \in \mathcal{C}_1 \text{ for every } u \in B^1 \ , \text{ and } (\boldsymbol{c})_{E(u)} \in \mathcal{C}_2 \text{ for every } u \in B^2 \Big\}$$

(for $\eta = 0, 1$ or, alternatively, for $\mathcal{C}_1 = \mathcal{C}_2$, $\mathbb{C}$ coincides with the code discussed before).

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- More sophisticated analysis leads to the bound on the minimum distance that coincides with the bound in [Barg Zémor '04].

Conclusion

The presented codes are a generalization of the known expander codes (yet they are different), and have parameters as good as those of the best known expander codes. Could the generalized expander codes have better parameters than the known expander codes have?

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- Decoding time complexity is linear in the length and polynomial in $1/\varepsilon$.

LDPC Codes

- √ Attain the capacity of BEC [Luby Mitzenmacher Shokrollahi Spielman '01], [Oswald Shokrollahi '02], and a variety of other communication channels [Richardson Shokrollahi Urbanke '01].

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- × Decoding error probability is believed to decrease polynomially with the code length.
- ✓ Decoding complexity per bit:
  - Conjectured in [Khandekar McEliece '01] for any 'typical' channel as $O\left(\log(1/\pi) + 1/\varepsilon \cdot \log(1/\varepsilon)\right)$, where π is a decoded error probability.
  - LDPC over BEC: $O\left(\log(1/\varepsilon)\right)$ [Luby Mitzenmacher Shokrollahi Spielman '01], [Oswald Shokrollahi '02].
  - IRA over BEC: a bounded constant [Pfister Sason Urbanke '04].

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- ? Decoding time complexity
  - is linear in a code length;
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Our Approach

Memoryless BSC with crossover probability p, and capacity $C = 1 - \mathsf{H}_2(p)$.

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We consider concatenated codes $\mathbb{C}_{cont}$ with:

- A family of the nearly-MDS expander codes $\mathbb{C}_\Phi$ as outer codes.
- A 'typical' binary LDPC codes $\mathcal{C}_{in}$ of (constant for a fixed ε) length n_{in} as an inner code.

We derive a condition on the parameters of LDPC codes, sufficient for exponential decay of error probability of $\mathbb{C}_{cont}$.

Characteristics of 'Typical' LDPC Codes

Decoding Complexity

We assume that for LDPC (or other) codes over BSC it is given by:

$$O\left(n_{in}^{s} \cdot \frac{1}{\varepsilon^{r}}\right) .$$

(r is a positive constant, for LDPC codes essentially $s = 1$).

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Decoding error probability $\mathsf{Prob}_e(\mathcal{C}_{in})$

- No satisfying results on asymptotic behavior for LDPC codes over BEC or other channels.
- We obtain a sufficient condition to guarantee that $\mathsf{Prob}_e(\mathbb{C}_{cont})$ decreases exponentially.

Sufficient Condition

Notation $\mathcal{C}_{in}\,[\mathcal{R}_{in}, n_{in}]$ is for the code $\mathcal{C}_{in}$ of rate $\mathcal{R}_{in}$ and length n_{in}.

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Theorem

Consider a BSC, and let C be its capacity. Suppose that:

(i) *There exist constants $\mathsf{b} > 0$, $\vartheta > 0$, $\varepsilon_1 \in (0,1)$, such that for any ϵ, $0 < \epsilon < \varepsilon_1$, and for a sequence of alphabets $\{\Phi_i\}_{i=1}^{\infty}$ where the sequence $\{\log_2 |\Phi_i|\}_{i=1}^{\infty}$ is dense, there exists a family of codes $\mathbb{C}_\Phi$ of rate $1-\epsilon$ (with their respective decoders) that can correct a fraction $\vartheta\epsilon^{\mathsf{b}}$ of errors.*

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(ii) *There exist constants $\varepsilon_2 \in (0,1)$ and $h_0 > 0$, such that for any ϵ, $0 < \epsilon < \varepsilon_2$, the decoding error probability of a family of codes $\mathcal{C}_{in}$ satisfies*

$$\mathsf{Prob}_e\left(\mathcal{C}_{in}\left[(1-\epsilon)C,\ \frac{1}{\epsilon^{h_0}}\right]\right) < \epsilon^{\mathsf{b}}\ .$$

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Time Complexity

We show that over a BSC, when taking code $\mathbb{C}_{cont}$ with the outer code $\mathbb{C}_{\Phi}$ and the inner code $\mathcal{C}_{in}$ as assumed, the decoding time complexity of is given by

$$N \cdot \textsc{Poly}(1/\varepsilon) \ .$$

Decoding in [Barg Zémor '02] and [Barg Zémor '03]

We show that the codes in [Barg Zémor '02] and [Barg Zémor '03] cannot be tuned to have all three aforementioned properties.

Asymptotic Goodness

Definition

A family of codes $\{C_i\}_{i=0}^{\infty}$, where each C_i is a $[n_i, k_i, d_i]$ linear code, is said to be *asymptotically good* if it satisfies the following conditions:

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Problem Statement

How weak the constituent codes $\mathcal{C}_A$ and $\mathcal{C}_B$ could be such that the overall expander code will be asymptotically good?

Asymptotic Goodness – Some Answers

The known bound on the minimum distance of $\mathbb{C}$:

$$\delta \geq \frac{\delta_A \delta_B - \gamma_{\mathcal{G}} \sqrt{\delta_A \delta_B}}{1 - \gamma_{\mathcal{G}}} \, .$$

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[Barg Zémor '04]

If $d_A \geq 3$ and $d_B \geq 3$, then for the random bipartite graph $\mathcal{G}$ with probability close to 1, the resulting code $\mathbb{C}$ is good asymptotically.

Codes of Minimum Distance 2

Theorem

Let $\mathcal{C}_A$ and $\mathcal{C}_B$ be codes of minimum distance 2, and let $\mathcal{G}$ be any Δ-regular bipartite graph. Then, the minimum distance of such code $\mathbb{C}$ is bounded from above by

$$D \leq O\left(\log_{\Delta-1}(n)\right) .$$

Moreover, if the underlying graph $\mathcal{G}$ is a Ramanujan graph as in [Lubotsky Philips Sarnak '88] or [Margulis '88], then the minimum distance of $\mathbb{C}$ is bounded from below by

$$D \geq \frac{4}{3}\log_{\Delta-1}(2n) .$$

Simple Lower Bound

Theorem

Consider the code $\mathbb{C}$ with the constituent codes $\mathcal{C}_A$ and $\mathcal{C}_B$ of minimum distance $d_A \geq 2$ and $d_B \geq 2$, respectively, with the underlying graph $\mathcal{G}$ as in [Lubotsky Philips Sarnak '88] or [Margulis '88]. Then, its relative minimum distance is bounded from below by

$$D \geq (2n)^{1/3 \cdot \log_{\Delta-1}(d_A-1)(d_B-1)} - 1 \, .$$

Sufficient Condition

Theorem

Let $\mathcal{C}_A$ and $\mathcal{C}_B(u)$ (for every $u \in B$) be linear codes with the minimum distance $d_A = \delta_A \Delta$ and d_B, respectively. Let $\mathcal{G}$ be a bipartite (α, ζ)-expander such that the degree of every $u \in A$ is Δ. If

$$\frac{\delta_A}{\zeta + \delta_A - 1} < d_B \ ,$$

then the relative minimum distance of $\mathbb{C}$ is $\geq \alpha \delta_A$.

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- Construction of constrained LDPC (expander) codes.

Conclusion

Combining classical techniques from coding theory, like GMD-decoding, concatenated code analysis, algebraic coding, and others, with expander-based constructions leads to interesting results, such as constructions of provably linear-time encodable and decodable LDPC codes that have better parameters.