WEAKLY SINGULAR INTEGRAL EQUATIONS

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Contents

1. Introduction.

This course is devoted to the smoothness/singularities of the solutions of weakly singular integral equations of the second kind, and to piecewise polynomial collocation type methods to solve such equations. In Section 5 we prove theorems which characterise the boundary singulatities of the derivatives of a solution and undertake a change of variables that kills these singularities. This enables to justify some new collocation type methods probably not considered in the literature. Since two of these methods are based on the spline interpolation or quasi-interpolation, we undertake also a study of this approximation tool, see Section 8 identical to Section 4 in lecture notes [34].

It is assumed that the reader has taken an elementary course of functional analysis. In Section 2 we remind all or almost all that we need about functional spaces and operator theory.

In the main text we minimise the quoting to literure. Bibliographical remarks and further comments on the central results of the lectures can be found in the end of the lecture notes.

Besides elementary training exercises, Exercises and Problems contain some more serious problem settings for possible master and doctoral theses.

Let us recall standard designations used during the present notes:

$\mathbb{R} = (-\infty, \infty)$ is the set of real numbers, $\mathbb{R}_+ = [0, \infty)$,

$\mathbb{C}$ is the set of complex numbers,

$\mathbb{N} = \{1, 2, 3, ...\}$ is the set of natural numbers,

$\mathbb{Z} = \{... - 1, 0, 1, 2, ...\}$ is the set of integers, $\mathbb{Z}_+ = \mathbb{N}_0 = \{0, 1, 2, ...\}$,

$\varphi(t) \asymp \psi(t)$ as $t \to 0$ means that $\frac{\varphi(t)}{\psi(t}$ and $\frac{\psi(t)}{\varphi(t}$ are bounded as $t \to 0$,

$\varphi(t) \sim \psi(t)$ as $t \to 0$ means that $\frac{\varphi(t)}{\psi(t} \to 1$ as $t \to 0$.

Sometimes we use abbreviated designations of partial derivatives:

$\partial_x = \frac{\partial}{\partial x}$, $\partial_x^k = (\frac{\partial}{\partial x})^k$.

By c we denote a generic constant that may have different values by different occurrences.

2. Requisites.

2.1. Spaces. Below $\mathbf{K}$ stand for $\mathbb{R}$ or $\mathbb{C}$; its elements are called *scalars*.

A *vector space* X is a non-empty set with two operations – addition ($u, v \in X \mapsto u + v \in X$) and multiplication to scalars ($u \in X$, $\alpha \in \mathbf{K} \mapsto \alpha u \in X$) such that that the following axioms are satisfied:

$u + v = v + u, \quad u + (v + w) = (u + v) + w,$

$\alpha(u + v) = \alpha u + \alpha v, \quad (\alpha + \beta)u = \alpha u + \beta u, \quad (\alpha\beta)u = \alpha(\beta u), \quad 1u = u;$

there is an element $\mathbf{0}$ in X such that $u + \mathbf{0} = u$, $0u = \mathbf{0}$ for all $u \in X$.

The elements (called also vectors) $u_1, ..., u_n$ of a vector space X are *linearly dependent* if there are scalars $\alpha_1, ...,\alpha_n$ not all of which are zero such that $\alpha_1 u_1 + ... + \alpha_n u_n = \mathbf{0}$; otherwise $u_1, ..., u_n$ are called *linearly independent*. The *dimension* of X is n (dim$X = n$) if there are n linearly independent elements in X and every set of $n+1$ elements is linearly dependent; the dimension of X is infinite (dim$X = \infty$) if for any natural number n, there are n linearly independent elements in X. A *subspace* X_0 of a vector space X is a non-empty subset of X which itself is a vector space with respect to the operations of X (thus $u, v \in X_0 \Rightarrow u + v \in X_0$; $u \in X_0$, $\alpha \in \mathbf{K} \Rightarrow \alpha u \in X_0$). By span$S$, the linear span of a subset $S \subset X$, is denoted the set of all linear combinations $\sum_{k=1}^{n} \alpha_k u_k$ with $\alpha_k \in \mathbf{K}$, $u_k \in S$, $n = 1, 2, ...$; clearly, span$S$ is a subspace of X.

A *normed space* X is a vector space which is equipped with a norm $\| \cdot \| = \| \cdot \|_X$, a function from X into $\mathbb{R}_+$, such that

$\| u \| = 0$ if and only if $u = \mathbf{0}$;

$\| \alpha u \| = | \alpha | \| u \|$ $\forall \alpha \in \mathbf{K}$, $u \in X$;

$\| u + v \| \le \| u \| + \| v \|$ $\forall u, v \in X$.

A sequence $(u_n) \subset X$ *converges* to $u \in X$ (one writes $u_n \to u$ or $\lim u_n = u$) if $\| u_n - u \| \to 0$ as $n \to \infty$. A sequence $(u_n) \subset X$ is a *Cauchy sequence* if $\| u_m - u_n \| \to 0$ as $m, n \to \infty$. Every convergent sequence $(u_n) \subset X$ is Cauchy but the inverse is not true in general. A normed space

X is called *complete* if every Cauchy sequence of its elemets converges to an element of X. A complete normed space is called *Banach space.*

For $u_0 \in X$ and $r > 0$, the set $B(u_0, r) := \{u \in X : \| u - u_0 \| \le r\}$ is called (closed) *ball* of X with the centre u_0 and radius r. A set $S \subset X$ is called:

bounded if it is contained in a ball of X;

open if for any $u_0 \in S$ there is an $r > 0$ such that $B(u_0, r) \subset S$;

closed if $(u_n) \subset S$, $u_n \to u$ implies $u \in S$;

relatively compact if every sequence $(u_n) \subset S$ contains a convergent subsequence (with a limit in X not necessarily belonging to S);

compact if S is closed and relatively compact.

The *closure* $\overline{S}$ of a set $S \subset X$ is the smallest closed set containing S. A set $S \subset X$ is said to be dense in X if $\overline{S} = X$. A relatively compact set is bounded; in finite dimensional spaces, also the inverse is true.

The *Kolmogorov n-width* $d_n(S, X)$ of a set $S \subset X$ is defined by

$$d_n(S,X) = \inf_{X_n \subset X:\, \dim X_n = n} \ \sup_{u \in S} \ \inf_{u_n \in X_n} \| u - u_n \|_X$$

where the infimum is taken over all subspaces $X_n \subset X$ of dimension n.

Examples of Banach spaces of functions on a bounded interval:

$C[0,1]$ consists of all continuous functions $u: [0,1] \to \mathbf{K}$,

$$\| u \|_{C[0,1]} = \| u \|_\infty = \max_{0 \le x \le 1} | u(x) |;$$

$C^m[0,1]$ consists of all m $(m \ge 1)$ times continuously differentiable functions $u: [0,1] \to \mathbf{K}$,

$$\| u \|_{C^m[0,1]} = \max_{0 \le k \le m} \| u^{(k)} \|_\infty;$$

$L^p(0,1)$, $1 \le p < \infty$, consists of all (equivalence classes of) measurable functions $u: (0,1) \to \mathbf{K}$ such that $\| u \|_p < \infty$,

$$\| u \|_{L^p(0,1)} = \| u \|_p = \left(\int_0^1 | u(x) |^p \, dx \right)^{1/p};$$

$L^\infty(0,1)$ consists of all (equivalence classes of) measurable functions $u: (0,1) \to \mathbf{K}$ such that $\| u \|_\infty < \infty$,

$$\| u \|_{L^\infty(0,1)} = \| u \|_\infty = \sup_{0 < x < 1} | u(x) |$$

(more precisely, $\| u \|_\infty = \inf_{\mathrm{meas}(E)=0} \sup_{x \in (0,1) \setminus E} | u(x) |$ where the infimum is taken over all measurable subsets $E \subset (0,1)$ of measure 0);

$W^{m,p}(0,1)$, $m \in \mathbb{N}$, $1 \le p \le \infty$, called *Sobolev space*, consists of $m-1$ times continuously differentiable functions $u: (0,1) \to \mathbf{K}$ such that $u^{(k)} \in L^p(0,1)$ for $k = 0, ..., m$ (the derivatives are understood in the sense of distributions),

$$\| u \|_{W^{m,p}(0,1)} = \| u \|_{m,p} = \left(\sum_{k=0}^m \int_0^1 | u^{(k)}(x) |^p \, dx \right)^{1/p} \quad \text{for } 1 \le p < \infty,$$

$$\| u \|_{W^{m,\infty}(0,1)} = \| u \|_{m,\infty} = \max_{0 \le k \le m} \| u^{(k)} \|_\infty .$$

Examples of vector and Banach spaces of functions on $\mathbb{R}$:

$C(\mathbb{R})$ is vector space consisting of all continuous functions $u : \mathbb{R} \to \mathbf{K}$;

$BC(\mathbb{R})$ is Banach space consisting of all bounded continuous functions $u : \mathbb{R} \to \mathbf{K}$,

$$\| u \|_{BC(\mathbb{R})} = \| u \|_\infty = \sup_{x \in \mathbb{R}} | u(x) |;$$

$C^m(\mathbb{R})$ is vector space consisting of all m ($m \geq 1$) times continuously differentiable functions $u : \mathbb{R} \to \mathbf{K}$;

$L^\infty(\mathbb{R})$ is Banach space consisting of all (equivalence classes of) measurable functions $u : \mathbb{R} \to \mathbf{K}$ such that $\| u \|_\infty < \infty$,

$$\| u \|_\infty = \mathrm{vraisup}_{x \in \mathbb{R}} | u(x) | := \inf_{\mathrm{meas}(E)=0} \sup_{x \in \mathbb{R} \setminus E} | u(x) |;$$

$V^{m,p}(\mathbb{R})$, $1 \leq p \leq \infty$, is vector space consisting of all $m-1$ times continuously differentiable functions $u : \mathbb{R} \to \mathbf{K}$ such that $u^{(m)} \in L^p(\mathbb{R})$ (the derivatives are understood in the sense of distributions);

$W^{m,p}(\mathbb{R})$, $1 \leq p \leq \infty$, is Banach space (called Sobolev space) consisting of all $m-1$ times continuously differentiable functions $u : \mathbb{R} \to \mathbf{K}$ such that $u^{(k)} \in L^p(\mathbb{R})$ for $k = 0, 1, ..., m$ (the derivatives are understood in the sense of distributions),

$$\| u \|_{W^{m,p}(\mathbb{R})} = \| u \|_{m,p} = \left(\sum_{k=0}^{m} \int_{\mathbb{R}} | u^{(k)}(x) |^p \right)^{1/p} \quad \text{for } 1 \leq p < \infty,$$

$$\| u \|_{W^{m,\infty}(\mathbb{R})} = \| u \|_{m,\infty} = \max_{0 \leq k \leq m} \| u^{(k)} \|_\infty;$$

$C_{\mathrm{per}}(\mathbb{R})$ is Banach space consisting of all functions $u \in C(\mathbb{R})$ that are periodic with period 1 (shortly, 1-periodic),

$$\| u \|_{C_{\mathrm{per}}(\mathbb{R})} = \| u \|_\infty = \max_{0 \leq x \leq 1} | u(x) | = \sup_{x \in \mathbb{R}} | u(x) |;$$

$C^m_{\mathrm{per}}(\mathbb{R})$ is Banach space consisting of all 1-periodic m times continuously differentiable functions $u : \mathbb{R} \to \mathbf{K}$,

$$\| u \|_{C^m_{\mathrm{per}}(\mathbb{R})} = \| u \|_{m,\infty} = \max_{0 \leq k \leq m} \| u^{(k)} \|_\infty;$$

$L^p_{\mathrm{per}}(\mathbb{R})$, $1 \leq p \leq \infty$, is Banach space consisting of all 1-periodic functions $u : \mathbb{R} \to \mathbf{K}$ such that the restriction of u to $(0, 1)$ belongs to $L^p(0, 1)$,

$$\| u \|_{L^p_{\mathrm{per}}(\mathbb{R})} = \| u \|_p = \left(\int_0^1 | u(x) |^p \, dx \right)^{1/p} \quad \text{for } 1 \leq p < \infty,$$

$$\| u \|_{L^\infty_{\mathrm{per}}(\mathbb{R})} = \| u \|_\infty;$$

$W^{m,p}_{\mathrm{per}}(\mathbb{R})$, $1 \leq p \leq \infty$, is Banach space consisting of all functions $u \in W^{m,p}(\mathbb{R}) \cap C_{\mathrm{per}}(\mathbb{R})$,

$$\| u \|_{W^{m,p}_{\mathrm{per}}(\mathbb{R})} = \| u \|_{m,p} = \left(\sum_{k=0}^{m} \int_0^1 | u^{(k)}(x) |^p \, dx \right)^{1/p} \quad \text{for } 1 \leq p < \infty,$$

$$\| u \|_{W^{m,\infty}_{\mathrm{per}}(\mathbb{R})} = \| u \|_{m,\infty} = \max_{0 \leq k \leq m} \| u^{(k)} \|_\infty .$$

All these spaces are infinite dimensional. The space $C[0,1]$ is a closed subspace of $BC(0,1)$; both are closed subspaces of $L^\infty(0,1)$.

Theorem 2.1 (Arzela). *A set $S \subset C[0,1]$ is relatively compact in $C[0,1]$ if and only if the following two conditions are fulfilled*:

(i) *the functions $u \in S$ are uniformly bounded, i.e., there is a constant c such that $| u(x) | \leq c$ for all $x \in [0,1]$, $u \in S$*;

(ii) *the functions $u \in S$ are equicontinuous, i.e., for every $\varepsilon > 0$ there is a $\delta > 0$ such that $x_1, x_2 \in [0,1]$, $| x_1 - x_2 | \leq \delta$ implies $| u(x_1) - u(x_2) | \leq \varepsilon$ for all $u \in S$.*

2.2. Linear operators. Let X and Y be two vector spaces. *Operator* $A : X \to Y$ is a function defined on X and with values in Y; operator A is *called linear* if

$$A(u+v) = Au + Av, \quad A(\alpha u) = \alpha Au$$

for all $u, v \in X$ and $\alpha \in \mathbf{K}$.

Assume now that X and Y are normed spaces. An operator $A : X \to Y$ is said to be continuous if $\| u_n - u \|_X \to 0$ implies $\| Au_n - Au \|_Y \to 0$. A linear operator $A : X \to Y$ occurs to be continuous if and only if it is be bounded, i.e., if there is a constant c such that

$$\| Au \|_Y \leq c \| u \|_X$$

for all $u \in X$. The smallest constant c in this inequality is called the *norm* of A,

$$\| A \|_{X \to Y} = \sup\{\| Au \|_Y : \; u \in X, \; \| u \|_X = 1\}.$$

A sequence of linear bounded operators $A_n : X \to Y$ is said to be *pointwise convergent* (or strongly convergent) if the sequence $(A_n u)$ is convergent in Y for any $u \in X$.

Theorem 2.2 (Banach–Steinhaus). *Let X and Y be Banach spaces. A sequence of linear bounded operators $A_n : X \to Y$ converges pointwise if and only if the following two conditions are fulfilled*:

(i) *there is a constant c such that $\| A_n \|_{X \to Y} \leq c$ for all n;*

(ii) *there is a dense set $S \subset X$ such that the sequence $(A_n u)$ is convergent in Y for every $u \in S$.*

For pointwise convergent $A_n : X \to Y$, the limit operator $A : X \to Y$, $Au = \lim A_n u$, is linear and bounded.

2.3. Inverse operator. Let X and Y be Banach spaces and $A : X \to Y$ a linear operator. Introduce the subspaces

$$\mathcal{N}(A) = \{u \in X : \; Au = \mathbf{0}\} \subset X \quad \text{(the null space of } A),$$

$$\mathcal{R}(A) = \{f \in Y : \; f = Au, \; x \in X\} \subset Y \quad \text{(the range of } A).$$

If $\mathcal{N}(A) = \{\mathbf{0}\}$ then the inverse operator $A^{-1} : \mathcal{R}(A) \subset Y \to X$ exists on $\mathcal{R}(A)$, i.e., $A^{-1}Au = u$ $\forall u \in X$, $AA^{-1}f = f \;\; \forall f \in \mathcal{R}(A)$; clearly also A^{-1} is linear. If $\mathcal{N}(A) = \{\mathbf{0}\}$ and $\mathcal{R}(A) = Y$ then the inverse operator $A^{-1} : Y \to X$ is defined on whole Y; a nontrivial fact is that A^{-1} is bounded if A is. This is the essence of the following theorem.

Theorem 2.3 (Banach). *Let X and Y be Banach spaces and let $A : X \to Y$ be a linear bounded operator with $\mathcal{N}(A) = \{\mathbf{0}\}$ and $\mathcal{R}(A) = Y$. Then the inverse operator $A^{-1} : Y \to X$ is linear and bounded.*

Theorem 2.4 (Banach). *Let X and Y be Banach spaces and $A : X \to Y$ a linear bounded operator having the inverse $A^{-1} : Y \to X$. Assume that the linear bounded operator $B : X \to Y$ satisfies the condition*

$$\| B \|_{X \to Y} \| A^{-1} \|_{Y \to X} < 1.$$

Then $A + B : X \to Y$ *has the inverse* $(A+B)^{-1} : Y \to X$ *(defined on whole* Y*) and*

$$\| (A+B)^{-1} \|_{Y \to X} \leq \frac{\| A^{-1} \|_{Y\to X}}{1 - \| B \|_{X \to Y} \| A^{-1} \|_{Y \to X}}.$$

2.4. Linear compact operators. Let X , Y, U, V be Banach spaces. A linear operator $T : X \to Y$ is said to be *compact* if it maps bounded subsets of X into relatively compact subsets of Y. Equivalently, $T : X \to Y$ is compact if for every bounded sequence $(u_n) \subset X$, the sequence (Tu_n) contains a subsequence that converges in Y. Linear compact operators are bounded. A linear bounded finite dimensional operator (i.e., a linear bounded operator with finite dimensional range) is compact. For linear compact operators $T_1, T_2 : X \to Y$, $\alpha_1, \alpha_2 \in \mathbf{K}$, the operator $\alpha_1 T_1 + \alpha_2 T_2 : X \to Y$ is compact. For a linear compact operator $T : X \to Y$ and linear bounded operators $A : U \to X$ and $B : Y \to V$, the operators $TA : U \to Y$ and $BT : X \to V$ are compact.

Theorem 2.5. *Let* $T_n : X \to Y$, $n = 1, 2, ...$, *be linear compact operators,* $T : X \to Y$ *a linear bounded operator, and let* $\| T_n - T \|_{X \to Y} \to 0$ *as* $n \to \infty$. *Then* $T : X \to Y$ *is compact.*

Theorem 2.6. *Let* $T : X \to Y$ *be a linear compact operator and let the linear bounded operators* $B_n : Y \to V$ *converge pointwise to* $B : Y \to V$ as $n \to \infty$. *Then*

$$\| B_n T - BT \|_{X \to V} \to 0 \quad \text{as} \quad n \to \infty.$$

(Similar claim about $\| TA_n - TA \|_{U \to Y}$ is wrong in general.)

Denote by $I = I_X$ the identity operator in X, i.e., $Iu = u$ for every $u \in X$.

Theorem 2.7 (Fredholm alternative)**.** *Let* $T : X \to X$ *be a linear compact operator and let*

$$\mathcal{N}(I - T) = \{\mathbf{0}\}.$$

Then $I - T$ *has the bounded inverse* $(I - T)^{-1} : X \to X$.

Theorem 2.8. *Let* X *and* Y *be Banach spaces such that* $Y \subset X$, Y *is dense in* X *and* $\| u \|_X \leq c \| u \|_Y$ *for every* $u \in Y$. *Let* $T : X \to X$ *be a linear compact operator that maps* Y *into* Y, *and let also* $T : Y \to Y$ be *compact. Assume that the equation* $u = Tu + f$ *with given* $f \in Y$ *has a solution* $u \in X$. *Then* $u \in Y$.

The only claim $u \in Y$ of Theorem 2.8 will be trivial if we add the assumption that $\mathcal{N}(I - T) = \{\mathbf{0}\}$, since then by Theorem 2.7 equation $u = Tu + f$ is uniquely solvable in X as well as in Y. Actually this additional assumption is acceptable for our needs in the sequel so far as we do not treat eigenvalue problems.

Examples of linear compact integral operators. With the help of Theorem 2.1 it easy to see that the *Fredholm integral operator*

$$T : C[0,1] \to C[0,1], \ (Tu)(x) = \int_0^1 K(x,y) u(y) dy, \ 0 \leq x \leq 1,$$

is compact provided that its *kernel* $K(x, y)$ is continuous on the square $[0, 1] \times [0, 1]$. Similarly, the *Volterra integral operator*

$$T : C[0,1] \to C[0,1], \ (Tu)(x) = \int_0^x K(x,y) u(y) dy, \ 0 \leq x \leq 1,$$

is compact provided that the kernel $K(x, y)$ is continuous on the triangle $\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq 1\}$.

2.5. Differentiation of composite functions. Theorem 2.9 (Faa di Bruno). *Let u be an m times continuously differentiable function on an interval which contains the values of $\varphi \in C^m[0,1]$. Then the composite function $u(\varphi(x))$ is m times continuously differentiable on $[0,1]$ and the differentiation formula*

$$\left(\frac{d}{dx}\right)^j u(\varphi(x)) = \sum_{k_1+2k_2+...+jk_j=j} \frac{j!}{k_1!...k_j!} u^{(k_1+...+k_j)}(\varphi(x)) \left(\frac{\varphi'(x)}{1!}\right)^{k_1} ... \left(\frac{\varphi^{(j)}(x)}{j!}\right)^{k_j}$$

holds for $j = 1, ..., m$; the sum is taken over all non-negative integers $k_1, ..., k_j$ such that $k_1 + 2k_2 + ... + jk_j = j$.

3. Weakly singular integral operators.

3.1. Weakly singular kernels. Consider the integral operator T defined by its kernel function $K(x,y)$ via the formula

$$(Tu)(x) = \int_0^1 K(x,y)u(y)dy, \quad 0 \leq x \leq 1,$$

where u is taken from some set of functions, for example, from $C[0,1]$. In the literature, the weak singularity of the kernel K and of the corresponding operator T may have different senses. A tight understanding is that K has the form

$$K(x,y) = a(x,y) \mid x - y \mid^{-\nu} \tag{3.1}$$

where a is a continuous function on $[0,1] \times [0,1]$ and $0 < \nu < 1$. This kernel has the property

$$\sup_{0 \leq x \leq 1} \int_0^1 \mid K(x,y) \mid dy < \infty \tag{3.2}$$

often used to define the weak singularity in the wide sense: a kernel K is weakly singular if it is absolutely integrable w.r.t. y and satisfies (3.2). The kernels we will consider in the sequel are somewhere in the middle of these two extremal understandings of the weak singularity: we assume that K is continuous on $([0,1] \times [0,1]) \setminus \text{diag}$ and

$$\mid K(x,y) \mid \leq c_K(1+ \mid x - y \mid^{-\nu}) \quad \text{for} \quad (x,y) \in ([0,1] \times [0,1]) \setminus \text{diag} \tag{3.3}$$

where $\nu < 1$. Here diag means the diagonal of $\mathbb{R}^2$:

$$\text{diag} = \text{diag}(\mathbb{R}^2) = \{(x,y) \in \mathbb{R}^2 : x = y\}.$$

For instance, the kernels

$$K(x,y) = a(x,y) \log \mid x - y \mid, \quad K(x,y) = a(x,y) \mid x - y \mid^{-\nu} \log^k \mid x - y \mid$$

with $a \in C([0,1] \times [0,1])$ and many others are weakly singular in this sense.

3.2. The smoothness-singularity class $\mathcal{S}^{m,\nu}$ of kernels. We are interested in kernels that are C^m-smooth outside the diagonal. Introduce the following smoothness-singularity class $\mathcal{S}^{m,\nu}$ of kernels. For given $m \in \mathbb{N}_0$ and $\nu \in \mathbb{R}$, denote by $\mathcal{S}^{m,\nu} = \mathcal{S}^{m,\nu}(([0,1] \times [0,1]) \setminus \text{diag})$ the set of m times continuously differentiable kernels K on $([0,1] \times [0,1]) \setminus \text{diag}$ that satisfy there for all $k, l \in \mathbb{N}_0$, $k + l \leq m$, the inequality

$$\left| \left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^l K(x,y) \right| \leq c_{K,m} \left\{ \begin{array}{cc} 1, & \nu + k < 0 \\ 1+ \mid \log \mid x - y \mid\mid, & \nu + k = 0 \\ \mid x - y \mid^{-\nu-k}, & \nu + k > 0 \end{array} \right\}. \tag{3.4}$$

Note that for $k = l = 0$, $\nu > 0$, condition (3.4) coincides with (3.3). A kernel $K \in \mathcal{S}^{m,\nu}$ is weakly singular if $\nu < 1$. A kernel $K \in \mathcal{S}^{m,\nu}$ with $\nu < 0$ is bounded but its derivatives may have singularities on the diagonal; $\nu = 0$ corresponds to a logarithmically singular kernel. A consequene of (3.4) is that

$$\left|\left(\frac{\partial}{\partial y}\right)^k \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^l K(x,y)\right| \leq c'_{K,m} \left\{ \begin{array}{cc} 1, & \nu + k < 0 \\ 1 + \mid \log \mid x - y \mid\mid, & \nu + k = 0 \\ \mid x - y \mid^{-\nu-k}, & \nu + k > 0 \end{array} \right\}. \tag{3.5}$$

Indeed, using the equality $\partial_y = (\partial_x + \partial_y) - \partial_x$, we can obtain (3.5) from (3.4) first for $k = 1$, then for $k = 2$ etc.

Observe also that the differentiation $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^l$ does not influence on the r.h.s. of (3.4). This tells us that (3.4) is somehow related to kernels that depend on the difference $x - y$ of arguments. For example, kernel (3.1) belongs to $\mathcal{S}^{m,\nu}$ if $a \in C^m([0,1] \times [0,1])$; actually the condition on a can be weakened, see Exercise 4. A further important example is given by $K(x,y) = a(x,y) \log \mid x - y \mid$ with an $a \in C^m([0,1] \times [0,1])$ – this kernel K belongs to $\mathcal{S}^{m,0}$.

Lemma 3.1. (i) *If $K \in \mathcal{S}^{m,\nu}$ with an $m \geq 1$ then $\partial_x K(x,y)$ and $\partial_y K(x,y)$ belong to $\mathcal{S}^{m-1,\nu+1}$ whereas $(\partial_x + \partial_y)K(x,y)$ belongs to $\mathcal{S}^{m-1,\nu}$.*

(ii) *If $K \in \mathcal{S}^{m,\nu}$ then $(x - y)K(x,y)$ belongs to $\mathcal{S}^{m,\nu-1}$.*

Proof. These claims are elementary consequences of the definition of $\mathcal{S}^{m,\nu}$. □

3.3. Compactness of a weakly singular integral operator in $C[0,1]$. A weak singularity of the kernel implies that the corresponding integral operator is compact in the space $C[0,1]$. More precisely, the following statement holds true.

Lemma 3.2. *A kernel $K \in \mathcal{S}^{m,\nu}$ with $m \geq 0$, $\nu < 1$ defines a compact operator $T : L^\infty(0,1) \to C[0,1]$, hence also a compact operator $T : C[0,1] \to C[0,1]$ and a compact operator $T : L^\infty(0,1) \to L^\infty(0,1)$.*

Proof. Take a smooth "cutting" function $e : [0,\infty) \to \mathbb{R}$ satisfying the conditions $e(r) = 0$ for $0 \leq r \leq \frac{1}{2}$, $e(r) = 1$ for $r \geq 1$ and $0 \leq e(r) \leq 1$ for all $r \geq 0$. Define

$$K_n(x,y) = e(n \mid x - y \mid)K(x,y), \quad (x,y) \in [0,1] \times [0,1],$$

and

$$(T_n u)(x) = \int_0^1 K_n(x,y)u(y)dy, \quad n \in \mathbb{N}.$$

The kernels $K_n(x,y)$ are continuous on $[0,1] \times [0,1]$ – the possible diagonal singularity is "cut" off by the factor $e(n \mid x - y \mid)$, $K_n(x,y) = 0$ in a neighborhood of the diagonal. Hence the operators $T_n : L^\infty(0,1) \to C[0,1]$ are compact. Further, for $u \in L^\infty(0,1)$, $0 \leq x \leq 1$, we have

$$(Tu - T_n u)(x) = \int_0^1 [K(x,y) - K_n(x,y)]u(y)dy = \int_0^1 K(x,y)[1 - e(n \mid x - y \mid)]u(y)dy,$$

$$\mid (Tu - T_n u)(x) \mid \leq c_K \int_0^1 \mid x - y \mid^{-\nu} [1 - e(n \mid x - y \mid)]dy \parallel u \parallel_\infty$$

$$\leq c_K \int_{|x-y| \leq 1/n} \mid x - y \mid^{-\nu} dy \parallel u \parallel_\infty = 2c_K \int_0^{1/n} z^{-\nu} dz \parallel u \parallel_\infty = 2c_K \frac{(1/n)^{1-\nu}}{1-\nu} \parallel u \parallel_\infty$$

that implies $Tu \in C[0,1]$ as a uniform limit of $T_n u \in C[0,1]$, and

$$\parallel T - T_n \parallel_{L^\infty(0,1) \to C[0,1]} \leq 2c_K \frac{(1/n)^{1-\nu}}{1-\nu} \to 0 \text{ as } n \to \infty.$$

Thus T maps $L^\infty(0,1)$ into $C[0,1]$ and $T: L^\infty(0,1) \to C[0,1]$ is compact as a norm limit of compact operators $T_n: L^\infty(0,1) \to C[0,1]$, see Theorem 2.5. $\square$

4. Differentiation of weakly singular integrals.

First we recall a well known result about the closedness of the graph of the differentiation operator; the proof is left as an exercise.

Lemma 4.1. *Let* $v_n \in C^1(0,1)$ and $v_n \to v$, $v_n' \to w$ *uniformly on every closed subinterval* $[\delta, 1-\delta]$, $\delta > 0$. *Then* $v \in C^1(0,1)$ *and* $v' = w$.

We are ready to establish a differentiation formulae for weakly singular integrals with respect to a parameter.

Theorem 4.1. *Let* $g(x,y)$ *be a continuously differentiable function on* $((0,1)\times[0,1]) \setminus \text{diag}$ *satisfiying there the inequalities*

$$\mid g(x,y) \mid \le c \mid x-y \mid^{-\nu}, \quad \left|\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right) g(x,y)\right| \le c \mid x-y \mid^{-\nu}, \;\; \nu < 1. \tag{4.1}$$

Then the function $x \mapsto \int_0^1 g(x,y)dy$ *is continuously differentiable in* $(0,1)$ *and*

$$\frac{d}{dx}\int_0^1 g(x,y)dy = \int_0^1 \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right) g(x,y)dy + g(x,0) - g(x,1), \;\; 0<x<1. \tag{4.2}$$

Proof. For functions g that are continuously differentiable on $(0,1)\times[0,1]$ including the diagonal, formula (4.2) is obvious. Let g satisfy the conditons of the Lemma. Take a cutting function $e\in C^1[0,\infty)$ satisfying $e(r)=0$ for $0\le r\le \frac{1}{2}$, $e(r)=1$ for $r\ge 1$ and $0\le e(r)\le 1$ for all $r\ge 0$; we already used this cutting function in the proof of Lemma 3.2. Introduce the functions $g_n(x,y) = e(n \mid x-y \mid)g(x,y)$, $n=1,2,\ldots$. They are are continuously differentiable on $(0,1)\times[0,1]$ and (4.2) holds for them: denoting $v_n(x) = \int_0^1 e(n \mid x-y \mid)g(x,y)dy$, we have

$$v_n'(x) = \frac{d}{dx}\int_0^1 e(n \mid x-y \mid)g(x,y)dy = \int_0^1 e(n \mid x-y \mid)(\partial_x+\partial_y)g(x,y)dy$$

$$+e(nx)g(x,0) - e(n(1-x))g(x,1), \;\; 0<x<1.$$

We took into account that $(\partial_x+\partial_y)e(n \mid x-y \mid) = 0$. With the help of (4.1) we find that

$$v_n(x) \to \int_0^1 g(x,y)dy, \;\; v_n'(x) \to \int_0^1 (\partial_x+\partial_y)g(x,y)dy + g(x,0) - g(x,1) \;\text{ as }\; n\to\infty$$

uniformly on every closed subinterval $[\delta, 1-\delta]$, $\delta>0$. By Lemma 4.1, the function $\int_0^1 g(x,y)dy$ is continuously differentiable on $(0,1)$ and (4.2) holds true for it. $\square$

Theorem 4.2. *Let* $g(x,y)$ *be a continuously differentiable function for* $0\le y<x<1$ *satisfiying there the inequalities*

$$\mid g(x,y) \mid \le c(x-y)^{-\nu}, \quad \left|\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right) g(x,y)\right| \le c(x-y)^{-\nu}, \;\; \nu < 1. \tag{4.3}$$

Then the function $\int_0^x g(x,y)dy$ *is continuously differentiable in* $(0,1)$ *and*

$$\frac{d}{dx}\int_0^x g(x,y)dy = \int_0^x \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right) g(x,y)dy + g(x,0), \;\; 0<x<1. \tag{4.4}$$

Proof. This can be proved by the same idea as Theorem 4.1. Alternatively, we can derive (4.4) from (4.2) extending g by the zero values to $((0,1)\times[0,1]) \setminus \text{diag}$ and noticing that (4.3) implies (4.1) for the extended g. The details of the argument are proposed as an exercise. $\square$

5. Boundary singularities of the solution.

5.1. Boundary singularities of a solution to w.s.i.e. is a usual phenominon. Consider the integral equation

$$u(x) = \int_0^1 K(x,y)u(y)dy + f(x), \quad 0 \le x \le 1, \tag{5.1}$$

where $K \in \mathcal{S}^{m,\nu}$ with $m \ge 1$, $\nu < 1$, $f \in C^m[0,1]$. Let us demonstrate that in general $u \notin C^1[0,1]$. Indeed, supposing that $u \in C^1[0,1]$, we can differentiate (5.1) as an equality and we obtain on the basis of Theorem 4.2

$$u'(x) = \int_0^1 [(\partial_x + \partial_y)K(x,y)]u(y)]dy + \int_0^1 K(x,y)u'(y)dy$$

$$+K(x,0)u(0) - K(x,1)u(1) + f'(x).$$

Since the integral operators with the kernels $K(x,y)$ and $(\partial_x + \partial_y)K(x,y)$ are weakly singular and $u, u' \in C[0,1]$, the first two terms on the r.h.s. are on the basis of Theorem 3.2 continuous on $[0,1]$; the same is true for the term $f'(x)$. On the other hand, the term $K(x,0)u(0)$ has a singularity at $x = 0$ provided that $u(0) \ne 0$ and $K(x,0)$ really has a singularity allowed by inequality (3.3), and similarly the term $K(x,1)u(1)$ has a singularity at $x = 1$ if $u(1) \ne 0$ and $K(x,1)$ has a singularity. Thus the assumption $u \in C^1[0,1]$ leads to a contradiction if $K(x,0)$ or $K(x,1)$ is singular and $u(0) \ne 0$, $u(1) \ne 0$; these inequalities hold for most of $f \in C^m[0,1]$.

5.2. Weighted space $C^{m,\nu}(0,1)$. For $m \ge 1$, $\nu < 1$, denote by $C^{m,\nu}(0,1)$ the space of functions $f \in C^m(0,1)$ that satisfy the inequalities

$$| f^{(j)}(x) | \le c_f \left\{ \begin{array}{cc} 1 & j+\nu-1<0 \\ 1+ | \log \rho(x) |, & j+\nu-1=0 \\ \rho(x)^{-j-\nu+1}, & j+\nu-1>0 \end{array} \right\}, \quad 0<x<1, \ \ j=0,...,m, \tag{5.2}$$

where

$$\rho(x) = \min\{x, 1-x\}$$

is the distance from $x \in (0,1)$ to the boundary of the interval $(0,1)$. Introduce the weight functions

$$w_\lambda(x) = \left\{ \begin{array}{cc} 1, & \lambda<0 \\ 1/(1+ | \log \rho(x) |), & \lambda=0 \\ \rho(x)^\lambda & \lambda>0 \end{array} \right\}, \quad 0<x<1, \ \ \lambda \in \mathbb{R}.$$

Equipped with the norm

$$\| f \|_{C^{m,\nu}(0,1)} = \sum_{j=0}^{m} \sup_{0<x<1} w_{j+\nu-1}(x) \mid f^{(j)}(x) \mid,$$

$C^{m,\nu}(0,1)$ becomes a Banach space.

For $j = 0$ (5.2) yields $| f(x) | \le c_f$ telling us that a function $f \in C^{m,\nu}(0,1)$ is bounded on $(0,1)$. For $j = 1$ (5.2) yields

$$| f'(x) | \le c_f \rho(x)^{-\nu}, \ \ 0 < x < 1,$$

if $0 < \nu < 1$; for $\nu \le 0$ we have a less restrictive inequality. This implies $f' \in L^q(0,1)$ for a $q > 1$ such that $q\nu < 1$. Hence, for any $x_1, x_2 \in (0,1)$, we have

$$| f(x_1) - f(x_2) | = \left| \int_{x_1}^{x_2} f'(x)dx \right| \le \left(\int_{x_1}^{x_2} | f'(x) |^q \, dx \right)^{\frac{1}{q}} \left(\int_{x_1}^{x_2} dx \right)^{\frac{1}{q'}} = \| f' \|_{L^q} | x_1 - x_2 |^{\frac{1}{q'}}$$

where $\frac{1}{q}+\frac{1}{q'}=1$. We see that f is uniformly continuous on $(0,1)$. A uniformly continuous function f on $(0,1)$ has the boundary limits

$$f(0) := \lim_{x\to 0} f(x), \quad f(1) := \lim_{x\to 1} f(x),$$

hence f has a continuous extension to $[0,1]$. So we established a natural imbedding

$$C^{m,\nu}(0,1) \subset C[0,1], \quad m \geq 1, \ \nu < 1. \tag{5.3}$$

Moreover, with the help of Arzela theorem (Theorem 2.1) we obtain that the imbedding operator is compact.

If $\nu < 0$ we can apply the same argument for f' and so on. We obtain the following generalization of imbedding (5.3):

$$C^{m,\nu}(0,1) \subset C^{l}[0,1], \quad m \geq 1, \ \nu < 1, \ l = \min\{m-1, \mid \mathrm{int}\nu \mid\} \tag{5.4}$$

where intν, the integer part of ν, is the greatest integer not exceeding ν. Imbedding (5.4) is compact.

5.3. Compactness of integral operators in weighted spaces. The following theorem is crucial in the smoothness considerations for the solutions of (5.1). It has a simple formulation but not so simple proof.

Theorem 5.1. *Let $K \in \mathcal{S}^{m,\nu}$, $m \geq 1$, $\nu < 1$. Then the Fredholm integral operator T defined by $(Tu)(x) = \int_0^1 K(x,y)u(y)dy$ maps $C^{m,\nu}(0,1)$ into itself and $T : C^{m,\nu}(0,1) \to C^{m,\nu}(0,1)$ is compact.*

Proof. (i) *A technical formulation of what we have to prove.* First of all, taking a function $u \in C^{m,\nu}(0,1)$, we have to ensure that $Tu \in C^{m,\nu}(0,1)$, or equivalently, $Tu \in C^m(0,1)$ and $w_{i+\nu-1}D^iTu \in BC(0,1)$, $i = 0,...,m$, where $D = \frac{d}{dx}$ is the differentiation operator and $w_{i+\nu-1}$ are the weight functions introduced in Section 5.2. Second, we have to prove that the operators $w_{i+\nu-1}D^iT : C^{m,\nu}(0,1) \to BC(0,1)$, $i = 0,...,m$, are compact. Then for a given bounded sequence $(u_n)_{n\in\mathbb{N}} \subset C^{m,\nu}(0,1)$, the sequences $(w_{i+\nu-1}D^iTu_n)$, $i = 0,...,m$, are relatively compact in $BC(0,1)$, and repeatedly extracting convergent subsequences from the preceding subsequences, first for $j = 0$, after that for $j = 1$ etc., we can arrive to a subsequence determined by an infinite set $N' \subset \mathbb{N}$ such that all $(w_{i+\nu-1}D^iTu_n)_{n\in N'}$, $i = 0,...,m$, converge uniformly in $(0,1)$, or equivalently, the sequence $(Tu_n)_{n\in N'}$ converges in $C^{m,\nu}(0,1)$ that means the compactness of $T : C^{m,\nu}(0,1) \to C^{m,\nu}(0,1)$. (A fastidious reader can use Lemma 4.1 to ensure that the limits of $(D^iTu_n)_{n\in N'}$, $i = 0,...,m$, are consistent in $(0,1)$.)

For $i = 0$, we have $w_{i+\nu-1}(x) \equiv 1$, and $w_{i+\nu-1}D^iT = T : C^{m,\nu}(0,1) \subset C[0,1] \to C[0,1]$ is compact by Lemma 3.2. Thus *we have to prove the compactness of $w_{i+\nu-1}D^iT : C^{m,\nu}(0,1) \to BC(0,1)$ for $i = 1,...,m$.*

(ii) *Differentiation of Tu.* Take an arbitrary $u \in C^{m,\nu}(0,1)$ and a "cutting" function $e \in C^m[0,\infty)$ that satisfies

$$0 \leq e(r) \leq 1 \text{ for } r \geq 0, \ \ e(r) = 0 \text{ for } 0 \leq r \leq \frac{1}{2}, \ \ e(r) = 1 \text{ for } r \geq 1.$$

Fix an arbitrary point $x' \in (0,1)$ and denote $r' = \frac{1}{2}\rho(x') = \frac{1}{2}\min\{x', 1-x'\}$. For x satisfying $\mid x - x' \mid \leq \frac{1}{2}r'$, we split

$$\int_0^1 K(x,y)u(y)dy$$

$$= \int_0^1 e\left(\frac{\mid x-y \mid}{r'}\right) K(x,y)u(y)dy + \int_0^1 \left\{1 - e\left(\frac{\mid x-y\mid}{r'}\right)\right\} K(x,y)u(y)dy.$$

In the first integral on r.h.s., the diagonal singularity is cut off by the factor $e\left(\frac{|x-y|}{r'}\right)$; we may differentiate this integral m times under the integral sign. In the second integral on r.h.s., the coefficient function $1-e(|x-y|/r')$ vanishes for $y=0$ and $y=1$. Due to estimate (3.4) and Theorem 4.1, this integral is also differentiable; differentiation formula (4.2) yields

$$\frac{\partial}{\partial x}\int_0^1\left\{1-e\left(\frac{|x-y|}{r'}\right)\right\}K(x,y)u(y)dy$$

$$=\int_0^1\left\{1-e\left(\frac{|x-y|}{r'}\right)\right\}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\{K(x,y)u(y)\}\,dy,\quad |x-x'|\leq r'/2$$

(the boundary terms of (4.2) vanish in our case; $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)e(|x-y|/r')=0$). In its turn, the last integral can be differentiated in a similar manner. By repeated differentiation we obtain

$$(D^iTu)(x)=\int_0^1\left(\frac{\partial}{\partial x}\right)^i\left\{e\left(\frac{|x-y|}{r'}\right)K(x,y)\right\}u(y)dy$$

$$+\int_0^1\left\{1-e\left(\frac{|x-y|}{r'}\right)\right\}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^i\{K(x,y)u(y)\}\,dy,\quad |x-x'|\leq\frac{1}{2}r',\ \ 1\leq i\leq m.$$

Differentiating the product of functions under the integrals by the Leibnitz rule, setting after that $x=x'$ but writing again x instead of x', we arrive at the formula

$$(5.5)\qquad (D^iTu)(x)=\sum_{j=0}^{i}\binom{i}{j}\int_0^1 e_j(x,y)\left(\frac{\partial}{\partial x}\right)^{i-j}K(x,y)u(y)dy$$

$$+\sum_{j=0}^{i}\binom{i}{j}\int_0^1\left\{1-e\left(\frac{2|x-y|}{\rho(x)}\right)\right\}\left\{\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^{i-j}K(x,y)\right\}u^{(j)}(y)dy,\quad 0<x<1,$$

$$e_j(x,y):=\left[\left(\frac{\partial}{\partial x}\right)^j e\left(\frac{|x-y|}{r}\right)\right]_{r=\rho(x)/2},\quad 0\leq j\leq i,\ \ 1\leq i\leq m.$$

Multiplying both sides of (5.5) to the weight function $w_{i+\nu-1}(x)$, the result can be rewritten in the form

$$(5.6)\qquad w_{i+\nu-1}D^iTu=\sum_{j=0}^{i}\binom{i}{j}\left(T_{i,j}u+S_{i,j}(w_{j+\nu-1}D^ju)\right),\quad 1\leq i\leq m,$$

where for $j=0,...,i$, $i=1,...,m$,

$$(5.7)\qquad (T_{i,j}u)(x)=\int_0^1 w_{i+\nu-1}(x)e_j(x,y)\left(\frac{\partial}{\partial x}\right)^{i-j}K(x,y)u(y)dy,$$

$$(5.8)\quad (S_{i,j}v)(x)=\int_0^1\frac{w_{i+\nu-1}(x)}{w_{j+\nu-1}(y)}\left\{1-e\left(\frac{2|x-y|}{\rho(x)}\right)\right\}\left\{\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)^{i-j}K(x,y)\right\}v(y)dy.$$

Now the proof of the compactness of the operators $w_{i+\nu-1}D^iT: C^{m,\nu}(0,1)\to BC(0,1)$, $i=1,...,m$, can be reduced to the study of the mapping properties of $T_{i,j}$ and $S_{i,j}$. In (5.6),

$$\sup_{0<y<1} w_{j+\nu-1}(y)\,|(D^ju)(y)|\leq\|u\|_{C^{m,\nu}(0,1)},\quad j=0,...,i.$$

To prove compactness of the operators $w_{i+\nu-1}D^iT : C^{m,\nu}(0,1) \to BC(0,1)$, $i = 1, ..., m$, it is sufficient to establish that

$$(5.9) \qquad T_{i,j} : C^{m,\nu}(0,1) \to BC(0,1), \;\; i = 1, ..., m, \;\; j = 0, ..., i, \;\; \text{are compact},$$

$$(5.10) \qquad S_{i,j} : BC(0,1) \to BC(0,1), \;\; i = 1, ..., m, \;\; j = 0, ..., i, \;\; \text{are compact}.$$

(iii) *Proof of* (5.10). Denote by $H_{i,j}$ the kernel function of the integral operator $S_{i,j}$, $1 \le i \le m$, $0 \le j \le i$,

$$H_{i,j}(x,y) = \frac{w_{i+\nu-1}(x)}{w_{j+\nu-1}(y)} \left\{ 1 - e\left(\frac{2 \mid x-y \mid}{\rho(x)}\right)\right\} \left\{ \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^{i-j} K(x,y) \right\}.$$

Clearly $H_{i,j}$ is continuous on $([(0,1) \times (0,1))/\text{diag}$, actually even on $([0,1] \times [0,1])/\text{diag}$ since $1 - e\left(\frac{2|x-y|}{\rho(x)}\right) = 0$ for $\mid x - y \mid \ge \frac{\rho(x)}{2}$ that implies

$$\text{supp}\, H_{i,j} \subset \left\{ (x,y) \in [0,1] \times [0,1] : \mid x - y \mid \le \frac{\rho(x)}{2} \right\}.$$

Note that

$$\frac{\rho(x)}{2} \le \rho(y) \le \frac{3}{2}\rho(x) \;\; \text{for} \;\; \mid x-y \mid \le \frac{\rho(x)}{2}, \;\; \text{i.e., for } y \in \left(x - \frac{\rho(x)}{2}, x + \frac{\rho(x)}{2}\right),$$

hence similar relations hold for the weight functions: with some positive constant $d_j \ge 1$,

$$(5.11) \qquad \frac{1}{d_j} w_{j+\nu-1}(x) \le w_{j+\nu-1}(y) \le d_j w_{j+\nu-1}(x) \;\; \text{for} \;\; \mid x - y \mid \le \frac{\rho(x)}{2}, \;\; j = 0, ..., m.$$

Thus

$$\mid H_{i,j}(x,y) \mid \le d_j \frac{w_{i+\nu-1}(x)}{w_{j+\nu-1}(x)} \left| \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^{i-j} K(x,y) \right| \le d_j \left| \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^{i-j} K(x,y) \right|.$$

Now (3.4) tells us that the kernels $H_{i,j}$, $i = 1, ..., m$, $j = 0, ..., i$, are weakly singular, and (5.10) holds due to Lemma 3.2.

(iv) *Proof of* (5.9): *case* $0 < \nu < 1$. The following argument holds also for $\nu = 0$, except for $j = i$. Denote by $K_{i,j}(x,y)$ the kernel of the integral operator $T_{i,j}$,

$$K_{i,j}(x,y) = w_{i+\nu-1}(x) e_j(x,y) \partial_x^{i-j} K(x,y), \;\; 1 \le i \le m, \;\; 0 \le j \le i.$$

Observe that

$$\text{supp}\, e_0 \subset \{(x,y) \in [0,1] \times [0,1] : \mid x - y \mid \ge \rho(x)/4\},$$

whereas for $j > 0$ the support of e_j is smaller,

$$(5.12) \qquad \text{supp}\, e_j \subset \{(x,y) \in [0,1] \times [0,1] : \;\rho(x)/4 \le \mid x-y \mid \le \rho(x)/2\}, \;\; 0 < j \le m.$$

Further,

$$\mid e_j(x,y) \mid \le c_j (\rho(x)/2)^{-j}, \;\; c_j := \max_{r \ge 0} \mid e^{(j)}(r) \mid, \;\; 0 \le j \le m.$$

Hence

$$(5.13) \qquad \mid e_j(x,y) \mid \le c \mid x - y \mid^{-j}, \;\; 0 \le j \le m,$$

(for $j=0$ this inequality is trivially true). For $0<\nu<1$, $0\le j\le i$, and for $\nu=0$, $0\le j<i$, (3.4) yields $\mid \partial_x^{i-j}K(x,y)\mid\le c\mid x-y\mid^{-(i-j)-\nu}$, and we obtain

$$\mid K_{i,j}(x,y)\mid\le cw_{i+\nu-1}(x)\mid x-y\mid^{-i-\nu},$$

$$\int_0^1 \mid K_{i,j}(x,y)\mid dy\le cw_{i+\nu-1}(x)\int_{\{y:\,|x-y|\ge\frac{\rho(x)}{4}\}}\mid x-y\mid^{-i-\nu}dy \tag{5.14}$$

$$\le 2cw_{i+\nu-1}(x)\int_{\frac{\rho(x)}{4}}^1 z^{-i-\nu}dz\le c'w_{i+\nu-1}(x)\left\{\begin{array}{ll}\rho(x)^{-i-\nu+1}, & i+\nu>1\\ 1+\mid\log\rho(x)\mid, & i+\nu=1\end{array}\right\}=c',\quad 0<x<1.$$

This means that for $0<\nu<1$, $0\le j\le i$, and for $\nu=0$, $0\le j<i$, the operators $T_{i,j}:C[0,1]\to BC(0,1)$ are bounded that together with the compact imbedding $C^{m,\nu}(0,1)\subset C[0,1]$ implies the compactness of $T_{i,j}:C^{m,\nu}(0,1)\to BC(0,1)$. Thus (5.9) holds true for $0<\nu<1$, whereas for $\nu=0$ we yet have to prove that also $T_{i,i}:C^{m,\nu}(0,1)\to BC(0,1)$ is compact.

(v) ***Proof of*** (5.9): *case* $\nu=0$. To prove (5.9) for $\nu=0$, it remains to establish that the operators $T_{i,i}:C^{m,\nu}(0,1)\to BC(0,1)$, $1\le i\le m$, are compact. Let us try to follow the estimation idea of the proof part (iv): now (for $j=i$) we have by (3.4) $\mid\partial_x^{i-j}K(x,y)\mid\le c(1+\log\mid x-y\mid)$,

$$\mid K_{i,i}(x,y)\mid\le cw_{i-1}(x)\mid x-y\mid^{-i}(1+\log\mid x-y\mid),$$

and instead of (5.14) we obtain

$$\int_0^1\mid K_{i,i}(x,y)\mid dy\le cw_{i-1}(x)\int_{\{y:\,|x-y|\ge\frac{\rho(x)}{4}\}}\mid x-y\mid^{-i}(1+\log\mid x-y\mid)dy$$

$$\le c'\left\{\begin{array}{ll}1/(1+\mid\log\rho(x)\mid), & i=1\\ \rho(x)^{i-1}, & i>1\end{array}\right\}\rho(x)^{-i+1}(1+\mid\log\rho(x)\mid)=c'\left\{\begin{array}{ll}1, & i=1\\ 1+\mid\log\rho(x)\mid, & i>1\end{array}\right\}.$$

We see that $T_{i,i}:BC(0,1)\to BC(0,1)$ is till bounded (and hence $T_{i,i}:C^{m,\nu}(0,1)\to BC(0,1)$ is compact) for $i=1$ but not for $i>1$. To prove the compactness of $T_{i,i}:C^{m,\nu}(0,1)\to BC(0,1)$ for $i\ge 2$ we need new ideas. Observe that

$$e_i(x,y):=-\partial_y e_{i-1}(x,y)$$

and integrate in (5.7) by parts. Clearly $e_{i-1}(x,0)=e_{i-1}(x,1)=0$, so we obtain

$$(T_{i,i}u)(x)=\int_0^1 w_{i-1}(x)\left[-\partial_y e_{i-1}(x,y)\right]K(x,y)u(y)dy$$

$$=\int_0^1 w_{i-1}(x)e_{i-1}(x,y)[\partial_y K(x,y)]u(y)dy+\int_0^1 w_{i-1}(x)e_{i-1}(x,y)K(x,y)u'(y)dy,$$

$$T_{i,i}=T'_{i,i}+T''_{i,i}.$$

Due to (5.13) and (3.5), the kernel of the operator $T'_{i,i}$ has the estimate

$$w_{i-1}(x)\mid e_{i-1}(x,y)[\partial_y K(x,y)]\mid\le cw_{i-1}(x)\mid x-y\mid^{-i+1}\mid x-y\mid^{-1}=c\rho(x)^{i-1}\mid x-y\mid^{-i},$$

and similarly as in (5.14) we obtain that $T'_{i,i}:BC(0,1)\to BC(0,1)$ is bounded, hence $T'_{i,i}:C^{m,\nu}(0,1)\to BC(0,1)$ is compact. To prove the compactness of $T''_{i,i}:C^{m,\nu}(0,1)\to BC(0,1)$, we present it in the form

$$(T''_{i,i}u)(x)=\int_0^1\frac{w_{i-1}(x)}{w_0(y)}e_{i-1}(x,y)K(x,y)[w_0(y)u'(y)]dy.$$

Here $\| w_0 u' \|_\infty \leq \| u \|_{C^{m,\nu}(0,1)}$, so it suffices to observe that

$$T'''_{i,i} : BC(0,1) \to BC(0,1) \text{ is compact for } (T'''_{i,i}v)(x) := \int_0^1 \frac{w_{i-1}(x)}{w_0(y)} e_{i-1}(x,y)K(x,y)v(y)dy$$

as for an integral operator with a weakly singular kernel. Indeed, taking into account (5.11) and (5.12) we can estimate the kernel of $T'''_{i,i}$ as follows:

$$\frac{w_{i-1}(x)}{w_0(y)} \mid e_{i-1}(x,y)K(x,y) \mid \leq d_0 \rho(x)^{i-1}(1+ \mid \log \rho(x) \mid) \mid x-y \mid^{-i+1} (1+ \mid \log \mid x-y \mid\mid)$$

$$\leq c(1+ \mid \log \mid x-y \mid\mid)^2, \quad (x,y) \in \operatorname{supp} e_{i-1}.$$

This completes the proof of the Theorem in the most important case $0 \leq \nu < 1$. In the case of $\nu < 0$, (5.9) could be established by same ideas; more terms and more times must be integrated by parts in (5.7). We do not go into details which are somewhat inconveniet. Instead we demonstrate another idea, how the proof of the Theorem for $\nu < 0$ can be obtained from the case $0 \leq \nu < 1$.

(vi) *Extending the proof for negative ν.* Let $\nu \in [-1,0)$. Then $u \in C^{m,\nu}(0,1)$ is continuously differentiable on $[0,1]$ and Theorem 4.1 yields

$$\frac{d}{dx}\int_0^1 K(x,y)u(y)dy = \int_0^1 (\partial_x + \partial_y)[K(x,y)u(y)]dy + u(0)K(x,0) - u(1)K(x,1)$$

$$= \int_0^1 [(\partial_x + \partial_y)K(x,y)]u(y)]dy + \int_0^1 K(x,y)u'(y)dy + u(0)K(x,0) - u(1)K(x,1),$$

or

$$DTu = T^{(1)}u + TDu + R_1 u,$$

$$w_{i+\nu-1}D^iTu = w_{i+\nu-1}D^{i-1}T^{(1)}u + w_{i+\nu-1}D^{i-1}TDu + w_{i+\nu-1}D^{i-1}R_1u, \quad 1 \leq i \leq m,$$

where $T^{(1)}$ is the integral operator with the kernel $K^{(1)}(x,y) = (\partial_x + \partial_y)K(x,y)$ and $(R_1u)(x) = u(0)K(x,0) - u(1)K(x,1)$ is a finite dimensional (a two dimensional) operator. For $u \in C^{m,\nu}(0,1)$ it holds $Du \in C^{m-1,\nu+1}(0,1)$ with $\nu+1 \in [0,1)$,

$$\| Du \|_{C^{m-1,\nu+1}(0,1)} \leq \| u \|_{C^{m,\nu}(0,1)}.$$

The operator $w_{i+\nu-1}D^{i-1}T = w_{(i-1)+(\nu+1)-1}D^{i-1}T : C^{m-1,\nu+1}(0,1) \to BC(0,1)$ is compact on the basis of (i)–(v), hence $w_{i+\nu-1}D^{i-1}TD : C^{m,\nu}(0,1) \to BC(0,1)$ is compact. The same is true for $w_{i+\nu-1}D^{i-1}T^{(1)} : C^{m,\nu}(0,1) \to BC(0,1)$ since $K^{(1)}(x,y)$ satisfies same inequalities as $K(x,y)$. Finally, the compactness of $w_{i+\nu-1}D^{i-1}R_1 : C^{m,\nu}(0,1) \to BC(0,1)$ is a consequence of the boundedness of this finite dimensional operator. As a summary, we obtain that $w_{i+\nu-1}D^iT : C^{m,\nu}(0,1) \to BC(0,1)$ is compact for $i = 1,...,m$. This implies the claim of the Theorem for $\nu \in [-1,0)$, see (i).

Having stablished the compactness of $w_{i+\nu-1}D^iT : C^{m,\nu}(0,1) \to BC(0,1)$ for $\nu \in [-1,0)$, we in similar way extend the claim for $\nu \in [-2,-1)$ etc. □

5.4. Smoothness and singularities of the solutions. We are ready to present the basic result about the smoothness and singularities of the solutions to weakly singular integral equations.

Theorem 5.2. *Let $K \in \mathcal{S}^{m,\nu}$, $f \in C^{m,\nu}(0,1)$, $m \geq 1$, $\nu < 1$, and let $u \in C[0,1]$ be a solution of equation* (5.1). *Then $u \in C^{m,\nu}(0,1)$.*

Proof. By Lemma 3.2, the integral operator $T : C[0,1] \to C[0,1]$ is compact. By Theorem 5.1 T maps $C^{m,\nu}(0,1)$ into itself and $T : C^{m,\nu}(0,1) \to C^{m,\nu}(0,1)$ is compact. With $X = C[0,1]$ and $Y = C^{m,\nu}(0,1)$, Theorem 2.8 yields that $u \in C^{m,\nu}(0,1)$ for the solutions $u \in C[0,1]$ of (5.1). □

5.5. A smoothing change of variables. The derivatives of a solution $u \in C^{m,\nu}(0,1)$ to equation (5.1) may have boundary singularities. Now we undertake a change of variables that kills the singularities – the solution of the transformed equation will be C^m-smooth on $[0,1]$ including the boundary points.

Let $\varphi : [0,1] \to [0,1]$ be a smooth strictly increasing function such that $\varphi(0) = 0$, $\varphi(1) = 1$. Introducing the change of variables

$$x = \varphi(t), \;\; y = \varphi(s),$$

equation (5.1) takes the form

$$v(t) = \int_0^1 K_\varphi(t,s)v(s)ds + f_\varphi(t), \quad 0 \le t \le 1, \tag{5.15}$$

where

$$f_\varphi(t) := f(\varphi(t)), \;\; K_\varphi(t,s) := K(\varphi(t), \varphi(s))\varphi'(s);$$

the solutions of equations (5.15) and (5.1) are in the relations

$$v(t) = u(\varphi(t)), \;\; u(x) = v(\varphi^{-1}(x)).$$

Under conditions we have set on $\varphi : [0,1] \to [0,1]$, the inverse function $\varphi^{-1} : [0,1] \to [0,1]$ exists and is continuous.

Theorem 5.3. *Given $m \ge 1$, $\nu < 1$, let $p \in \mathbb{N}$ satisfy*

$$p > \left\{ \begin{array}{ll} m, & \nu \le 0 \\ \frac{m}{1-\nu}, & 0 < \nu < 1 \end{array} \right\}. \tag{5.16}$$

Let $\varphi \in C^p[0,1]$ satisfy the conditions $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi'(t) > 0$ for $0 < t < 1$ and

$$\varphi^{(j)}(0) = \varphi^{(j)}(1) = 0, \;\; j = 1, ..., p-1, \;\; \varphi^{(p)}(0) \ne 0, \;\; \varphi^{(p)}(1) \ne 0. \tag{5.17}$$

Then the following claims hold true.

(i) *For $f \in C^{m,\nu}(0,1)$, the function $f_\varphi(t) = f(\varphi(t))$ belongs to $C^m[0,1]$ and*

$$f_\varphi^{(j)}(0) = f_\varphi^{(j)}(1) = 0, \quad j = 1, ..., m. \tag{5.18}$$

(ii) *For $K \in \mathcal{S}^{0,\nu}$, the kernel $K_\varphi(t,s) = K(\varphi(t), \varphi(s))\varphi'(s)$ belongs to $\mathcal{S}^{0,\nu}$ and hence defines a compact integral operator*

$$T_\varphi : C[0,1] \to C[0,1], \;\; (T_\varphi v)(t) = \int_0^1 K_\varphi(t,s)v(s)ds.$$

Proof. (i) Clearly $f_\varphi \in C^m(0,1)$, thus claim (i) concerns only the boundary behaviour of f_φ. Due to the imbedding (5.3), after the extension of f_φ by continuity to pints 0 and 1, we have $f_\varphi \in C[0,1]$. It remains to show that

$$f_\varphi^{(j)}(0) := \lim_{t\to 0} f_\varphi^{(j)}(t) = 0, \;\; f_\varphi^{(j)}(1) := \lim_{t\to 1} f_\varphi^{(j)}(t) = 0, \quad j = 1, ..., m.$$

We establish these relations for $t \to 0$; for $t \to 1$ the argument is similar. By the formula of Faa di Bruno (see Theorem 2.9),

$$f_\varphi^{(j)}(t) = \sum_{k_1+2k_2...+jk_j=j} c_{k_1,...,k_j} f^{(k_1+...+k_j)}(\varphi(t))\, \varphi'(t)^{k_1} ... \varphi^{(j)}(t)^{k_j}, \quad 0 < t < 1,$$

with some constants $c_{k_1,...,k_j}$. In a vicinity of 0, the inclusion $f \in C^{m,\nu}(0,1)$ yields

$$| f^{(k)}(\varphi(t)) | \leq c \left\{ \begin{array}{cc} 1, & k < 1-\nu \\ 1+ | \log \varphi(t) |, & k = 1-\nu \\ \varphi(t)^{1-\nu-k}, & k > 1-\nu \end{array} \right\}.$$

Due to (5.17),

$$\varphi(t) \asymp t^p, \;\; \varphi^{(i)}(t) \asymp t^{p-i} \;\text{ as }\; t \to 0, \;\; i = 0, ..., p,$$

hence

$$| f_\varphi^{(j)}(t) |$$

$$\leq c \sum_{k_1+2k_2...+jk_j=j} \left\{ \begin{array}{cc} 1, & k_1+...+k_j < 1-\nu \\ 1+ | \log t |, & k_1+...+k_j = 1-\nu \\ t^{p(1-\nu-k_1-...-k_j)}, & k_1+...+k_j > 1-\nu \end{array} \right\} t^{(p-1)k_1} t^{(p-2)k_2} ... t^{(p-j)k_j}$$

$$= c \sum_{k_1+2k_2...+jk_j=j} \left\{ \begin{array}{cc} t^{p(k_1+...+k_j)-j}, & k_1+...+k_j < 1-\nu \\ (1+ | \log t |) t^{p(k_1+...+k_j)-j}, & k_1+...+k_j = 1-\nu \\ t^{p(1-\nu)-j}, & k_1+...+k_j > 1-\nu \end{array} \right\}, \;\; 1 \leq j \leq m.$$

For $\nu > 0$, we have $k_1 + ... + k_j > 1 - \nu$ and $| f_\varphi^{(j)}(t) | \leq ct^{p(1-\nu)-j}$ in accordance to lower line. For $\nu = 0$, there is one combination of $k_1, ..., k_j$ such that $k_1+2k_2...+jk_j = j$ and $k_1+...+k_j = 1-\nu$, namely $k_1 = ... = k_{j-1} = 0$, $k_j = 1$, yielding $| f_\varphi^{(j)}(t) | \leq ct^{p-j}(1+ | \log t |)$. For $\nu < 0$, the smallest exponent $p(k_1 + ... + k_j) - j$ with restrictions $k_1 + 2k_2... + jk_j = j$ and $k_1 + ... + k_j < 1 - \nu$ again corresponds to the combination $k_1 = ... = k_{j-1} = 0$, $k_j = 1$, yielding $| f_\varphi^{(j)}(t) | \leq t^{p-j}$ from the upper line which dominates over terms in in the lower and central lines. As a summary, in a neighborhood of 0, it holds

$$| f_\varphi^{(j)}(t) | \leq c \left\{ \begin{array}{cc} t^{p-j}, & \nu < 0 \\ t^{p-j}(1+ | \log t |), & \nu = 0 \\ t^{p(1-\nu)-j} & \nu > 0 \end{array} \right\}, \;\; j = 1, ..., m.$$

Now condition (5.16) implies that $\lim_{t\to 0} f_\varphi^{(j)}(t) = 0$ for $j = 1, ..., m$.

(ii) Claim (ii) is trivial for $\nu < 0$ since then $K_\varphi(t,s)$ is bounded together with $K(x,y)$. To prove claim (ii) for $0 \leq \nu < 1$, we first examine the properties of the function

$$\Phi(t,s) := \left\{ \begin{array}{cc} \frac{\varphi(t)-\varphi(s)}{t-s}, & t \neq s \\ \varphi'(t), & t = s \end{array} \right\}, \;\; 0 \leq t, s \leq 1.$$

Due to the conditions on φ, we have $\Phi \in C^{p-1}([0,1] \times [0,1])$, $\Phi(t,s) > 0$ for $(t,s) \in ([0,1] \times [0,1]) \setminus \{(0,0), \, (1,1)\}$; we show that there exists a positive constant c_0 such that

$$\Phi(t,s) \geq c_0 \min\{(t+s)^{p-1}, \, [(1-t)+(1-s)]^{p-1}\}, \;\; 0 \leq t, s \leq 1. \tag{5.19}$$

It suffices to establish estimate (5.19) in a neighborhood of the points $(0,0)$; for a neighborhood of the point (1.1) the estimate follows by the symmetry; on the rest part of $[0,1] \times [0,1]$ function Φ is greater than a positive constant implying (5.19) also there, possibly with a smaller but still positive constant c_0. We choose a neighborhood $U_\delta \subset [0,1] \times [0,1]$ of $(0,0)$ of a sufficiently small radius $\delta > 0$ such that $\varphi^{(p)}(t) \neq 0$ for $0 \leq t \leq \delta$, see (5.17). Then $\varphi^{(p)}(t) > 0$ for $0 \leq t \leq \delta$, since $\varphi^{(p)}(t) < 0$ for $0 \leq t \leq \delta$ together with the conditions $\varphi'(0) = ... = \varphi^{(p-1)}(0) = 0$ should imply $\varphi'(t) < 0$ for $0 \le t \le \delta$ (use the Taylor formula!). Denote $d_0 := \min_{0\le t\le\delta} \varphi^{(p)}(t) > 0$. Let $0 < s < t \le \delta$. Due to (5.17), the Taylor formula with the integral form of the rest term yields

$$\varphi(t) - \varphi(s) = \frac{1}{(p-1)!}\int_0^t (t-\tau)^{p-1}\varphi^{(p)}(\tau)d\tau - \frac{1}{(p-1)!}\int_0^s (s-\tau)^{p-1}\varphi^{(p)}(\tau)d\tau$$

$$= \frac{1}{(p-1)!}\int_0^s [(t-\tau)^{p-1} - (s-\tau)^{p-1}]\varphi^{(p)}(\tau)d\tau + \frac{1}{(p-1)!}\int_s^t (t-\tau)^{p-1}\varphi^{(p)}(\tau)d\tau.$$

The functions $(t-\tau)^{p-1}-(s-\tau)^{p-1}$ and $(t-\tau)^{p-1}$ under last two integrals are positive. Estimating $\varphi^{(p)}(\tau) \ge d_0 > 0$ we obtain

$$\varphi(t) - \varphi(s) \ge \frac{d_0}{(p-1)!}\left(\int_0^s [(t-\tau)^{p-1} - (s-\tau)^{p-1}]d\tau + \int_s^t (t-\tau)^{p-1}d\tau\right)$$

$$= \frac{d_0}{(p-1)!}\left(\int_0^t (t-\tau)^{p-1}d\tau - \int_0^s (s-\tau)^{p-1}d\tau\right) = \frac{d_0}{p!}(t^p - s^p),$$

and (5.19) follows for $0 < s < t \le \delta$:

$$\frac{\varphi(t)-\varphi(s)}{t-s} \ge \frac{d_0}{p!}\frac{t^p - s^p}{t-s} = \frac{d_0}{p!}\sum_{j=0}^{p-1} t^j s^{p-1-j} \ge c_0 \sum_{j=0}^{p-1}\left(\begin{array}{c} p-1 \\ j \end{array}\right) t^j s^{p-1-j} = c_0(t+s)^{p-1}.$$

The case $0 < t < s \le \delta$ is symmetrical to the treated case $0 < s < t \le \delta$. For $0 < s = t \le \delta$, (5.19) follows by a limit argument. This completes the proof of (5.19).

Let us return to claim (ii) of the Theorem for $0 \le \nu < 1$. Consider the case $0 < \nu < 1$. Due to (3.4) and (5.19),

$$| K_\varphi(t,s) | \le c_K \mid \varphi(t) - \varphi(s) \mid^{-\nu} \varphi'(s) = c_K \left(\frac{\varphi(t)-\varphi(s)}{t-s}\right)^{-\nu} \mid t-s \mid^{-\nu} \varphi'(s)$$

$$\le c_K c_0^{-\nu} \mid t-s \mid^{-\nu} \frac{\varphi'(s)}{[\min\{(t+s)^{p-1},\ ((1-t)+(1-s))^{p-1}\}]^\nu} \le c \mid t-s \mid^{-\nu};$$

on the last step we took into account that $\varphi'(s) \asymp s^{p-1}$ as $s \to 0$, $\varphi'(s) \asymp (1-s)^{p-1}$ as $s \to 1$. Thus $K_\varphi \in \mathcal{S}^{0,\nu}$. In the case $\nu = 0$,

$$| K_\varphi(t,s) | \le c_K(1 + \mid \log \mid \varphi(t) - \varphi(s) \mid\mid \varphi'(s)$$

$$= c_K(1 + \mid \log \frac{\varphi(t)-\varphi(s)}{t-s} \mid + \mid \log \mid t-s \mid\mid)\varphi'(s)$$

$$\le c(1 + \mid \log \min\{(t+s,\ [(1-t)+(1-s)]\} \mid + \mid \log \mid t-s \mid\mid)\varphi'(s)$$

$$\le c_1 + c_2(1 + \mid \log \mid t-s \mid\mid), \text{ i.e., } K_\varphi \in \mathcal{S}^{0,0}.$$

Having established that $K_\varphi \in \mathcal{S}^{0,\nu}$ for $\nu < 1$, the compactness of the operator $T_\varphi : C[0,1] \to C[0,1]$ follows by Lemma 3.2. □

Corollary 5.1. *Assume the conditions of Theorems* 5.2 *and* 5.3. *Then* $v \in C^m[0,1]$,

$$v^{(j)}(0) = v^{(j)}(1) = 0, \quad j = 1, ..., m, \tag{5.20}$$

for the solutions of equation (5.15).

Remark 5.1. One can conjecture that, under condition of Theorem 5.3, $K \in \mathcal{S}^{m,\nu}$ implies $K_\varphi \in \mathcal{S}^{m,\nu}$. The argument becomes rather technical to justify this. For us the relation $K_\varphi \in \mathcal{S}^{0,\nu}$ established in Theorem 5.3 is sufficient in the sequel.

Example 5.1. Let us present an example of function φ that satisfies the conditions of Theorem 5.3:

$$\varphi(t) = c_p \int_0^t \tau^{p-1}(1-\tau)^{p-1}d\tau, \;\; c_p = \frac{1}{\int_0^1 \tau^{p-1}(1-\tau)^{p-1}d\tau} = \frac{(2p-1)!}{[(p-1)!]^2}, \;\; p \in \mathbb{N}. \tag{5.21}$$

Clearly, $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi'(t) = c_p t^{p-1}(1-t)^{p-1} > 0$ for $0 < t < 1$, $\varphi^{(j)}(0) = \varphi^{(j)}(1) = 0$ for $j = 1, ..., p-1$, $\varphi^{(p)}(0) = (p-1)!c_p = (2p-1)!/(p-1)!$, $\varphi^{(p)}(1) = (-1)^{p-1}(2p-1)!/(p-1)!$. In this example, φ is a polynomial of degree $2p-1$.

6. Specification for the Volterra integral equation.

The Volterra integral equation

$$u(x) = \int_0^x K(x,y)u(y)dy + f(x), \quad 0 \le x \le 1, \tag{6.1}$$

can be considered as a special case of the Fredholm integral equation (5.1) in which $K(x,y) = 0$ for $0 \le x < y \le 1$. The class $\mathcal{S}^{m,\nu}(([0,1]\times[0,1]) \setminus \text{diag})$ is well defined for such kernels, hence the results of Section 5 hold for equation (6.1). Nevertheless, it is worth to revisit the results of Section 5 since normally the derivatives of a solution $u(x)$ to (6.1) may have singularities only at $x = 0$. We "project" the formulations of the main concepts and results of Section 5 to the needs of Volterra equation (6.1). The proofs are omitted since they contain no new ideas, conversely, they are some simplifications of the arguments in Section 5.

Denote

$$\triangle = \{(x,y) : 0 \le y < x \le 1\}$$

and introduce the class $\mathcal{S}^{m,\nu}(\triangle)$ of kernels $K(x,y)$ that are defined and m times continuously differentiable on $\triangle$ and satisfy for $(x,y) \in \triangle$ and for all $k, l \in \mathbb{N}_0$, $k + l \le m$, the inequality

$$\left| \left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^l K(x,y) \right| \le c_{K,m} \left\{ \begin{array}{cc} 1, & \nu + k < 0 \\ 1 + |\log(x-y)|, & \nu + k = 0 \\ (x-y)^{-\nu-k}, & \nu + k > 0 \end{array} \right\}.$$

After an extension of $K \in \mathcal{S}^{m,\nu}(\triangle)$ by the zero value outside $\triangle$ we obtain a kernel $K \in \mathcal{S}^{m,\nu}(([0,1]\times[0,1]) \setminus \text{diag})$.

For $m \ge 1$, $\nu < 1$, denote by $C^{m,\nu}(0,1]$ the space of functions $f \in C^m(0,1]$ that satisfy the inequalities

$$| f^{(j)}(x) | \le c_f \left\{ \begin{array}{cc} 1 & j + \nu - 1 < 0 \\ 1 + |\log x|, & j + \nu - 1 = 0 \\ x^{-j-\nu+1}, & j + \nu - 1 > 0 \end{array} \right\}, \quad 0 < x \le 1, \;\; j = 0, ..., m.$$

Introduce the weight functions

$$w_\lambda^0(x) = \left\{ \begin{array}{cc} 1, & \lambda < 0 \\ 1/(1 + |\log x|), & \lambda = 0 \\ x^\lambda & \lambda > 0 \end{array} \right\}, \quad 0 < x \le 1, \;\; \lambda \in \mathbb{R}.$$

The norm in $C^{m,\nu}(0,1]$ is given by

$$\| f \|_{C^{m,\nu}(0,1]} = \sum_{j=0}^m \sup_{0<x\le 1} w_{j+\nu-1}^0(x) \, | f^{(j)}(x) | \, .$$

There holds the compact imbeding

$$C^{m,\nu}(0,1] \subset C[0,1], \quad m \geq 1, \ \nu < 1.$$

Theorem 6.1. *Let $K \in \mathcal{S}^{m,\nu}(\Delta)$, $m \geq 1$, $\nu < 1$. Then the Volterra integral operator T defined by $(Tu)(x) = \int_0^x K(x,y)u(y)dy$ maps $C^{m,\nu}(0,1]$ into itself and $T : C^{m,\nu}(0,1] \to C^{m,\nu}(0,1]$ is compact.*

Theorem 6.2. *Let $K \in \mathcal{S}^{m,\nu}(\Delta)$, $f \in C^{m,\nu}(0,1]$, $m \geq 1$, $\nu < 1$, and let $u \in C[0,1]$ be the (unique) solution of equation* (6.1). *Then $u \in C^{m,\nu}(0,1]$.*

See Exercise 14 about the existence and uniqueness of the solution.

An alternative proof of Theorem 6.2. Again, Theorem 6.2 is an elementary consequence ot Theorem 6.1 (similarly as Theorem 5.2 was an elementary consequence of Theorem 5.1). Alternatively, Theorem 6.2 can be derived from Theorem 5.2 by a prolongation argument, and we demonstrate how this can be done. First of all, we extend $K \in \mathcal{S}^{m,\nu}(\triangle)$ by the zero values from $\triangle$ to $([0,1] \times [0,1])\backslash$diag obtaining $K \in \mathcal{S}^{m,\nu}(([0,1] \times [0,1]) \setminus \text{diag})$. The Volterra equation (6.1) is equivalent to the Fredholm equation (5.1) with the extended kernel. By Theorem 5.2, we know about the solution of (5.1) that $u \in C^{m,\nu}(0,1)$. It remains to show that actually no singularities of the derivatives of $u(x)$, the solution of (6.1) and (5.1), occur at $x = 1$. To show this, we extend f from $[0,1]$ to $[0,1+\delta]$, $0 < \delta < 1/m$, using the reflection formula

$$f(x) = \sum_{j=0}^{m} d_j f(1 - j(x-1)), \ \ 1 < x \leq 1 + \delta, \tag{6.2}$$

where d_j are chosen so that the C^m-smooth joining takes place at $x = 1$. Namely, differentiating (6.2) k times we have

$$f^{(k)}(x) = \sum_{j=0}^{m} (-j)^k d_j f^{(k)}(1 - j(x-1)), \ \ 1 < x \leq 1 + \delta,$$

and the C^m-smooth joining at $x = 1$ happens if

$$\sum_{j=0}^{m} (-j)^k d_j = 1, \ \ k = 0, ..., m.$$

We obtained a uniquely solvable $(m+1) \times (m+1)$ Vandermonde system to determine $d_0, ..., d_m$. Using the reflection formula we extend also the kernel $K(x,y)$ along the lines $y = \gamma x$, $0 < \gamma < 1$, from the triangle $\triangle = \{(x,y) : \ 0 \leq y < x \leq 1\}$ onto the triangle $\triangle_\delta = \{(x,y) : \ 0 \leq y < x \leq 1+\delta\}$ with a $\delta > 0$. The extension procedure preserves f in $C^{m,\nu}(0,1+\delta]$ and K in $\mathcal{S}^{m,\nu}(\triangle_\delta)$. Introduce the extended equation

$$u(x) = \int_0^x K(x,y)u(y)dy + f(x), \quad 0 \leq x \leq 1 + \delta;$$

for $0 \leq x \leq 1$ this equation coincides with (6.1). By Theorem 5.2 applied to the extended equation, u is C^m-smooth for $0 < x < 1 + \delta$, hence no singularities of the derivatives of u at $x = 1$ are possible. □

Let $\varphi : \ [0,1] \to [0,1]$ be a smooth strictly increasing function such that $\varphi(0) = 0$, $\varphi(1) = 1$. Introducing the change of variables

$$x = \varphi(t), \ \ y = \varphi(s),$$

we rewrite the equation (6.1) in the form

$$v(t) = \int_0^t K_\varphi(t,s)v(s)ds + f_\varphi(t), \quad 0 \leq t \leq 1, \tag{6.3}$$

where

$$f_\varphi(t) := f(\varphi(t)), \ \ K_\varphi(t,s) := K(\varphi(t), \varphi(s))\varphi'(s);$$

the solutions of equations (6.1) and (6.3) are in the relations

$$v(t) = u(\varphi(t)), \ \ u(x) = v(\varphi^{-1}(x)).$$

An obtrusive mistake is to write "formally" $\int_0^{\varphi(t)} K(\varphi(t), \varphi(s))u(\varphi(s))\varphi'(s)ds$ as the result of the change of variables in the integral $\int_0^x K(x,y)u(y)dy$. We must be more careful! Actually the change of variables $y = \varphi(s)$ yields

$$\int_0^x K(x,y)u(y)dy = \int_0^{\varphi^{-1}(x)} K(x, \varphi(s))u(\varphi(s))\varphi'(s)ds,$$

and after that the change of variables $x = \varphi(t)$ completes the result as

$$\int_0^x K(x,y)u(y)dy \mid_{x=\varphi(t)} = \int_0^t K(\varphi(t), \varphi(s))u(\varphi(s))\varphi'(s)ds.$$

So we really obtain the transformed equation in the Volterra form (6.3).

Theorem 6.3. *Given* $m \geq 1$, $\nu < 1$, *let*

$$p > \left\{ \begin{array}{cc} m, & \nu \leq 0 \\ \frac{m}{1-\nu}, & 0 < \nu < 1 \end{array} \right\}.$$

Let $\varphi \in C^p[0,1]$ *satisfy* $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi'(t) > 0$ *for* $0 < t < 1$ *and*

$$\varphi^{(j)}(0) = 0, \ \ j = 1, ..., p-1, \ \ \varphi^{(p)}(0) \neq 0.$$

Then the following claims hold true.

(i) *For* $f \in C^{m,\nu}(0,1]$, *the function* $f_\varphi(t) := f(\varphi(t))$ *belongs* to $C^m[0,1]$ *and*

$$f_\varphi^{(j)}(0) = 0, \ \ \ j = 1, ..., m.$$

(ii) *For* $K \in \mathcal{S}^{0,\nu}(\Delta)$, *the kernel* $K_\varphi(t,s) := K(\varphi(t), \varphi(s))\varphi'(s)$ *belongs to* $\mathcal{S}^{0,\nu}(\Delta)$ *and defines a compact Volterra integral operator*

$$T_\varphi : C[0,1] \to C[0,1], \ \ (T_\varphi v)(t) = \int_0^t K_\varphi(t,s)v(s)ds.$$

An example of function φ satisfying the conditions of Theorem 6.3 is given by $\varphi(t) = t^p$.

Corollary 6.1. *Assume the conditions of Theorems* 6.2 *and* 6.3. *Then* $v \in C^m[0,1]$,

$$v^{(j)}(0) = 0, \ \ \ j = 1, ..., m,$$

for the solution of equation (6.3).

7. A collocation method for weakly singular integral equations.

7.1. Interpolation by polynomials on a uniform grid. Given an interval $[a,b]$ and $m \in \mathbb{N}$, introduce the uniform grid consisting of m points

(7.1) $$x_i = a + (i - \frac{1}{2})h, \ \ i = 1, ..., m, \ \ h = \frac{b-a}{m}.$$

Denote by Π_m the Lagrange interpolation projection operator assigning to any $u \in C[a,b]$ the polynomial $\Pi_m u \in \mathcal{P}_{m-1}$ that interpolates f at points (7.1). Here $\mathcal{P}_{m-1}$ denotes the set of polynomials of degree not exceeding $m-1$.

Lemma 7.1. *In the case of interpolation knots* (7.1)*, for $f \in C^m[a,b]$ it holds*

$$\max_{a\le x\le b} \mid f(x) - (\Pi_m f)(x) \mid \le \theta_m h^m \max_{a\le x\le b} \mid f^{(m)}(x) \mid, \tag{7.2}$$

$$\theta_m = \frac{1\cdot 3\cdot \ldots \cdot (2m-1)}{2\cdot 4\cdot \ldots \cdot 2m} \sim (\pi m)^{-\frac{1}{2}} \text{ as } m\to\infty.$$

Further, for $m = 2k$, $k \ge 1$,

$$\max_{x_k\le x\le x_{k+1}} \mid f(x) - (\Pi_m f)(x) \mid \le \vartheta_m h^m \max_{a\le x\le b} \mid f^{(m)}(x) \mid, \tag{7.3}$$

$$\vartheta_m = 2^{-2m}\frac{m!}{((m/2)!)^2} \sim \sqrt{2/\pi}\, m^{-\frac{1}{2}} 2^{-m},$$

whereas for $m = 2k+1$, $k \ge 1$,

$$\max_{x_k\le x\le x_{k+2}} \mid f(x) - (\Pi_{m-1} f)(x) \mid \le \vartheta_m h^m \max_{a\le x\le b} \mid f^{(m)}(x) \mid, \tag{7.4}$$

$$\vartheta_m = \frac{2\sqrt{3}}{9}\frac{(k!)^2}{(2k+1)!} \sim \frac{2\sqrt{3}}{9}\sqrt{2\pi}\, m^{-\frac{1}{2}} 2^{-m}.$$

These estimates are elementary consequences of the error formula (see e.g. [4], [18] or [24])

$$f(x) - (\Pi_m f)(x) = \frac{f^{(m)}(\xi)}{m!}(x-x_1)\ldots(x-x_m), \;\; x\in[a,b],\; \xi = \xi(x) \in (a,b)$$

that holds for the interpolation with arbitrary pairwise different knots $x_1, \ldots, x_m$ of $[a,b]$. Namely, for points (7.1), the maximum of $\mid (x-x_1)\ldots(x-x_m) \mid$ on $[a,b]$ is attained at the end points of the interval, whereas the maximum of $\mid (x-x_1)\ldots(x-x_{2k}) \mid$ on $[x_k, x_{k+1}]$ is attained at the centre of $[x_k, x_{k+1}]$ (which is also the centre of $[a,b]$). To establish (7.4), we take into account that the maximum of $\mid (x-x_k)(x-x_{k+1})(x-x_{k+2}) \mid$ on $[x_k, x_{k+2}]$ equals $\frac{2\sqrt{3}}{9}h^3$ and elementarily estimate the remaining product on $[x_k, x_{k+2}]$.

Comparing estimates (7.2)–(7.4) we observe that in the central parts of $[a,b]$, the estimates are approximately 2^m times preciser than on the whole interval. Surprisingly estimates (7.3) and (7.4) are for $m \ge 2$ preciser even than the error estimate of the Chebyshev interpolant of the same order on $[a,b]$, see estimate (7.7) below. In the central parts of $[a,b]$, the interpolation process on the uniform grid has also good stability properties: in contrast to an exponential growth of $\| \Pi_{m-1} \|_{C[a,b]\to C[a,b]}$ as $m\to\infty$, it holds

$$\| \Pi_{m-1} \|_{C[a,b]\to C[\frac{a+b}{2}-rh^{1/2},\, \frac{a+b}{2}+rh^{1/2}]} \le c_r(1+\log m) \tag{7.5}$$

with a constant c_r which depends only on $r > 0$. This is the Runck's theorem (see [4], p. 170).

7.2. Chebyshev interpolation. For the interpolation of functions on $[a,b]$ by polynomials of degree $m-1$, the best choice of interpolation knots is the *Chebyshev knots*

$$x_i = \frac{b-a}{2}\left(-\cos\frac{2i-1}{2m}\pi\right) + \frac{a+b}{2} \in (a,b), \quad i=1,...,m. \tag{7.6}$$

The *Chebyshev interpolant* $\Pi_m f$ is a polynomial of degree $m-1$ that interpolates f at these knots. The proof of the following estimate can be found in most of text books on numerical analysis, see e.g. [4], [18] or [24].

Lemma 7.2 *In the case of Chebyshev knots* (7.6)*, for* $f \in C^m[a,b]$*, it holds*

$$\max_{a\le x\le b} \mid f(x) - (\Pi_m f)(x) \mid \le \frac{(b-a)^m}{m!\,2^{2m-1}} \max_{a\le x\le b} \mid f^{(m)}(x) \mid . \tag{7.7}$$

The Chebyshev interpolant $\Pi_m f$ occurs to be the best approximation to the function $f(x) = x^m$ with respect to the uniform norm on $[a,b]$, and (7.7) turns into equality for this function. Computations with Chebyshev interpolant are numerically (relatively) stable, since (see [4])

$$\| \Pi_m \|_{C[a,b]\to C[a,b]} \le 8 + \frac{4}{\pi}\log m, \quad m \in \mathbb{N}. \tag{7.8}$$

It is known that for any projection operator $P_m : C[a,b] \to \mathcal{P}_{m-1}$, i.e., for any operator $P_m : C[a,b] \to C[a,b]$ such that $P_m^2 = P_m$ and the range $\mathcal{R}(P_m) = \mathcal{P}_{m-1}$, it holds

$$\| P_m \|_{C[a,b]\to C[a,b]} \ge c_0(1+\log m), \quad m \in \mathbb{N},$$

where $c_0 > 0$ is independent of m. Thus in the case of Chebyshev knots, $\| \Pi_m \|_{C[a,b]\to C[a,b]}$ of is of minimal possible growth order as $m \to \infty$.

7.3. Piecewise polynomial interpolation. Introduce in $\mathbb{R}$ the uniform grid $\mathbb{R}_h := \{jh : j \in \mathbb{Z}\}$ where $h = 1/n$, $n \in \mathbb{N}$. Let $m \ge 2$ be fixed. Given a function $f \in C[-\delta, 1+\delta]$, $\delta > 0$, we introduce a piecewise polynomial interpolant $\Pi_{h,m} f \in C[0,1]$ as follows. On every subinterval $[jh, (j+1)h]$, $0 \le j \le n-1$, the function $\Pi_{h,m} f$ is defined independently from other subintervals as a polynomial of degree $\le m-1$ that interpolates f at m points lh neighboring jh from two sides:

$$(\Pi_{h,m} f)(lh) = f(lh), \quad l = j - \frac{m}{2} + 1, ..., j + \frac{m}{2} \text{ if } m \text{ is even,}$$

$$(\Pi_{h,m} f)(lh) = f(lh), \quad l = j - \frac{m-1}{2}, ..., j + \frac{m-1}{2} \text{ if } m \text{ is odd.}$$

A unified writing form of these interpolation conditions is

$$(\Pi_{h,m} f)(lh) = f(lh) \text{ for } l \in \mathbb{Z} \text{ such that } l - j \in \mathbb{Z}_m \tag{7.9}$$

where

$$\mathbb{Z}_m = \{k \in \mathbb{Z} : -\frac{m}{2} < k \le \frac{m}{2}\}.$$

To see this, observe that $\mathbb{Z}_m$ contains the following m elements (integers):

$$\mathbb{Z}_m = \{-\frac{m}{2}+1, -\frac{m}{2}+2, ...\frac{m}{2}\} \text{ if } m \text{ is even,}$$

$$\mathbb{Z}_m = \{-\frac{m-1}{2}, -\frac{m-1}{2}+1, ..., \frac{m-1}{2}\} \text{ if } m \text{ is odd.}$$

Note that $\Pi_{h,m}f$ is continuous on $[0,1]$ since for the "interior" knots jh, $1 \le j \le n-1$, interpolation conditions (7.9) yield $(\Pi_{h,m}f)(jh) = f(jh)$ for $\Pi_{h,m}f$ as a function on $[(j-1)h, jh]$ as well as a function on $[jh,(j+1)h]$; the one side derivatives of the interpolant $\Pi_{h,m}f$ at the interior knots may be different.

Introduce the Lagrange fundamental polynomials $L_k \in \mathcal{P}_{m-1}$, $k \in \mathbb{Z}_m$, satisfying $L_k(l) = \delta_{k,l}$, $k,l \in \mathbb{Z}_m$, where $\delta_{k,l}$ is the Kronecker symbol, $\delta_{k,l} = 0$ for $k \ne l$ and $\delta_{k,k} = 1$. An explicit formula for L_k is given by

$$L_k(t) = \prod_{l \in \mathbb{Z}_m \setminus \{k\}} \frac{t-l}{k-l}\,, \quad k \in \mathbb{Z}_m. \tag{7.10}$$

It is easy to see that

$$(\Pi_{h,m}f)(x) = \sum_{k \in \mathbb{Z}_m} f((j+k)h) L_k(nx-j) \ \text{ for } x \in [jh,(j+1)h],\ j = 0,..,n-1. \tag{7.11}$$

Indeed, $\Pi_{h,m}f$ given by (7.11) is really a polynomial of degree $\le m-1$ on every interval $[jh,(j+1)h]$, and it satisfies interpolation conditions (7.9): for l with $l-j \in \mathbb{Z}_m$ we have

$$(\Pi_{h,m}f)(lh) = \sum_{k \in \mathbb{Z}_m} f((j+k)h) L_k(l-j) = \sum_{k \in \mathbb{Z}_m} f((j+k)h)\delta_{k,l-j} = f((j+l-j)h) = f(lh).$$

The interpolant $\Pi_{h,m}f$ could be defined on $[0,1]$ also for $m=1$ as a piecewise constant function with possible jumps at jh, $j = 1,...,n-1$. In this case we loose the continuity of the interpolant at the interior knots jh, $j=1,...,n-1$, but the real reason why we exclude the case $m=1$ from our consideration is that the interpolation points jh, $j=0,...,n-1$, are not properly located; a natural location of an interpolation point is the centre of the interval $[jh,(j+1)h)$ on which the interpolant is constant. The case $m=1$ with interpolation points $(j+\frac{1}{2})h$, $j=0,...,n-1$, can be examined independently in an elementary way.

For $m=2$, the interpolant $\Pi_{h,m}f$ is the usual piecewise linear function joining for $0 \le j \le n-1$ the pair of points $(jh, f(jh)) \in \mathbb{R}^2$ and $((j+1)h, f((j+1)h)) \in \mathbb{R}^2$ by a straight line. For $m=2$, $\Pi_{h,m}f$ does not need values of f outside $[0,1]$, and $\Pi_{h,m}$ is a projection operator in $C[0,1]$, i.e. $\Pi_{h,m}^2 = \Pi_{h,m}$.

For $m \ge 3$, $\Pi_{h,m}f$ uses values of f outside of $[0,1]$. For $f \in C[0,1]$, $\Pi_{h,m}f$ obtains a sense after an extension of f onto $[-\delta, 1+\delta]$. In the general case, the reflection formulae of the type (6.2) can be exploited to extend f, then $f(kh)$ for $k<0$ and $k>n$ is a linear combination of $f(jh)$, $j=0,...,n$, and the extended function maintains the C^m-smoothness of f. We are in a lucky situation if $f \in C^m[0,1]$ satisfies the boundary conditions $f^{(j)}(0) = f^{(j)}(1) = 0$, $j=1,...,m$, then the simplest extension operator

$$E_\delta : C[0,1] \to C[-\delta, 1+\delta], \quad (E_\delta f)(t) = \left\{ \begin{array}{ll} f(0), & -\delta \le t \le 0 \\ f(t), & 0 \le t \le 1 \\ f(1), & 1 \le t \le 1+\delta \end{array} \right\}.$$

maintains the smoothness of f. The operator

$$P_{h,m} := \Pi_{h,m} E_\delta : C[0,1] \to C[0,1] \tag{7.12}$$

is well defined and $P_{h,m}^2 = P_{h,m}$, i.e., $P_{h,m}$ is a projection operator in $C[0,1]$. The range $\mathcal{R}(P_{h,m})$ is a subspace of dimension $n+1$ in $C[0,1]$. A function $w_h \in \mathcal{R}(P_{h,m})$ is uniquely determined by its knot values $w_h(jh)$, $j=0,...,n$. Indeed, due to (7.11), for $w_h = P_{h,m}w_h = \Pi_{h,m}E_\delta w_h$ we have

$$w_h(t) = \sum_{k \in \mathbb{Z}_m} (E_\delta w_h)((j+k)h) L_k(nt-j) \ \text{ for } t \in [jh,(j+1)h],\ j=0,..,n-1, \tag{7.13}$$

with $(E_\delta w_h)(ih) = w_h(ih)$ for $i = 0,...,n$, $(E_\delta w_h)(ih) = w_h(0)$ for $i<0$ and $(E_\delta w_h)(ih) = w_h(1)$ for $i>n$. Clearly, for $w_h \in \mathcal{R}(P_{h,m})$ we have $w_h = 0$ if and only if $w_h(ih) = 0$, $i=0,...,n$.

For $f \in C[-\delta, 1+\delta]$, the interpolant $\Pi_{h,m} f$ is closely related to the "central" part interpolant of f on the uniform grid treated in Section 7.1. On $[jh, (j+1)h]$, $\Pi_{h,m} f$ coincides with the polynomial interpolant $\Pi_m f$ constructed for f on the interval $[a_j, b_j]$ where $a_j = (j - \frac{m-1}{2})h$, $b_j = (j + \frac{m+1}{2})h$ in the case of even m and $a_j = (j - \frac{m}{2})h$, $b_j = (i + \frac{m}{2})h$ in the case of odd m; moreover, $[jh, (j+1)h]$ is a "central" part of $[a_j, b_j]$ on which the interpolation error can be estimated by (7.3) and (7.4). On this way we obtain the following result.

Lemma 7.3. (i) *For* $f \in C^m[-\delta, 1+\delta]$,

$$\max_{0 \le t \le 1} \mid f(t) - (\Pi_{h,m} f)(t) \mid \le \vartheta_m h^m \max_{-\delta \le t \le 1+\delta} \mid f^{(m)}(t) \mid \tag{7.14}$$

with ϑ_m *defined in* (7.3) *and* (7.4) *respectively for even and odd* m.

(ii) *For* $f \in V^{(m)} := \{v \in C^m[0,1] : v^{(j)}(0) = v^{(j)}(1) = 0,\ j = 1, ..., m\}$, *it holds*

$$\max_{0 \le t \le 1} \mid f(t) - (P_{h,m} f)(t) \mid \le \vartheta_m h^m \max_{0 \le t \le 1} \mid f^{(m)}(t) \mid . \tag{7.15}$$

Proof. The claim (i) is a direct consequence of Lemma 7.1. Further, for $f \in V^{(m)}$, we have $E_\delta f \in C^m[-\delta, 1+\delta]$,

$$\max_{-\delta \le t \le 1+\delta} \mid (E_\delta f)^{(m)}(t) = \max_{0 \le t \le 1} \mid f^{(m)}(t) \mid, \quad (E_\delta f)(t) = f(t) \text{ for } 0 \le t \le 1,$$

and (7.14) applied to $E_\delta f$ takes the form (7.15). □

From (7.5) we obtain

$$\| P_{h,m} \|_{C[0,1 \to C[0,1]} = \| \Pi_{h,m} \|_{C[-\delta,1+\delta] \to C[0,1]} \le c(1 + \log m).$$

Thus the norms of projection operators are uniformly bounded with respect to n. Together with (7.15), noticing that $V^{(m)}$ is dense in $C[0,1]$, the Banach–Steinhaus theorem (Theorem 2.2) yields the following result.

Corollary 7.1. *For any* $f \in C[0,1]$, $\max_{0 \le t \le 1} \mid f(t) - (P_{n,m} f)(t) \mid \to 0$ *as* $n \to \infty$.

7.4. A piecewise polynomial collocation method: error estimate. Having a weakly singular integral equation (5.1), $u = Tu + f$, with $K \in \mathcal{S}^{m,\nu}$, $f \in C^{m,\nu}(0,1)$, $m \ge 2$, $0 < \nu < 1$, we rewrite it with the help of a smoothing change of variables in the form (5.15), $v = T_\varphi v + f_\varphi$, and after that approximate (5.15) by the $n+1$ dimensional equation

$$v_h = P_{h,m} T_\varphi v_h + P_{h,m} f_\varphi. \tag{7.16}$$

This is the operator form of the *piecewise polynomial collocation method* corresponding to the interpolation projection operator $P_{h,m}$ which is defined in (7.12). (A *collocation method* is always related with a "projection" of a given equation with the help of an *interpolation* projection operator. In this sense, collocation methods can be treated as a subclass of Galerkin methods. Galerkin methods correspond to general class of projection operators, not necessarily interpolation ones.)

Theorem 7.1. *Let* $K \in \mathcal{S}^{m,\nu}$, $f \in C^{m,\nu}(0,1)$, $m \ge 2$, $\nu < 1$, *and let* $\varphi : [0,1] \to [0,1]$ *satisfy the conditions of Theorem* 5.3. *Further, assume that* $\mathcal{N}(I - T) = \{\mathbf{0}\}$ (*or equivalently,* $\mathcal{N}(I - T_\varphi) = \{\mathbf{0}\}$). *Then there exists an* n_0 *such that for* $n \ge n_0$, *the collocation equation* (7.16) *has a unique solution* v_h. *The accuracy of* v_h *can be estimated by*

$$\| v - v_h \|_\infty \le c h^m \| v^{(m)} \|_\infty \tag{7.17}$$

where $v(t) = u_\varphi(t) = u(\varphi(t))$ *is the solution of* (5.15), $u(x)$ *is the solution of* (5.1). *The constant* c *in* (7.17) *is independent of* n *and* f (*it depends on* K, m *and* φ).

Proof. The following proof argument is typical for Galerkin (including collocation) methods.

By Theorems 5.3, $T_\varphi : C[0,1] \to C[0,1]$ is compact. Since $\mathcal{N}(I - T_\varphi) = \{\mathbf{0}\}$, the bounded inverse $(I - T_\varphi)^{-1} : C[0,1] \to C[0,1]$ exists due the Fredholm alternative (Theorem 2.7); denote

$$\kappa := \| (I - T_\varphi)^{-1} \|_{C[0,1] \to C[0,1]} .$$

Further, the compactness of $T_\varphi : C[0,1] \to C[0,1]$ and the pointwise convergence $P_{h,m}$ to I in $C[0,1]$ (see Corollary 7.1) imply by Theorem 2.6 the norm convergence

$$\varepsilon_h := \| P_{h,m}T_\varphi - T_\varphi \|_{C[0,1]\to C[0,1]} \to 0 \quad \text{as } n \to \infty \ \ (\text{as } h = 1/n \to 0).$$

Hence there is an n_0 such that $\kappa\varepsilon_h < 1$ for $n \geq n_0$. With the help of Theorem 2.4 we conclude that $I - P_{h,m}T_\varphi$ is invertible in $C[0,1]$ for $n \geq n_0$ and

$$\kappa_h := \| (I - P_{h,m}T_\varphi)^{-1} \|_{C[0,1]\to C[0,1]} \leq \frac{\kappa}{1-\kappa\varepsilon_h} \to \kappa \ \text{as } n \to \infty. \tag{7.18}$$

This proves the unique solvability of the collocation equation (7.16) for $n \geq n_0$.

Let v and v_h be the solutions of (5.15) and (7.16), respectively. Then

$$(I - P_{h,m}T_\varphi)(v - v_h) = v - P_{h,m}T_\varphi v - P_{h,m}f_\varphi = v - P_{h,m}v,$$

$$v - v_h = (I - P_{n,m}T_\varphi)^{-1}(v - P_{n,m}v)$$

and

$$\| v - v_h \|_\infty \leq \kappa_n \| v - P_{h,m}v \|_\infty, \ \ n \geq n_0. \tag{7.19}$$

By Theorem 5.2, for the solution u of (5.1) we have $u \in C^{m,\nu}(0,1)$; by Corollary 5.1, for $v(t) = u_\varphi(t) = u(\varphi(t))$ we have $v \in C^m[0,1]$ and $v^{(j)}(0) = v^{(j)}(1) = 0$, $j = 1, ..., m$; by Lemma 7.3(ii),

$$\| v - P_{h,m}v \|_\infty \leq \vartheta_m h^m \| v^{(m)} \|_\infty .$$

Now (7.19) yields

$$\| v - v_h \|_\infty \leq \kappa_h \vartheta_m h^m \| v^{(m)} \|_\infty$$

that together with (7.18) implies (7.17). □

Proving the convergence of the method, without the convergence speed, the assumptions of Theorem 7.1 can be relaxed, see Exercise 16.

7.5. The matrix form of the collocation method. The solution v_h of equation (7.16) belongs to $\mathcal{R}(P_{h,m})$, so the knot values $v_h(ih)$, $i = 0, ..., n$, determine v_h uniquely. Equation (7.16) is equivalent to a system of linear algebraic equation with respect to $v_h(ih)$, $i = 0, ..., n$, and our task is to write down this system.

Recall that for $w_h \in \mathcal{R}(P_{h,m})$, we have $w_h = 0$ if and only if $w_h(ih) = 0$, $i = 0, ..., n$. Since $(P_{h,m}w)(ih) = w(ih)$, $i = 0, ..., n$, equation (7.16) is equivalent to the (so-called collocation) conditions

$$v_h(ih) = (T_\varphi v_h)(ih) + f(ih), \ \ i = 0, ..., n,$$

i.e. $v_h \in \mathcal{R}(P_{h,m})$ satisfies equation (5.15) at the knots ih, $i = 0, ..., n$. (Actually collocation methods are usually a priori described by conditions of such type and after that an operator form of the method is derived; we follow the inverse way.) Using the representation (7.13) for v_h we obtain

$$(T_\varphi v_h)(ih) = \int_0^1 K_\varphi(ih,s)v_h(s)ds = \sum_{j=0}^{n-1} \int_{jh}^{(j+1)h} K_\varphi(ih,s)v_h(s)ds$$

$$= \sum_{j=0}^{n-1} \sum_{k\in\mathbb{Z}_m} \int_{jh}^{(j+1)h} K_\varphi(ih,s)L_k(ns-j)ds\,(E_\delta v_h)((j+k)h)$$

$$= \sum_{j=0}^{n-1} \sum_{k\in\mathbb{Z}_m} \alpha_{i,j,k} \cdot \left\{ \begin{array}{cc} v_h(0), & j+k\le 0 \\ v_h((j+k)h), & 1\le j+k\le n \ -1 \\ v_h(1), & j+k\ge n \end{array} \right\} = \sum_{l=0}^{n} b_{i,l} v_h(lh), \ \ i=0,...,n,$$

where we denoted

$$\alpha_{i,j,k} = \int_{jh}^{(j+1)h} K_\varphi(ih,s) L_k(ns-j)ds, \ \ i=0,...,n, \ \ j=0,...,n-1, \ \ k\in\mathbb{Z}_m, \tag{7.20}$$

$$b_{i,l} = \left\{ \begin{array}{cc} \sum_{k\in\mathbb{Z}_m} \sum_{\{j:\, 0\le j\le n-1,\, j+k\le 0\}} \alpha_{i,j,k}\,, & l=0 \\ \sum_{k\in\mathbb{Z}_m} \sum_{\{j:\, 0\le j\le n-1,\, j+k=l\}} \alpha_{i,j,k}\,, & l=1,...,n-1 \\ \sum_{k\in\mathbb{Z}_m} \sum_{\{j:\, 0\le j\le n-1,\, j+k\ge n\}} \alpha_{i,j,k}\,, & l=n \end{array} \right\}, \ \ i,l=0,...,n. \tag{7.21}$$

Thus the matrix form of the collocation method (7.16) is given by

$$v_h(ih) = \sum_{l=0}^{n} b_{i,l} v_h(lh) + f(ih), \quad i=0,...,n, \tag{7.22}$$

with $b_{i,l}$ defined by (7.20), (7.21). Having determined $v_h(ih)$, $i=0,...,n$, through solving the system (7.22), the collocation solution $v_h(t)$ at any intermediate point $t\in[jh,(j+1)h]$, $j=0,..,n-1$, is given by

$$v_h(t) = \sum_{k\in\mathbb{Z}_m} \left\{ \begin{array}{cc} v_h(0), & j+k\le 0 \\ v_h((j+k)h), & 1\le j+k\le n-1 \\ v_h(1), & j+k\ge n \end{array} \right\} \cdot L_k(nt-j) \tag{7.23}$$

where L_k, $k\in\mathbb{Z}_m$, are the Lagrange fundamental polynomials defined in (7.10).

8. Approximation by splines.

8.1. Cardinal B-splines. We present two equivalent definitions of the *father* B-*spline* B_m *of degree* $m-1$ (or, of *order* m, in other terminology).

Definition 8.1 (explicit formula):

$$B_m(x) = \frac{1}{(m-1)!} \sum_{i=0}^{m} (-1)^i \binom{m}{i} (x-i)_+^{m-1}, \quad x\in\mathbb{R}, \ \ m\in\mathbb{N},$$

where, as usual, $0!=1$, $0^0 := \lim_{x\downarrow 0} x^x = 1$,

$$\binom{m}{i} = \frac{m!}{i!\,(m-i)!}, \quad (x-i)_+^{m-1} := \left\{ \begin{array}{cc} (x-i)^{m-1}, & x-i\ge 0 \\ 0, & x-i<0 \end{array} \right\}.$$

Definition 8.2 (recursion): for $x\in\mathbb{R}$,

$$B_1(x) = \left\{ \begin{array}{cc} 1, & 0\le x<1 \\ 0, & x\in\mathbb{R}\setminus[0,1) \end{array} \right\}, \ \ B_m(x) = \int_{x-1}^{x} B_{m-1}(y)dy, \quad m=2,3,\dots .$$

In particular, linear, quadratic and cubic B-splines correspond respectively to $m=2,3,4$ and are given by the formulae

$$B_2(x) = \left\{ \begin{array}{cc} x, & 0\le x\le 1 \\ 2-x, & 1\le x\le 2 \\ 0, & \text{otherwise} \end{array} \right\},$$

$$B_3(x) = \left\{ \begin{array}{cc} \frac{1}{2}x^2, & 0 \le x \le 1 \\ \frac{1}{2}(-2x^2+6x-3), & 1 \le x \le 2 \\ \frac{1}{2}(3-x)^2, & 2 \le x \le 3 \\ 0, & \text{otherwise} \end{array} \right\},$$

$$B_4(x) = \left\{ \begin{array}{cc} \frac{1}{6}x^3, & 0 \le x \le 1 \\ \frac{1}{6}(-3x^3+12x^2-12x+4), & 1 \le x \le 2 \\ B_4(4-x), & 2 \le x \le 4 \\ 0, & \text{otherwise} \end{array} \right\}.$$

The equivalence of Definitions 8.1 and 8.2 can be easily established checking that both definitions yield the same $B_1(x)$ and imply $B_m(x) = 0$ for $x \le 0$, $B'_m(x) = B_{m-1}(x) - B_{m-1}(x-1)$ for $x \ge 0$, $m \ge 2$.

The following properties of B_m, $m \ge 2$, can be seen using suitable definition:

$$B_m \,|_{[i,i+1]} \in \mathcal{P}_{m-1}, \; i \in \mathbb{Z}, \quad B_m \in C^{(m-2)}(\mathbb{R}),$$

$$\text{supp} B_m = [0, m], \quad B_m(x) > 0 \text{ for } 0 < x < m, \quad \int_{\mathbb{R}} B_m(x)dx = 1,$$

$$B_m^{(m-1)}(x) = (-1)^i \begin{pmatrix} m-1 \\ i \end{pmatrix} \text{ for } i < x < i+1, \; i = 0, ..., m-1,$$

$$B_m(\frac{m}{2} - x) = B_m(\frac{m}{2} + x), \;\; x \in \mathbb{R}, \quad B_m(\frac{m}{2}) = \max_{x \in \mathbb{R}} B_m(x)$$

(moreover, $B'_m(x) > 0$ for $0 < x < \frac{m}{2}$ and $B'_m(x) < 0$ for $\frac{m}{2} < x < m$),

$$\sum_{j \in \mathbb{Z}} B_m(x-j) = 1, \;\; x \in \mathbb{R}.$$

The first line in this list says that B_m is a spline (=piecewise polynomial function) of degree $m-1$ and defect 1 with the "cardinal" knot set $\mathbb{Z}$; the defect of a spline is the difference between its degree and global smoothness number, in our case $(m-1) - (m-2) = 1$ that is the minimal possible defect of a spline not degenerated into a polynomial globally.

8.2. The Wiener interpolant. Introduce in $\mathbb{R}$ the uniform grid $h\mathbb{Z} = \{ih : i \in \mathbb{Z}\}$ of the step size $h > 0$. Denote by $S_{h,m}$, $m \in \mathbb{N}$, the space of splines of degree $m-1$ and defect 1 with the knot set $h\mathbb{Z}$. For $m = 1$, $S_{h,1}$ consists of piecewise constant functions with possible breaks at ih, $i \in \mathbb{Z}$. For $m \ge 2$, $S_{h,m}$ consists of the functions $g \in C^{m-2}(\mathbb{R})$ such that $g \,|_{[ih,(i+1)h]} \in \mathcal{P}_{m-1}$, $i \ni \mathbb{Z}$. Clearly the dilated and shifted B-splines $B_m(h^{-1}x - j)$, $j \in \mathbb{Z}$, belong to $S_{h.m}$, and the same is true for $\sum_{j \in \mathbb{Z}} d_j B_m(h^{-1}x - j)$ with arbitrary coefficients d_j; note that there are no problems with the convergence of the series since it is locally finite:

$$\sum_{j \in \mathbb{Z}} d_j B_m(h^{-1}x - j) = \sum_{j=i-m+1}^{i} d_j B_m(h^{-1}x - j) \;\text{ for }\; x \in [ih, (i+1)h), \; i \in \mathbb{Z}.$$

Indeed, supp $B_m(h^{-1}x - j) = [jh, (j+m)h]$, and only the terms with $(ih, (i+1)h) \cap (jh, (j+m)h) \ne \emptyset$ must be taken into account.

Given a function $f \in C(\mathbb{R})$, we look for its interpolant $Q_{h,m}f \in S_{h,m}$ in the form

(8.1) $$(Q_{h,m}f)(x) = \sum_{j\in\mathbb{Z}} d_j B_m(h^{-1}x - j), \quad x \in \mathbb{R},$$

and determine the coefficients d_j from the interpolation conditions

(8.2) $$(Q_{h,m}f)((k+\frac{m}{2})h) = f((k+\frac{m}{2})h), \quad k \in \mathbb{Z}.$$

This leads to the bi-infinite system of linear equations

$$\sum_{j\in\mathbb{Z}} B_m(k+\frac{m}{2}-j)d_j = f((k+\frac{m}{2})h), \quad k \in \mathbb{Z},$$

or

(8.3) $$\sum_{j\in\mathbb{Z}} b_{k-j}d_j = f_k, \quad k \in \mathbb{Z},$$

where

(8.4) $$b_k = k_{k,m} = B_m(k+\frac{m}{2}), \;\; f_k = f_{k,h,m} = f((k+\frac{m}{2})h), \;\; k \in \mathbb{Z}.$$

Denote

$$\mu := \text{int }((m-1)/2) = \left\{ \begin{array}{ll} (m-2)/2, & m \text{ even} \\ (m-1)/2, & m \text{ odd} \end{array} \right\}.$$

It follows from properties of B_m listed in Section 8.1 that

$$b_k = b_{-k} > 0 \text{ for } \mid k \mid \leq \mu, \;\; b_k = 0 \text{ for } \mid k \mid > \mu, \;\; \sum_{|k|\leq\mu} b_k = 1.$$

Thus (8.3) is a bi-infinite system with the symmetric Toeplitz band matrix $\mathfrak{B} = (b_{k-j})_{k,j\in\mathbb{Z}}$ of the band width $2\mu+1$. For $m=2$, system (8.3) reduces to relations $d_k = f((k+1)h))$, $k \in \mathbb{Z}$, and $(Q_{h,2}f)(x) = \sum_{j\in\mathbb{Z}} f((j+1)h))B_2(nx-j)$ is the usual piecewise linear interpolant which can be constructed on every subinterval $[ih,(i+1)h]$ independently from other subintervals. All is clear in the cases $m = 1, 2$ and we focus our attention to the case $m \geq 3$. A delicate problem appears that the solution of system (8.3) always exists but is nonunique for $m \geq 3$ if we allow an exponential growth of $\mid d_j \mid$ as $\mid j \mid \to \infty$. Indeed, we can arbitrarily fix 2μ consecutive unknowns d_j, for instance $d_1, ..., d_{2\mu}$, and after that (8.3) enables a recursive determination of d_j for $j = 0, -1, ...$ and for $j = 2\mu+1, 2\mu+2, ...$ In particular, solutions of the homogenous system $\sum_{j\in\mathbb{Z}} b_{k-j}d_j = 0$, $k \in \mathbb{Z}$, constitute in the vector space of all bisequences (d_j) a 2μ dimensional subspace; later we describe a basis of this subspace.

Only one of the solutions of system (8.3) is reasonable for an $f \in BC(\mathbb{R})$ or an $f \in C(\mathbb{R})$ of a polynomial growth as $\mid x \mid\to \infty$. This solution is related to the Wiener theorem (see [40]) which states the following:

$$\text{if } \sum_{k\in\mathbb{Z}} \mid b_k \mid < \infty \text{ and } \beta(x) := \sum_{k\in\mathbb{Z}} b_k e^{\mathrm{i}kx} \neq 0 \text{ for all } x \in \mathbb{R},$$

$$\text{then } \alpha(x) := 1/\beta(x) = \sum_{k\in\mathbb{Z}} a_k e^{\mathrm{i}kx} \text{ with } \sum_{k\in\mathbb{Z}} \mid a_k \mid < \infty.$$

Setting $z = e^{\mathrm{i}x}$ we reformulate the Wiener theorem for Laurent series on the unit circle $\mid z \mid = 1$ of the complex plane as follows: if real or complex numbers b_k, $k \in \mathbb{Z}$, are such that

(8.5) $$\sum_{k\in\mathbb{Z}} \mid b_k \mid < \infty, \;\; b(z) := \sum_{k\in\mathbb{Z}} b_k z^k \neq 0 \text{ for all } z \in \mathbb{C} \text{ with } \mid z \mid = 1,$$

then also the function $a(z) := 1/b(z)$ has an expansion $a(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ with $a_k \in \mathbb{C}$, $k \in \mathbb{Z}$, such that $\sum_{k\in\mathbb{Z}} | a_k |< \infty$. (Hence the expansion of a converges uniformly on the circle $| z |= 1$ of the complex plane, similarly as the expansion of b does.) It is easy to understand (the argument is presented in Section 8.3 in more details) that the Toeplitz matrix $\mathfrak{A} = (a_{k-j})_{k,j\in\mathbb{Z}}$ is the inverse to $\mathfrak{B} = (b_{k-j})_{k,j\in\mathbb{Z}}$, i.e., $\mathfrak{B}\mathfrak{A} = \mathfrak{A}\mathfrak{B} = \mathfrak{I}$. We call $\mathfrak{A}$ the Wiener inverse of $\mathfrak{B}$. Condition (8.5) occurs to be fulfilled in our interpolation problem (8.1)–(8.4), so we can use the Wiener inverse $\mathfrak{A}$ of $\mathfrak{B}$ to compute $d_k = \sum_{j\in\mathbb{Z}} a_{k-j} f_j$ and to define the *Wiener interpolant* $Q_{h,m}f$ by

$$(Q_{h,m}f)(x) = \sum_{k\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} a_{k-j} f((j+\frac{m}{2})h) \right) B_m(h^{-1}x - k), \quad x \in \mathbb{R}. \tag{8.6}$$

Only a finite number of b_k do not vanish in the interpolation system (8.3). As we will see Section 8.3, this enables an elementary construction of the numbers a_k, $k \in \mathbb{Z}$, and it occurs that a_k are real and decay exponentially as $| k |\to \infty$.

Due to the exponential decay of a_k, we may truncate the series in (8.6) to $O(\log n)$ terms to compute $Q_{h,m}f$ at a fixed point x with an accuracy $O(h^m)$. A further consequence of the exponential decay of a_k is that the series in (8.6) converges not only for bounded but also for polynomially growing functions $f(x)$ as $| x |\to \infty$, and $Q_{h,m}f$ is well defined for such functions f.

8.3. Construction of the Wiener interpolant. Using the values $b_k = b_{k,m} = B_m(k + \frac{m}{2})$ (see (8.4)), introduce the function

$$b(z) = b^m(z) := \sum_{|k|\le\mu} b_k z^k = b_0 + \sum_{k=1}^{\mu} b_k(z^k + z^{-k}), \ 0 \ne z \in \mathbb{C}, \tag{8.7}$$

the *characteristic polynomial* $P_{2\mu} = P^m_{2\mu} \in \mathcal{P}_{2\mu}$ of B_m defined by

$$P_{2\mu}(z) = z^\mu b(z),$$

and the function

$$a(z) = a^m(z) := 1/b^m(z) = z^\mu/P^m_{2\mu}(z), \ \ z \in \mathbb{C}, \ \ z \ne z_\nu, \ \ \nu = 1, ..., 2\mu, \tag{8.8}$$

where z_ν, $\nu = 1, ..., 2\mu$, are the roots of the $P_{2\mu} \in \mathcal{P}_{2\mu}$ (called the *characteristic roots*). From (8.7) we observe that together with z_ν also $1/z_\nu$ is a characteristic root. It occurs that all characteristic roots are real and simple; then clearly $z_\nu < 0$, $\nu = 1, ..., 2\mu$ and $z_\nu \ne -1$, $\nu = 1, ..., 2\mu$, thus there is exactly μ characteristic roots in the interval $(-1, 0)$ and μ characteristic roots in $(-\infty, -1)$. We omit a relatively sophisticated proof of this statement and quote to the monograph by Stechkin and Subbotin [23]. It is possible to check the statement when the interpolant (8.1), (8.6) is constructed in the practice, since the algorithm needs the values of z_ν, $\nu = 1, ..., 2\mu$, so they must be computed in any case. Let us turn to examples:

$$m = 3: \ \ \mu = 1, \ \ b_{-1} = b_1 = \frac{1}{8}, \ \ b_0 = \frac{3}{4}, \ \ P^3_2(z) = \frac{1}{8}(z^2 + 6z + 1), \ \ z_{1,2} = -3 \pm \sqrt{8};$$

$$m = 4: \ \ \mu = 1, \ \ b_{-1} = b_1 = \frac{1}{6}, \ \ b_0 = \frac{2}{3}, \ \ P^4_2(z) = \frac{1}{6}(z^2 + 4z + 1), \ \ z_{1,2} = -2 \pm \sqrt{3};$$

$$m = 6: \ \ \mu = 2, \ \ P^6_4(z) = \frac{1}{5!}(z^4 + 26z^3 + 66z^2 + 26z + 1), \ \ w_{1,2} := -13 \pm \sqrt{105},$$

$$z_{1,2,3,4} = \frac{w_{1,2} \pm \sqrt{w^2_{1,2} - 4}}{2}, \ \ z_1 \approx -0,043096, \ \ z_2 \approx -0,430575,$$

$$z_3 = \frac{1}{z_1}, \quad z_4 = \frac{1}{z_2};$$

$$m = 10: \quad \mu = 4, \quad P_8^{10}(z) = \frac{1}{9!}(z^8 + 502z^7 + 14608z^6 + 88234z^5 + 156190z^4$$

$$+88234z^3 + 14608z^2 + 502z + 1), \quad z_5 = \frac{1}{z_1}, \ z_6 = \frac{1}{z_2}, \ z_7 = \frac{1}{z_3}, \ z_8 = \frac{1}{z_4},$$

$$z_1 = -2.121307 \cdot 10^{-3}, \quad z_2 = -0,043223, \quad z_3 = -0,201751, \quad z_4 = -0,607997.$$

All computations of the present Section 8 are performed by Evely Leetma.

For functions $a(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ and $b(z) = \sum_{k\in\mathbb{Z}} b_k z^k$ defined by the absolutely convergent Laurent series on the unit circle $\mid z \mid = 1$, it is easily seen that $a(z)b(z) = \sum_{k\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} a_{k-j} b_j\right) z^k$. For a and b defined in (8.7), (8.8), this yields $\sum_{j\in\mathbb{Z}} a_{k-j} b_j = \delta_{k,0}$, $k \in \mathbb{Z}$, where $\delta_{k,l}$ is the Kronecker symbol. Replacing here k by $k-l$ we rewrite it as $\sum_{j\in\mathbb{Z}} a_{k-l-j} b_j = \delta_{k-l,0}$, $k, l \in \mathbb{Z}$, or introducing the new summation index $j' = j + l$, as $\sum_{j'\in\mathbb{Z}} a_{k-j'} b_{j'-l} = \delta_{k,l}$. Finally, writing j instead j', the equality takes the form

$$\sum_{j\in\mathbb{Z}} a_{k-j} b_{j-l} = \delta_{k,l}, \quad k, l \in \mathbb{Z}.$$

Similarly (or simply by a symmetry argument),

$$\sum_{j\in\mathbb{Z}} b_{k-j} a_{j-l} = \delta_{k,l}, \quad k, l \in \mathbb{Z}.$$

The last two equalities mean that the matrix $\mathfrak{B} = (b_{k-j})_{k,j\in\mathbb{Z}}$ of system (8.3) has the inverse $\mathfrak{B}^{-1} = \mathfrak{A} = (a_{k-j})_{k,j\in\mathbb{Z}}$. Our task takes the form: find the coefficients a_k of the Laurent series $a(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ for the function a defined by (8.7), (8.8).

Let us arrange the characteristic roots $z_1, ..., z_{2\mu}$ so that $z_1, ..., z_\mu$ are in the interval $(-1, 0)$ and $z_{\mu+\nu} = 1/z_\nu$, $\nu = 1, ..., \mu$. Since all roots are simple, the function $a(z) := 1/b(z) = \frac{z^\mu}{P_{2\mu}(z)}$ has a representation

$$\frac{z^\mu}{P_{2\mu}(z)} = \sum_{\nu=1}^{2\mu} \frac{c_\nu}{z - z_\nu}.$$

Multiplying by $\prod_{\lambda=1}^{2\mu}(z - z_\lambda) = P_{2\mu}(z)/b_\mu$ we rewrite it as

$$\frac{z^\mu}{b_\mu} = \sum_{\nu=1}^{2\mu} c_\nu \prod_{\nu\neq\lambda=1}^{2\mu} (z - z_\lambda) = c_1 \prod_{\lambda=2}^{2\mu} (z - z_\lambda) + ... + c_{2\mu} \prod_{\lambda=1}^{2\mu-1} (z - z_\lambda).$$

Setting $z = z_\nu$ we determine the coefficients c_ν:

$$c_\nu = \frac{z_\nu^\mu}{b_\mu \prod_{\nu\neq\lambda=1}^{2\mu}(z_\nu - z_\lambda)} = \frac{z_\nu^\mu}{P'_{2\mu}(z_\nu)}, \quad \nu = 1, ..., 2\mu.$$

Thus

$$a(z) = \sum_{\nu=1}^{2\mu} \frac{z_\nu^\mu}{P'_{2\mu}(z_\nu)} \frac{1}{z - z_\nu} = \sum_{\nu=1}^{\mu} \left(\frac{z_\nu^\mu}{P'_{2\mu}(z_\nu)} \frac{1}{z - z_\nu} + \frac{z_\nu^{-\mu}}{P'_{2\mu}(z_\nu^{-1})} \frac{1}{z - z_\nu^{-1}} \right).$$

It follows from (8.7) that $P_{2\mu}(z^{-1}) = z^{-2\mu}P_{2\mu}(z)$. Differentiating this equality and setting then $z = z_\nu$ we find that $-P'_{2\mu}(z_\nu^{-1})z_\nu^{-2} = z_\nu^{-2\mu}P'_{2\mu}(z_\nu)$, or $\frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} = -\frac{z_\nu^{-\mu+1}}{P'_{2\mu}(z_\nu^{-1})}$. Now we can rewrite

$$a(z) = \sum_{\nu=1}^{\mu} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \left(\frac{z_\nu}{z - z_\nu} - \frac{z_\nu^{-1}}{z - z_\nu^{-1}} \right) = \sum_{\nu=1}^{\mu} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \left(\frac{z_\nu z^{-1}}{1 - z_\nu z^{-1}} + \frac{1}{1 - z_\nu z} \right).$$

Expanding

$$\frac{z_\nu z^{-1}}{1 - z_\nu z^{-1}} = \sum_{k=1}^{\infty} z_\nu^k z^{-k} \quad \text{for} \quad | z | > | z_\nu |, \quad \nu = 1, ..., \mu,$$

$$\frac{1}{1 - z_\nu z} = \sum_{k=0}^{\infty} z_\nu^k z^k \quad \text{for} \quad | z | < | z_\nu |^{-1}, \quad \nu = 1, ..., \mu,$$

we arrive at the desired expansion

$$a(z) = \sum_{\nu=1}^{\mu} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \left(\sum_{k=1}^{\infty} z_\nu^k z^{-k} + \sum_{k=0}^{\infty} z_\nu^k z^k \right) = \sum_{k \in \mathbb{Z}} a_k z^k$$

which converges for $\theta_m <| z |< \theta_m^{-1}$ where

$$\theta_m = \max_{1 \le \nu \le \mu} | z_\nu |< 1.$$

Thus

$$a_k = \sum_{\nu=1}^{\mu} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} z_\nu^{|k|} = a_{-k}, \ | a_k | \le c_m \theta_m^{|k|}, \ k \in \mathbb{Z}, \ c_m = \sum_{\nu=1}^{\mu} \frac{| z_\nu |^{\mu-1}}{| P'_{2\mu}(z_\nu) |}. \tag{8.9}$$

As we see, a_k decays exponentially. Moreover,

$$\sum_{k \in \mathbb{Z}} a_k = 1, \ \sum_{k \in \mathbb{Z}} | a_k |= \frac{(-1)^\mu}{P_{2\mu}(-1)}, \quad a_k = (-1)^k | a_k |\neq 0, \ k \in \mathbb{Z}. \tag{8.10}$$

Indeed, (8.7), (8.8) immediately yield the first of the claims:

$$\sum_{k \in \mathbb{Z}} a_k = a(1) = 1/b(1) = 1/ \sum_{k=-\mu}^{\mu} b_k = 1.$$

Next we prove the third one of claims (8.10). We start from equalities

$$a(z) = 1/b(z) = \frac{z^\mu}{P_{2\mu}(z)} = \frac{z^\mu}{b_\mu \prod_{\lambda=1}^{2\mu}(z - z_\lambda)} = \frac{z^\mu}{b_\mu \prod_{\lambda=1}^{2\mu}(z+ | z_\lambda |)}.$$

Setting $-z$ into the place of z, we have

$$a(-z) = \frac{(-z)^\mu}{b_\mu \prod_{\lambda=1}^{2\mu}(-z+ | z_\lambda |)} = \frac{(-z)^\mu}{b_\mu \prod_{\lambda=1}^{\mu}(-z+ | z_\lambda |) \prod_{\lambda=1}^{\mu}(-z+ | z_\lambda^{-1} |)}$$

$$= \frac{z^\mu(-1)^\mu}{b_\mu \prod_{\lambda=1}^{\mu}(z- | z_\lambda |) \prod_{\lambda=1}^{\mu}(z- | z_\lambda^{-1} |)}$$

$$= \frac{1}{b_\mu \prod_{\lambda=1}^{\mu}(1- | z_\lambda | z^{-1}) \prod_{\lambda=1}^{\mu}(| z_\lambda^{-1} | -z)}$$

$$= \frac{\prod_{\lambda=1}^{\mu} | z_\lambda |}{b_\mu \prod_{\lambda=1}^{\mu}(1- | z_\lambda | z^{-1}) \prod_{\lambda=1}^{\mu}(1- | z_\lambda | z)}$$

$$= \frac{\prod_{\lambda=1}^{\mu} | z_\lambda |}{b_\mu} \prod_{\lambda=1}^{\mu} \left(\sum_{k=0}^{\infty} | z_\lambda |^k z^{-k} \right) \prod_{\lambda=1}^{\mu} \left(\sum_{k=0}^{\infty} | z_\lambda |^k z^k \right) = \sum_{k\in\mathbb{Z}} c_k z^k, \quad | z |= 1,$$

with some $c_k > 0$, $k \in \mathbb{Z}$. Returning to z instead of $-z$, we obtain $a(z) = \sum_{k\in\mathbb{Z}}(-1)^k c_k z^k$. The Laurent expansion $a(z) = \sum_{k\in\mathbb{Z}} a_k z^k$ is unique, therefore $a_k = (-1)^k c_k = (-1)^k \mid a_k \mid$, $k \in \mathbb{Z}$, as asserted in (8.10). Equivalently, it holds $\mid a_k \mid= (-1)^k a_k$, $k \in \mathbb{Z}$. Now the second one of claims (8.10) follows by the same argument as the first one:

$$\sum_{k\in\mathbb{Z}} | a_k | = \sum_{k\in\mathbb{Z}} a_k (-1)^k = a(-1) = 1/b(-1) = (-1)^\mu / P_{2\mu}(-1).$$

Let us summarise the main results and present some further comments.

Theorem 8.1. *For a bounded or polynomially growing $f \in C(\mathbb{R})$, the Wiener interpolant $Q_{h,m} f$ is well defined by formula (8.6) with $a_k = a_{k,m}$ given in (8.9) where $z_\nu = z_{\nu,m}$, $\nu = 1, ...\mu$, are the roots of the characteristic polynomial $P_{2\mu} = P_{2\mu}^m$ in the interval $(-1, 0)$; further properties of a_k are listed in (8.9) and (8.10).*

Together with considerations of Section 8.2, it is easily seen that the null space $\mathcal{N}(\mathfrak{B})$ of the matrix $\mathfrak{B} = (b_{k-j})_{k,j\in\mathbb{Z}}$ in the vector space X of all bi-infinite vectors $(d_j)_{j\in\mathbb{Z}}$ is of dimension 2μ and is spanned by bisequences $(z_\nu^j)_{j\in\mathbb{Z}}$, $\nu = 1, ..., 2\mu$. For $\nu = 1, ...\mu$, it holds $\mid z_\nu^j \mid\to \infty$ as $j \to -\infty$, whereas for $\nu = \mu + 1, ..., 2\mu$, it holds $\mid z_\nu^j \mid\to \infty$ as $j \to \infty$. Hence any nontrivial element of $\mathcal{N}(\mathfrak{B})$ is a bisequence which grows exponentially either as $j \to \infty$ or as $j \to -\infty$.

Clearly, $\| Q_{h,m} \|_{BC(\mathbb{R})\to BC(\mathbb{R})} \le \sum_{k\in\mathbb{Z}} | a_{k,m} | =: \alpha_m$ but this estimate is be coarse; in Section 8.7 we present an exact formula for $\| Q_{h,m} \|_{BC(\mathbb{R})\to BC(\mathbb{R})}$.

The numerical values of $\alpha_m = \sum_{k\in\mathbb{Z}} | a_{k,m} |= 1/ | P_{2\mu}^m(-1) |$ for $m = 2, ..., 10$ are presented in the following table.

m	α_m	α_{m+1}/α_m
2	1	2
3	2	1.5
4	3	1.6
5	4.8	1.5625
6	7.5	1.5738
7	11.803279	1.5699
8	18.529412	1.5711
9	29.111913	1.5707
10	45.725806	1.5708
20	4181.841275	1.570796327

We can observe that $\alpha_{m+1}/\alpha_m \to \pi/2 = 1.5707963268...$ as $m \to \infty$. It is challenging to confirm this empiric guess analytically.

Finally, let us mention that for any bisequence d_j, $j \in \mathbb{Z}$, such that $\sup_j | d_j |< \infty$, it holds

$$\frac{\sup_j | d_j |}{\alpha_m} \le \sup_{x\in\mathbb{R}} | \sum_{j\in\mathbb{Z}} d_j B(h^{-1}x - j) | \le \sup_j | d_j | .$$

8.4. Euler splines. A spline $E \in S_{h,m}$ is called *perfect* if

$$E^{(m-1)}(x) = (-1)^i \ \text{for} \ ih < x < (i + 1)h, \ i \in \mathbb{Z}.$$

If $E \in S_{h,m}$ is perfect then so is $E + g$ with any $g \in \mathcal{P}_{m-2}$.

For $m = 1$, the *Euler perfect spline* $E_{h,1} \in S_{h,1}$ is defined by the formula

$$E_{h,1}(x) = \text{sign } \sin h^{-1}\pi x = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2k+1)h^{-1}\pi x}{2k+1}. \tag{8.11}$$

Here, as usual,

$$\text{sign} z = \left\{ \begin{array}{ll} -1, & z < 0 \\ 0, & z = 0 \\ 1, & z > 0 \end{array} \right\}, \ z \in \mathbb{R};$$

the series in (8.11) is the Fourier expansion of the $2h$-periodic piecewise constant function $y = \text{sign } \sin h^{-1}\pi x$. For $m \geq 2$, the *Euler perfect spline* $E_{h,m} \in S_{h,m}$ is determined recursively as a special integral function of $E_{h,m-1}$, namely,

$$E_{h,m}(x) = \left\{ \begin{array}{ll} \int_{h/2}^{x} E_{h,m-1}(y)dy, & m = 2l \\ \int_{0}^{x} E_{h,m-1}(y)dy, & m = 2l+1 \end{array} \right\}, \ l = 1, 2, \ldots;$$

the lower bound of integration is chosen so that the zero mean value of $E_{h,m-1}$ over a period remains to be zero also for $E_{h,m}$. Starting from (8.11) we recursively find that

$$E_{h,m}(x) = \left\{ \begin{array}{ll} \frac{4}{\pi}\frac{(-1)^l h^{m-1}}{\pi^{m-1}} \sum_{k=0}^{\infty} \frac{\cos(2k+1)h^{-1}\pi x}{(2k+1)^m}, & m = 2l \\ \frac{4}{\pi}\frac{(-1)^l h^{m-1}}{\pi^{m-1}} \sum_{k=0}^{\infty} \frac{\sin(2k+1)h^{-1}\pi x}{(2k+1)^m}, & m = 2l+1 \end{array} \right\} \tag{8.12}$$

where $l = 1, 2, \ldots$. For instance, $E_{h,2} \in S_{h,2}$ is continuous piecewise linear function with the knot values $E_{h,2}(ih) = (-1)^{i+1}\frac{h}{2}$, $i \in \mathbb{Z}$; a consequence is that

$$E_{h,2}(x_1) - E_{h,2}(x_2) = (-1)^i(x_1 - x_2), \quad \text{for } x_1, x_2 \in (ih, (i+1)h), \ i \in \mathbb{Z}.$$

Clearly, $E'_{h,m} = E_{h,m-1}$ for $m \geq 2$. Further, from (8.12) we observe that $x = (i + \frac{1}{2})h$, $i \in \mathbb{Z}$, are the zeroes of $E_{h,m}$ for even m, and $x = ih$, $i \in \mathbb{Z}$, are the zeroes of $E_{h,m}$ for odd m. A unified formulation is that $x = (i + \frac{m-1}{2})h$, $i \in \mathbb{Z}$, are the zeroes of $E_{h,m}$ and $x = (i + \frac{m}{2})h$, $i \in \mathbb{Z}$, are the local extremums of $E_{h,m}$ (the zeroes of $E'_{h,m} = E_{h,m-1}$). There are no other zeroes and extrema of $E_{h,m}$ – this can be easily seen by recursion, since by Rolle's theorem an additional zero of $E_{h,m}$ gives rise to an additional zero of $E'_{h,m} = E_{h,m-1}$. It is clear also that the roots of $E_{h,m}$ are simple. Further, for $m = 2l$,

$$\| E_{h,m} \|_\infty = | E_{h,m}(0) | = \frac{4}{\pi}\frac{h^{m-1}}{\pi^{m-1}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^m}$$

(the absolute value of $E_{h,m}$ at other local extremum points $x = ih$ is same). Similarly, for $m = 2l+1$,

$$\| E_{h,m} \|_\infty = | E_{h,m}(\frac{h}{2}) | = \frac{4}{\pi}\frac{h^{m-1}}{\pi^{m-1}} \sum_{k=0}^{\infty} \frac{\sin(2k+1)\frac{\pi}{2}}{(2k+1)^m} = \frac{4}{\pi}\frac{h^{m-1}}{\pi^{m-1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^m}.$$

Unifying these two formulae, we can write

$$\| E_{h,m} \|_\infty = \Phi_m \pi^{-(m-1)} h^{m-1}, \ m \in \mathbb{N}, \tag{8.13}$$

where

$$\Phi_m = \frac{4}{\pi} \left\{ \begin{array}{ll} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^m}, & m = 2l \\ \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^m}, & m = 2l+1 \end{array} \right\} = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^{km}}{(2k+1)^m} \tag{8.14}$$

is for $m \in \mathbb{N}$ known as the *Favard constant.* In particular,

$$\Phi_1 = 1, \ \Phi_2 = \pi/2, \ \Phi_3 = \pi^2/8, \ \Phi_4 = \pi^3/24,$$

and it holds

$$\Phi_1 < \Phi_3 < \Phi_5 < \ldots < \frac{4}{\pi} < \ldots < \Phi_6 < \Phi_4 < \Phi_2, \quad \lim_{m\to\infty} \Phi_m = \frac{4}{\pi}.$$

8.5. Error bounds for the Wiener interpolant. Introduce the vector space $V^{m,\infty}(\mathbb{R})$, $m \in \mathbb{N}$, consisting of functions $f \in C^{m-1}(\mathbb{R})$ such that $f^{(m)} \in L^\infty(\mathbb{R})$ (the derivatives are understood in the sense of distributions). A function $f \in V^{m,\infty}(\mathbb{R})$ may grow as $| x |\to \infty$; with the help of the Taylor formula

$$f(x) = \sum_{l=0}^{m-1} \frac{f^{(l)}(0)}{l!} x^l + \frac{1}{(m-1)!} \int_0^x (x-t)^{m-1} f^{(m)}(t)dt, \ \ x \in \mathbb{R},$$

we observe that

$$| f(x) | \leq \| f^{(m)} \|_\infty \frac{1}{(m-1)!} | x |^m + O(x^{m-1}) \ \text{as} \ \ | x |\to \infty.$$

Hence, $Q_{h,m} f$ is well defined for $f \in V^{m,\infty}(\mathbb{R})$, see Sections 8.2, 8.3. Clearly, $W^{m,\infty}(\mathbb{R}) + \mathcal{P}_m \subset V^{m,\infty}(\mathbb{R})$; this inclusion is strict.

For $f \in V^{m,\infty}(\mathbb{R})$, $f^{(m-1)}$ is Lipschitz continuous:

$$| f^{(m-1)}(x_1) - f^{(m-1)}(x_2) | \leq \| f^{(m)} \|_\infty | x_1 - x_2 |, \ \ x_1, x_2 \in \mathbb{R}. \tag{8.15}$$

This immediately follows from the equality

$$f^{(m-1)}(x_2) - f^{(m-1)}(x_1) = \int_{x_1}^{x_2} f^{(m)}(x)dx, \ \ x_1, x_2 \in \mathbb{R}.$$

Theorem 8.2. *For $f \in V^{m,\infty}(\mathbb{R})$, $m \in \mathbb{N}$, there hold the pointwise estimate*

$$| f(x) - (Q_{h,m} f)(x) | \leq \| f^{(m)} \|_\infty | E_{h,m+1}(x) |, \ \ \ x \in \mathbb{R}, \tag{8.16}$$

and the uniform estimate

$$\| f - Q_{h,m} f \|_\infty \leq \Phi_{m+1} \pi^{-m} h^m \ \| f^{(m)} \|_\infty . \tag{8.17}$$

For $f = E_{h,m+1} \in W^{m,\infty}(\mathbb{R}) \subset V^{m,\infty}(\mathbb{R})$, inequalities (8.16) *and* (8.17) *turn into equalities.*

In the proof Theorem 8.2 we repeatedly exploit the following auxiliary result.

Lemma 8.1. *Suppose that for $g_\delta \in C(\mathbb{R})$, $\delta > 0$, the pointwise convergence $g_\delta(x) \to 0$, $x \in \mathbb{R}$, takes place as $\delta \to 0$, and let $| g_\delta(x) | \leq c(1 + | x |^r)$, $x \in \mathbb{R}$, where $c \geq 0$ and $r \geq 0$ are independent of δ. Then also $(Q_{h,m} g_\delta)(x) \to 0$ for any $x \in \mathbb{R}$ as $\delta \to 0$.*

Proof. Fix an arbitrary $x \in \mathbb{R}$ and take $i \in \mathbb{Z}$ such that $x \in [ih, (i+1)h)$. Then

$$(Q_{h,m} g_\delta)(x) = \sum_{k=i-m+1}^{i} \left(\sum_{j \in \mathbb{Z}} a_j g_\delta((k - j + \frac{m}{2})h) \right) B_m(h^{-1}x - k).$$

Since $B_m(h^{-1}x - k) \geq 0$ and $\sum_k B_m(h^{-1}x - k) = 1$, we obtain

$$| (Q_{h,m} g_\delta)(x) | \leq \max_{i-m+1 \leq k \leq i} \sum_{j \in \mathbb{Z}} | a_j || g_\delta((k - j + \frac{m}{2})h) | .$$

To complete the proof of the Lemma, it is sufficient to show that $\sum_{j \in \mathbb{Z}} | a_j || g_\delta((k - j + \frac{m}{2})h) |\to 0$ as $\delta \to 0$ for any fixed $k \in \mathbb{Z}$. Fixing an arbitrary small $\varepsilon > 0$, represent $\sum_{j \in \mathbb{Z}} = \sum_{|j| \leq N} + \sum_{|j| > N}$ with $N = N(\varepsilon, k)$ so large that

$$\sum_{|j| > N} | a_j | (1 + (| j - k | + \frac{m}{2})^r h^r) \leq \varepsilon;$$

such N exists since a_j decays exponentially as $| j |\to \infty$. Using the conditions of the Lemma we obtain

$$\sum_{|j| > N} | a_j || g_\delta((k - j + \frac{m}{2})h) | \leq c \sum_{|j| > N} | a_j | (1 + (| j - k | + \frac{m}{2})^r h^r) \leq c\varepsilon$$

and, for sufficiently small $\delta > 0$,

$$\sum_{|j|\le N} | a_j || g_\delta((i-j+\frac{m}{2})h) | < \varepsilon, \quad \sum_{j\in\mathbb{Z}} | a_j || g_\delta((i-j+\frac{m}{2})h) | < (c+1)\varepsilon.$$

This completes the proof of Lemma 8.1 . $\square$

Proof of Theorem 8.2. (i) *Proof scheme.* First of all we note that the claim about the sharpness of estimates (8.16) and (8.17) is elementary. Indeed, recall that $E_{h,m+1}$ vanishes at $(i+\frac{m}{2})h$, $i \in \mathbb{Z}$, which are the interpolation points for $Q_{h,m}$, hence $Q_{h,m}E_{h,m+1} = 0$; moreover,

$$\| E^{(m)}_{h,m+1} \|_\infty = \| E_{h,1} \|_\infty = 1.$$

Thus for $f = E_{h,m+1}$ (8.16) turns into equality $| E_{h,m+1}(x) | = | E_{h,m+1}(x) |$, whereas (8.17) turns into equality $\| E_{h,m+1} \|_\infty = \Phi_{m+1}\pi^{-m}h^m$ which holds due to (8.13).

Estimate (8.17) immediately follows from (8.16) and (8.13). So it remains to establish (8.16) only. For $m = 1$ and $m = 2$, usual local estimates of piecewise constant and piecewise linear interpolant can be presented in the form (8.16) remembering that $\Phi_2 = \pi/2$, $\Phi_3 = \pi^2/8$; the details are left as an exercise. In the sequel we assume that $m \ge 3$ and we prove (8.16) during four stages: in (ii) for periodic $f \in W^{m,\infty}(\mathbb{R})$, in (iii) for compactly supported $f \in W^{m,\infty}(\mathbb{R})$, in (iv) for $f \in V^{m,\infty}(\mathbb{R})$ of a special growth estimate, and in (v) for arbitrary $f \in V^{m,\infty}(\mathbb{R})$.

(ii) *Periodic case.* Here we prove (8.16) for $f \in W^{m,\infty}(\mathbb{R})$ which is periodic with a period $p = 2nh$, $n \in \mathbb{N}$. Then also $Q_{h,m}f$ is p-periodic, and so is $E_{h,m+1}$ (recall that $E_{h,m+1}$ has the period $2h$). We show that the violation of (8.16) for such f involves a contradiction. Let $\xi \in [0,p)$ be a point where the inverse to (8.16) holds: $| f(\xi) - (Q_{h,m}f)(\xi) | > \| f^{(m)} \|_\infty | E_{h,m+1}(\xi) |$; clearly $\xi \ne (i+\frac{m}{2})h$, $i \in \mathbb{Z}$, since $f - Q_{h,m}f$ vanishes at those points. Take $\theta \in \mathbb{R}$, $| \theta | < 1$, such that $\theta(f(\xi) - (Q_{h,m}f)(\xi)) = \| f^{(m)} \|_\infty E_{h,m+1}(\xi)$, and introduce the p-periodic function

$$g = \| f^{(m)} \|_\infty E_{h,m+1} - \theta(f - Q_{h,m}f) \in C^{m-2}(\mathbb{R}).$$

In the period interval $[0,p)$, g has at least $2n+1$ zeroes, namely ξ and $2n$ interpolation points $(i+\frac{m}{2})h$, $0 \le i+\frac{m}{2} < 2n$. It is easily seen that *if a continuous p-periodic function u has l zeroes in $[0,p)$ then it has at least l local extreme points in $[0,p)$* (claim 1); of course, those are zeroes of u' if u is continously differentiable. Applying claim 1 recursively to g and its derivatives we conclude that $v := g^{(m-2)} \in C(\mathbb{R})$ has at least $2n+1$ (local) extreme points in $[0,p)$. But next we show that actually *v may have at most $2n$ extreme points in* $[0,p)$ (claim 2) and thus we have the desired contradiction. Indeed, for $x_1, x_2 \in (ih,(i+1)h)$,

$$v'(x_1) - v'(x_2) = \| f^{(m)} \|_\infty (E_{h,2}(x_1) - E_{h,2}(x_2)) - \theta(f^{(m-1}(x_1) - f^{(m-1)}(x_2))$$

since $(Q_{h,m}f)^{(m-1)}(x)$ is a constant for $x \in (ih,(i+1)h)$. Further (see Section 8.4), $E_{h,2}(x_1) - E_{h,2}(x_2) = (-1)^i(x_1 - x_2)$, and together with (8.15) we obtain that

$$v'(x_1) - v'(x_2) \ge (1- | \theta |) \| f^{(m)} \|_\infty (x_1 - x_2) \text{ for } x_1, x_2 \in (ih,(i+1)h) \text{ if } i \text{ is even},$$

$$v'(x_1) - v'(x_2) \le (-1+ | \theta |) \| f^{(m)} \|_\infty (x_1 - x_2) \text{ for } x_1, x_2 \in (ih,(i+1)h) \text{ if } i \text{ is odd}.$$

We may assume that f is not identically constant since for a constant function f (8.16) holds trivially. Then due to periodicity, $\| f^{(m)} \|_\infty > 0$, and v' is strictly increasing in $(ih,(i+1)h)$ for even i and strictly decreasing in $(ih,(i+1)h)$ for odd i. Hence, inside an interval $(ih,(i+1)h)$, v may have at most one extreme point (claim 3). This leads us to an idea that perhaps different extreme points of v in the period interval $[0,p)$ can be removed into into different subintervals $(ih,(i+1)h)$ of $[0,p)$. This hypothesis occurs to be true. To show this, observe that v has the following further properties (claim 4):

$\star$ if ih, $i \in \mathbb{Z}$, is a minimum (respectively, maximum) point of v then this one of the adjacent intervals $((i-1)h,ih)$ and $(ih,(i+1)h)$ on which v' increases (respectively, decreases), is free of extreme points of v;

$\star$ for an interval $(jh,(j+1)h)$, $i\in\mathbb{Z}$, on which v' increases (respectively, decreases), at least one of the end points jh and $(j+1)h$ is not a minimum (respectively, a maximum) point of v.

Denote by $\mathcal{E}$ the set of the extreme points of v in the period interval $[0,p)$ and by $\mathcal{G}$ the set of the intervals $(ih,(i+1)h)$, $i=0,...,2n-1$. Define a "removing" map $\mu:\ \mathcal{E}\to\mathcal{G}$ as follows:

if $x\in\mathcal{E}$ belongs to an interval $(ih,(i+1)h)$ then $\mu(x)=(ih,(i+1)h)$;

if $x=ih\in\mathcal{E}$, $1\le i\le 2n-1$, and x is a minimum (respectively, maximum) point for v then $\mu(x)$ is one of the adjacent intervals $((i-1)h,ih)$ and $(ih,(i+1)h)$, namely, this one on which v' is increasing (respectively, decreasing); if $0\in\mathcal{E}$ (then by periodicity, $2nh=p\notin\mathcal{E}$ is an extreme point of the same type for v as 0), the choice is made by the same rule between the intervals $((2n-1)h,2nh)$ and $(0,h)$.

Due to claims 3 and 4, the "removing" map $\mu:\ \mathcal{E}\to\mathcal{G}$ is injective: for $x,x'\in\mathcal{E}$, $x\ne x'$, there holds $\mu(x)\ne\mu(x')$. Hence, for the cardinalities of sets $\mathcal{E}$ and $\mathcal{G}$ we have $\mathrm{card}(\mathcal{E})\le\mathrm{card}(\mathcal{G})=2n$. This completes the proof of claim 2 and the proof of (8.16) for periodic f.

(iii) *Case of compactly supported* f. Next we prove (8.16) for functions $f\in W^{m,\infty}(\mathbb{R})$ having a compact support. Assume that $f(x)=0$ for $x\ge r$ where $r>0$. Take a number $p=2nh$ with sufficiently large $n\in\mathbb{N}$ such that $p>2r$, and introduce the p-periodization f_p of f,

$$f_p(x)=\sum_{k\in\mathbb{Z}}f(x+kp)$$

(for a fixed x, this series contains maximally one nonzero term). The function f_p is p-periodic and still $f_p\in W^{m,\infty}(\mathbb{R})$, $\parallel f_p^{(m)}\parallel_\infty=\parallel f^{(m)}\parallel_\infty$. We have

$$f-Q_{h,m}f=f_p-Q_{h,m}f_p+(I-Q_{h,m})(f-f_p).$$

As proved in (ii), (8.16) holds true for f_p : for any $x\in\mathbb{R}$,

$$\mid f_p(x)-(Q_{h,m}f_p)(x)\mid\le\parallel f_p^{(m)}\parallel_\infty\mid E_{h,m+1}(x)\mid=\parallel f^{(m)}\parallel_\infty\mid E_{h,m+1}(x)\mid.$$

To establish (8.16) for f, it now suffices to show that for any fixed $x\in\mathbb{R}$,

$$((I-Q_{h,m})(f-f_p))(x)\to 0\ \text{ as }\ p\to\infty.$$

Clearly, $g_p(x):=f(x)-f_p(x)=0$ for fixed x and sufficiently large p, so it remains to observe that for any fixed $x\in\mathbb{R}$,

$$(Q_{h,m}g_p)(x)\to 0\ \text{ as }\ p\to\infty.$$

This follows by Lemma 8.1 (with $\delta=1/p$), since $\parallel g_p\parallel_\infty=\parallel f\parallel_\infty<\infty$.

(iv) *Case of* $f\in V^{m,\infty}(\mathbb{R})$ *of restricted growth.* Now we extend estimate (8.16) to $f\in V^{m,\infty}(\mathbb{R})$ satisfying the growth condition

$$f^{(k)}(x)/x^{m-k}\to 0\ \text{ as }\ \mid x\mid\to\infty,\quad k=0,...,m-1. \tag{8.18}$$

Condition (8.18) elementarily implies that

$$\delta^{m-k}\sup_{|x|\le 1/\delta}\mid f^{(k)}(x)\mid\to 0\ \text{ as }\ 0<\delta\to 0,\quad k=0,...,m-1. \tag{8.19}$$

Take a "cutting" function $e\in C^m(\mathbb{R})$ such that $0\le e(x)\le 1$ for all $x\in\mathbb{R}$, $e(x)=1$ for $\mid x\mid\le 1/2$, $e(x)=0$ for $\mid x\mid\ge 1$. Denote $f_\delta(x)=e(\delta x)f(x)$ and represent

$$f(x)-(Q_{h,m}f)(x)=f_\delta(x)-(Q_{h,m}f_\delta)(x)+((I-Q_{h,m})(f-f_\delta))(x). \tag{8.20}$$

Clearly $f_\delta\in V^m(\mathbb{R})$ and f_δ has a compact support $\mathrm{supp}f_\delta\subset[-1/\delta,1/\delta]$; as we proved in (iii), inequality (8.16) holds for f_δ:

$$\mid f_\delta(x)-(Q_{h,m}f_\delta)(x)\mid\le\parallel f_\delta^{(m)}\parallel_\infty\mid E_{h,m+1}(x)\mid,\quad x\in\mathbb{R}.$$

Denoting $c_k = \max_{x\in\mathbb{R}} \mid e^{(k)}(x) \mid$, we have due to (8.19)

$$\| f_\delta^{(m)} \|_\infty = \sup_{x\in\mathbb{R}} \mid (d/dx)^m [e(\delta x) f(x)] \mid$$

$$\leq \sum_{k=0}^{m} \binom{m}{k} \delta^{m-k} \sup_{x\in\mathbb{R}} \mid e^{(m-k)}(\delta x) \mid\mid f^{(k)}(x) \mid$$

$$\leq \| f^{(m)} \|_\infty + \sum_{k=0}^{m-1} \binom{m}{k} c_{m-k}\delta^{m-k} \sup_{|x|\leq 1/\delta} \mid f^{(k)}(x) \mid \to \| f^{(m)} \|_\infty \quad \text{as } \delta \to 0.$$

Concerning the second term in the r.h.s. of (8.20), we have $g_\delta(x) := f(x) - f_\delta(x) = 0$ for a fixed $x \in \mathbb{R}$ and sufficiently small $\delta > 0$; further, $\mid g_\delta(x) \mid \leq c(1+ \mid x \mid^m)$, $x \in \mathbb{R}$, due to condition (8.18). By Lemma 8.1, $(Q_{h,m}(f - f_\delta))(x) \to 0$ as $\delta \to 0$ for any $x \in \mathbb{R}$. With these considerations, (8.16) for f follows from (8.20) as $\delta \to 0$.

(v) *Case of arbitrary* $f \in V^{m,\infty}(\mathbb{R})$. Finally, we show that (8.16) holds for any $f \in V^{m,\infty}(\mathbb{R})$. Represent by the Taylor formula

$$f(x) = \sum_{l=0}^{m-1} \frac{f^{(l)}(0)}{l!} x^l + \frac{1}{(m-1)!} \int_0^x (x-t)^{m-1} f^{(m)}(t)dt, \quad x \in \mathbb{R},$$

and introduce the approximations

$$f_\delta(x) = \sum_{l=0}^{m-1} \frac{f^{(l)}(0)}{l!} x^l + \frac{1}{(m-1)!} \int_0^x (x-t)^{m-1} \theta(\delta t) f^{(m)}(t)dt, \quad x \in \mathbb{R}, \quad \delta > 0,$$

where $\theta(x) = 1$ for $\mid x \mid \leq 1$ and $\theta(x) = 0$ for $\mid x \mid > 1$. Clearly, $f_\delta \in V^{m,\infty}(\mathbb{R})$ and f_δ satisfies (8.18) for any $\delta > 0$. Further, and $f_\delta(x) = f(x)$ for a fixed $x \in \mathbb{R}$ if $\delta > 0$ is sufficiently small, and $\| f_\delta^{(m)} \|_\infty \leq \| f^{(m)} \|_\infty$ for $\delta > 0$. With this f_δ, we have the equality (8.20) in which, due to (iv),

$$\mid f_\delta(x) - (Q_{h,m} f_\delta)(x) \mid \leq \| f_\delta^{(m)} \|_\infty \mid E_{h,m+1}(x) \mid \leq \| f^{(m)} \|_\infty \mid E_{h,m+1}(x) \mid, \quad x \in \mathbb{R}.$$

Clearly,$\mid f(x) \mid + \mid f_\delta(x) \mid \leq c(1+ \mid x \mid^m)$ for $x \in \mathbb{R}$. Using Lemma 8.1 we obtain that $(Q_{h,m}(f - f_\delta))(x) \to 0$ as $\delta \to 0$, and (8.16) for f follows from (8.20) as $\delta \to 0$.

The proof of Theorem 8.2 is complete. □

Let us comment on Theorem 8.2.

Remark 8.1. A direct corollary of estimate (8.17) is that $Q_{h,m} f = f$ for $f \in \mathcal{P}_{m-1}$.

Remark 8.2. Using Banach–Steinhaus theorem and Theorem 8.2, it is easily seen that for any $f \in BUC(\mathbb{R})$ (for any bounded uniformly continuous function f on $\mathbb{R}$), $\| f - Q_{h,m} f \|_\infty \to 0$ as $h \to 0$.

Let us discuss **optimality properties of the spline interpolation** compared with other methods that use the same information about values f on the uniform grid $\Delta_h = \{(j + \frac{m}{2})h : j \in \mathbb{Z}\}$. Such a method can be identified with a mapping $M_h : C(\Delta_h) \to C(\mathbb{R})$ where $C(\Delta_h)$ is the vector space of grid functions defined on Δ_h and having values in $\mathbb{R}$ or $\mathbb{C}$.

Remark 8.3. For given $\gamma > 0$, we have in accordance to Theorem 8.2

$$\sup_{f\in V^{m,\infty}(\mathbb{R}),\ \|f^{(m)}\|_\infty \leq \gamma} \| f - Q_{h,m} f \|_\infty = \Phi_{m+1} \pi^{-m} h^m \gamma,$$

whereas for any mapping $M_h : C(\Delta_h) \to C(\mathbb{R})$ (linear or nonlinear, continuous or discontinuous), it holds

$$\sup_{f\in V^{m,\infty}(\mathbb{R}),\ \|f^{(m)}\|_\infty \leq \gamma} \| f - M_h(f \mid_{\Delta_h}) \|_\infty \geq \Phi_{m+1} \pi^{-m} h^m \gamma.$$

Indeed, introduce two functions $f_\pm = \pm\gamma E_{h,m+1} \in W^{m,\infty}(\mathbb{R}) \subset V^{m,\infty}(\mathbb{R})$. They satisfy $\| f_\pm^{(m)} \|_\infty = \gamma$, $E_{h,m+1} |_{\Delta_h} = \mathbf{0}$. If $M_h(\mathbf{0}) \notin BC(\mathbb{R})$, the claim is trivial, whereas in case $M_h(\mathbf{0}) \in BC(\mathbb{R})$ we have

$$\sup_{f\in V^{m,\infty}(\mathbb{R}),\ \|f^{(m)}\|_\infty\le\gamma} \| f - M_h(f |_{\Delta_h}) \|_\infty$$

$$\ge \max\{\| f_+ - M_h(f_+ |_{\Delta_h}) \|_\infty, \| f_- - M_h(f_- |_{\Delta_h}) \|_\infty\}$$

$$= \max\{\| f_+ - M_h(\mathbf{0}) \|_\infty, \| f_- - M_h(\mathbf{0}) \|_\infty\}$$

$$\ge \frac{1}{2}(\| f_+ - M_h(\mathbf{0}) \|_\infty + \| f_- - M_h(\mathbf{0}) \|_\infty)$$

$$\ge \frac{1}{2} \| f_+ - f_- \|_\infty = \| E_{h,m+1} \|_\infty \gamma = \Phi_{m+1}\pi^{-m}h^m\gamma.$$

Remark 8.4. Let $n \in \mathbb{N}$ be even and $h = 1/n$. Consider the subspace $C_{\text{per}}(\mathbb{R})$ of $C(\mathbb{R})$ consisting of 1-periodic continuous functions on $\mathbb{R}$, and denote $W^{m,\infty}_{\text{per}}(\mathbb{R}) = C_{\text{per}}(\mathbb{R}) \cap W^{m,\infty}(\mathbb{R})$; denote by $C_{\text{per}}(\Delta_h)$ the space of 1-periodic (grid) functions on the grid Δ_h, i.e., $f_h(ih) = f_h(1+ih)$, $i \in \mathbb{Z}$, for $f_h \in C_{\text{per}}(\Delta_h)$. Then for any mapping $M_h : C_{\text{per}}(\Delta_h) \to C_{\text{per}}(\mathbb{R})$, it holds

$$\sup_{f\in W^{m,\infty}_{\text{per}}(\mathbb{R}),\ \|f^{(m)}\|_\infty\le\gamma} \| f - M_h(f |_{\Delta_h}) \|_\infty \ge \Phi_{m+1}\pi^{-m}h^m\gamma.$$

The proof is same as in the case of Remark 8.3, we only need to observe that $E_{h,m+1} \in W^{m,\infty}_{\text{per}}(\mathbb{R})$ for even n.

Remark 8.5. For functions with compact supports, a similar result as in Remark 8.3 holds asymptotically as $h \to 0$. Denote by $W^{m,\infty}_0(\mathbb{R})$ the subspace of $W^{m,\infty}(\mathbb{R})$ consisting of functions $f \in W^{m,\infty}(\mathbb{R})$ with support in $[0, 1]$. Modifying the argument presented in the proof of Remark 8.2, it can be easily seen that for any mapping $M_h : C(\Delta_h) \to C(\mathbb{R})$, it holds

$$\liminf_{h\to 0} \sup_{f\in W^{m,\infty}_0(\mathbb{R}),\ \|f^{(m)}\|_\infty\le\gamma} \| f - M_h(f |_{\Delta_h}) \|_\infty /(\Phi_{m+1}\pi^{-m}h^m\gamma) \ge 1.$$

Namely, instead of $f_\pm = \pm\gamma E_{h,m+1}$, introduce $f_\pm = \pm\gamma e E_{h,m+1}$ where $e \in C^m(\mathbb{R})$ is supported in $(0, 1)$, $0 \le e(x) \le 1$ for all $x \in \mathbb{R}$ and $e(x) = 1$ for $\frac{1}{3} \le x \le \frac{2}{3}$. Then for sufficiently small $h > 0$, it still holds $\| eE_{h,m+1} \|_\infty = \| E_{h,m+1} \|_\infty = \Phi_{m+1}\pi^{-m}h^m$, and the Leibniz differentiation rule yields $\| (eE_{h,m+1})^{(m)} \|_\infty \to 1$ as $h \to 0$. The details are left as an exercise.

8.6. Further error estimates. (A) *Error estimates for the derivatives of the interpolant.* First we establish some estimates for the derivatives of the interpolant.

Lemma 8.2. *For $m \ge 2$, $f \in V^{l,\infty}(\mathbb{R})$, $l = 1, ..., m-1$, it holds*

$$\| (Q_{h,m}f)^{(l)} \|_\infty \le \alpha_m \| f^{(l)} \|_\infty,$$

$$\| (Q_{h,m}f)^{(l)} \|_\infty \le q_{m-l}\alpha_{m,l} \| f^{(l)} \|_\infty$$

where

$$\alpha_m = \sum_{k\in\mathbb{Z}} | a_{k,m} |, \quad \alpha_{m,l} = \sum_{k\in\mathbb{Z}} \left| \sum_{|j|\le\text{int}\{(m-l-1)/2\}} a_{k-j,m}b_{j,m-l} \right| < \alpha_m,$$

$q_{m-l} = \| Q_{h,m-l} \|_{BC(\mathbb{R})\to BC(\mathbb{R})}$, $b_{j,m-l} = B_{m-l}(j + \frac{m-l}{2})$ (*cf.* (8.4)), *and* $a_k = a_{k,m}$ *are defined in* (8.9).

Proof. By Definition 8.2, $B'_m(x) = B_{m-1}(x) - B_{m-1}(x-1)$, and (8.1), (8.6) yield

$$(Q_{h,m}f)'(x) = \frac{d}{dx}\sum_{j\in\mathbb{Z}} d_j B_m(h^{-1}x - j)$$

$$= h^{-1}\sum_{j\in\mathbb{Z}} d_j [B_{m-1}(h^{-1}x - j) - B_{m-1}(h^{-1}x - j - 1)]$$

$$= h^{-1}\sum_{j\in\mathbb{Z}} (d_j - d_{j-1}) B_{m-1}(h^{-1}x - j)$$

$$= h^{-1}\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} (a_{j-k,m} - a_{j-1-k,m}) f_k\right) B_{m-1}(h^{-1}x - j)$$

$$= h^{-1}\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} a_{j-k,m}(f_k - f_{k-1})\right) B_{m-1}(h^{-1}x - j)$$

$$= \sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} a_{j-k,m} h^{-1}\Delta_h f_k\right) B_{m-1}(h^{-1}x - j)$$

where $\Delta_h f_k := f_k - f_{k-1} = f((k+\frac{m}{2})h) - f((k-1+\frac{m}{2})h)$ is the backward difference of function f. Repeating the differentiations we obtain the formula

$$(Q_{h,m}f)^{(l)}(x) = \sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} a_{j-k,m} h^{-l}\Delta_h^l f_k\right) B_{m-l}(h^{-1}x - j), \quad x \in \mathbb{R}. \tag{8.21}$$

Since $B_{m-l}(x) \geq 0$, $\sum_{j\in\mathbb{Z}} B_{m-l}(h^{-1}x - j) = 1$, and $| h^{-l}\Delta^l_- f_k | \leq \| f^{(l)} \|_\infty$, the first claim of the Lemma now follows:

$$\| (Q_{h,m}f)^{(l)} \|_\infty \leq \sup_{j\in\mathbb{Z}} \left|\sum_{k\in\mathbb{Z}} a_{j-k,m} h^{-l}\Delta^l_- f_k\right| \leq \sup_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} | a_{j-k}, m | \ \sup_{k\in\mathbb{Z}} | h^{-l}\Delta^l_- f_k |$$

$$\leq \sum_{k\in\mathbb{Z}} | a_{k,m} | \| f^{(l)} \|_\infty = \alpha_m \| f^{(l)} \|_\infty .$$

According to (8.21), the spline $g := (Q_{h,m}f)^{(l)} \in S_{h,m-l}$ has the knot values

$$g((i+\frac{m}{2})h) = (Q_{h,m}f)^{(l)}((i+\frac{m}{2})h) = \sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} a_{j-k.m} h^{-l}\Delta_h^l f_k\right) b_{i-j,m-l}$$

$$= \sum_{k\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} a_{j-k,m} b_{i-j,m-l}\right) h^{-l}\Delta_h^l f_k = \sum_{k\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} a_{i-j-k,m} b_{j,m-l}\right) h^{-l}\Delta_h^l f_k,$$

and

$$| g((i+\frac{m}{2})h) | \leq \sum_{k\in\mathbb{Z}} \left| \sum_{j\in\mathbb{Z}} a_{i-j-k,} b_{j,m-l} \right| \parallel f^{(l)} \parallel_\infty, \;\; i\in\mathbb{Z}.$$

With the change of summation variable $k \mapsto k' = i-k$ we rewrite

$$\sum_{k\in\mathbb{Z}} \left| \sum_{j\in\mathbb{Z}} a_{i-j-k,m} b_{j,m-l} \right| = \sum_{k'\in\mathbb{Z}} \left| \sum_{j\in\mathbb{Z}} a_{k'-j,m} b_{j,m-l} \right| = \sum_{k\in\mathbb{Z}} \left| \sum_{j\in\mathbb{Z}} a_{k-j,m} b_{j,m-l} \right|.$$

Since $b_{j,m-l}=0$ for $| j |> \mathrm{int}((m-l-1)/2)$, we get

$$\sup_{i\in\mathbb{Z}} | g((i+\frac{m}{2})h) | \leq \sum_{k\in\mathbb{Z}} \left| \sum_{j\in\mathbb{Z}} a_{k-j,m} b_{j,m-l} \right| \parallel f^{(l)} \parallel_\infty = \alpha_{m,l} \parallel f^{(l)} \parallel_\infty .$$

Noticing that $Q_{h,m-l} g = g$, we arrive at the second claim of the Lemma:

$$\parallel (Q_{h,m}f)^{(l)} \parallel_\infty = \parallel g \parallel_\infty = \parallel Q_{h,m-l} g \parallel_\infty \leq \; q_{m-l} \sup_{i\in\mathbb{Z}} | g((i+\frac{m}{2})h) |$$

$$\leq q_{m-l}\alpha_{m,l} \parallel f^{(l)} \parallel_\infty .$$

Clearly,

$$\alpha_{m,l} = \sum_k \sum_j a_{k-j,m} b_{j,m-l} < \sum_k \sum_j | a_{k-j,m} | b_{j,m-l} = \sum_j \sum_k | a_{k-j,m} | b_{j,m-l}$$

$$= \sum_j \sum_k | a_{k,m} | b_{j,m-l} = \alpha_m \sum_j b_{j,m-l} = \alpha_m$$

since $a_{k,m}$ are of alternating sign (see (8.10)) whereas $b_{j,m-l} \geq 0$ and $\sum_j b_{j,m-l} = 1$. □

Theorem 8.3. *For $f \in V^{m,\infty}(\mathbb{R})$, $m \geq 2$, $l = 1, ..., m-1$, it holds*

$$\parallel f^{(l)} - (Q_{h,m}f)^{(l)} \parallel_\infty \leq \Phi_{m-l+1} \pi^{-(m-l)} h^{m-l} \left(1+\alpha_m\right) \parallel f^{(m)} \parallel_\infty,$$

$$\parallel f^{(l)} - (Q_{h,m}f)^{(l)} \parallel_\infty \leq \Phi_{m-l+1} \pi^{-(m-l)} h^{m-l} \left(1+q_{m-l}\alpha_{m,l}\right) \parallel f^{(m)} \parallel_\infty$$

with constants α_m and $\alpha_{m,i}$ defined in Lemma 8.2.

Proof. Introduce the operator $P_{h,m,l} := \partial^l Q_{h,m} K_l$ where $\partial^l = (d/dx)^l$ whereas $K_l : C(\mathbb{R}) \to C(\mathbb{R})$ is defined by

$$(K_l u)(x) = \frac{1}{(l-1)!} \int_0^x (x-t)^{l-1} u(t) dt, \;\; x \in \mathbb{R}.$$

Since $f - K_l \partial^l f$ is a polynomial of degree $l-1$ (it is the Taylor polynomial of f), we have $Q_{h,m}(f - K_l \partial^l f) = f - K_l \partial^l f$, $\;\partial^l Q_{h,m}(f - K_l \partial^l f) = 0$ and

$$f^{(l)} - (Q_{h,m}f)^{(l)} = f^{(l)} - \partial^l Q_{h,m} f = f^{(l)} - \partial^l Q_{h,m} K_l f^{(l)} = (I - P_{h,m,l}) f^{(l)}.$$

Further, since $Q_{h,m} g = g$ for $g \in S_{h,m}$, it holds $P_{h,m,l} g = g$ for $g \in S_{h,m-l}$ (hence $P_{h,m,l}$ is a projetor onto $S_{h,m}$). Indeed, $g \in S_{h,m-l}$ implies $K_l g \in S_{h,m}$ and $P_{h,m,l} g = \partial^l Q_{h,m} K_l g = \partial^l K_l g = g$. Now we can continue

$$f^{(l)} - (Q_{h,m}f)^{(l)} = (I - P_{h,m,l}) f^{(l)} = (I - P_{h,m,l})(f^{(l)} - Q_{h,m-l} f^{(l)}).$$

Estimating with the help of Theorem 8.2 we obtain

$$\| f^{(l)} - (Q_{h,m}f)^{(l)} \|_\infty$$

$$\leq (1+ \| P_{h,m,l} \|_{BC(\mathbb{R})\to BC(\mathbb{R})})\Phi_{m-l+1}\pi^{-(m-l)}h^{m-l} \| f^{(m)} \|_\infty .$$

For $g \in BC(\mathbb{R})$, it holds $K_l g \in V^{l,\infty}(\mathbb{R})$, $(K_l g)^{(l)} = g$, and Lemma 8.2 implies

$$\| P_{h,m,l} \|_{BC(\mathbb{R})\to BC(\mathbb{R})} \leq \alpha_m, \quad \| P_{h,m,l} \|_{BC(\mathbb{R})\to BC(\mathbb{R})} \leq \alpha_{m,l}$$

that completes the proof of the Theorem. □

Remark 8.6. Using Remark 8.2 we obtain that for $f \in V^{l,\infty}(\mathbb{R})$ with $f^{(l)} \in BUC(\mathbb{R})$, $0 < l < m$, it holds $\| f^{(l)} - (Q_{h,m}f)^{(l)} \|_\infty \to 0$ as $h \to 0$.

(B) *Interpolation of modestly smooth functions.* Lemma 8.2 is crucial also in the proof of the following estimates.

Theorem 8.4. *For* $m \geq 2$, $f \in V^{l,\infty}(\mathbb{R})$, $l = 1, ..., m-1$, it *holds*

$$\| f - Q_{h,m}f \|_\infty \leq \Phi_{l+1}\pi^{-l}h^l \,(1 + \alpha_m) \| f^{(l)} \|_\infty,$$

$$\| f - Q_{h,m}f \|_\infty \leq \Phi_{l+1}\pi^{-l}h^l \,(1 + q_{m-l}\alpha_{m,l}) \| f^{(l)} \|_\infty$$

with constants α_m *and* $\alpha_{m,l}$ *defined in Lemma* 8.2. *If, in addition,* $f^{(l)} \in BUC(\mathbb{R})$ *then*

$$\| f - Q_{h,m}f \|_\infty = o(h^l) \text{ as } h \to 0.$$

Proof. Let $f \in V^{l,\infty}(\mathbb{R})$, $0 < l < m$. Since $(Q_{h,m}f)((i+\frac{m}{2})h) = f((i+\frac{m}{2})h)$, $i \in \mathbb{Z}$, we have $Q_{h,l}Q_{h,m}f = Q_{h,l}f$,

$$f - Q_{h,m}f = (I - Q_{h,l})(f - Q_{h,m}f).$$

By Theorem 8.2,

$$\| f - Q_{h,m}f \|_\infty \leq \Phi_{l+1}\pi^{-l}h^l(\| f^{(l)} \|_\infty + \| (Q_{h,m}f)^{(l)} \|_\infty),$$

and Lemma 8.2 completes the proof of the error estimates formulated in the Theorem.

Assume in addition that $f^{(l)} \in BUC(\mathbb{R})$ and estimate again by Theorem 8.2

$$\| f - Q_{h,m}f \|_\infty \leq \Phi_{l+1}\pi^{-l}h^l \| f^{(l)} - (Q_{h,m}f)^{(l)} \|_\infty .$$

By Remark 8.6, $\| f^{(l)} - (Q_{h,m}f)^{(l)} \|_\infty \to 0$, hence $\| f - Q_{h,m}f \|_\infty = o(h^l)$ as $h \to 0$. □

(C) *Interpolation of functions* $g \in V_h^{m,\infty}(\mathbb{R})$. Theorem 8.2 admits an extension to a slightly wider class of functions than $V^{m,\infty}(\mathbb{R})$. Introduce the vector space $V_h^{m,\infty}(\mathbb{R})$, $m \geq 2$, which consists of functions $g \in C^{m-2}(\mathbb{R})$ such that $g^{(m-1)} |_{(ih,(i+1)h)} \in C((ih,(i+1)h))$, $g^{(m)} |_{(ih,(i+1)h)} \in L^\infty((ih,(i+1)h))$ for all $i \in \mathbb{Z}$, and

$$\sigma_{h,m}(g) := \sup_{i\in\mathbb{Z}} \sup_{ih<x<(i+1)h)} | g^{(m)}(x) | < \infty.$$

Clearly, $V^{m,\infty}(\mathbb{R}) \subset V_h^{m,\infty}(\mathbb{R})$ and $\| f^{(m)} \|_\infty = \sigma_{h,m}(f)$ for $f \in V^{m,\infty}(\mathbb{R})$.

Lemma 8.3. *For* $m \geq 2$, *it holds* $V_h^{m,\infty}(\mathbb{R}) = V^{m,\infty}(\mathbb{R}) + S_{h,m}$, *i.e., every* $g \in V_h^{m,\infty}(\mathbb{R})$ *has a representation*

$$g = f + g_h, \quad f \in V^{m,\infty}(\mathbb{R}), \quad g_h \in S_{h,m}. \tag{8.22}$$

In particular, (8.22) *holds with*

$$f(x) = \frac{1}{(m-1)!}\int_0^x (x-t)^{m-1}G_m(t)dt, \quad x \in \mathbb{R}, \tag{8.23}$$

where $G_m \in L^\infty(\mathbb{R})$ is defined by $G_m(x) = g^{(m)}(x)$ for $x \in (ih, (i+1)h)$, $i \in \mathbb{Z}$ (and other f differ from (8.23) by an additive polynomial of degree $m-1$).

Proof. Let $g \in V_h^{m,\infty}(\mathbb{R})$ and let f be defined by (8.23). Then $f^{(m)} = G_m \in L^\infty(\mathbb{R})$, hence $f \in V^{m,\infty}(\mathbb{R})$ and $f', ..., f^{(m-1)}$ are continuous in $\mathbb{R}$. Together with g, also $g_h := g - f \in C^{m-2}(\mathbb{R})$. Further, since $g^{(m-1)} \mid_{(ih,(i+1)h)} \in C((ih,(i+1)h))$, $g^{(m)} \mid_{(ih,(i+1)h)} = G_m \in L^\infty((ih,(i+1)h))$, we have

$$g_h^{(m-1)}(x) = g^{(m-1)}(x) - f^{(m-1)}(x)$$

$$= g^{(m-1)}((i+\frac{1}{2})h) + \int_{(i+\frac{1}{2})h}^{x} G_m(t)dt - \int_0^x G_m(t)dt$$

$$= g^{(m-1)}((i+\frac{1}{2})h) - \int_0^{(i+\frac{1}{2})h} G_m(t)dt, \ \ x \in (ih,(i+1)h), \ i \in \mathbb{Z}.$$

Thus $g_h^{(m-1)}(x)$ is constant on the intervals $(ih,(i+1)h)$, $i \in \mathbb{Z}$, i.e., $g_h \mid_{(ih,(i+1)h)} \in \mathcal{P}_{m-1}$, $i \in \mathbb{Z}$, and $g_h \in S_{h,m}$. □

Theorem 8.5. *Let $m \geq 2$. Assume that $g \in V_h^{m,\infty}(\mathbb{R})$ satisfies the inequality*

$$| g(x) | \leq c(1+ | x |^r), \ \ x \in \mathbb{R}, \tag{8.24}$$

where $r \geq 0$ and $c \geq 0$. Then

$$| g(x) - (Q_{h,m}g)(x) | \leq \sigma_{h,m}(g) \mid E_{h,m+1}(x) \mid, \ \ x \in \mathbb{R}, \tag{8.25}$$

$$\| g - Q_{h,m}g \|_\infty \leq \Phi_{m+1}\pi^{-m}h^m\sigma_{h,m}(g). \tag{8.26}$$

Proof. Let $f \in V^{m,\infty}(\mathbb{R})$ and $g_h \in S_{h,m}$ be defined by (8.22), (8.23). Then

$$| f(x) | \leq \| G_m \|_\infty | x |^m /m!, \ \ x \in \mathbb{R},$$

that together with (8.24) imply

$$| g_h(x) | \leq c'(1+ | x |^{\max\{m,r\}}), \ \ x \in \mathbb{R}.$$

Hence $Q_{h,m}f$, $Q_{h,m}g$ and $Q_{h,m}g_h$ are well defined and $Q_{h,m}g_h = g_h$. Equality (8.22) yields $(I - Q_{h,m})g = (I - Q_{h,m})f$. According to Theorem 8.2, $(I - Q_{h,m})f$ is estimated by (8.16), (8.17) which take the form (8.25), (8.26) since $\| f^{(m)} \|_\infty = \| G_m \|_\infty = \sigma_{h,m}(g)$. □

8.7. Stability of interpolation. Take a function $f \in C(\mathbb{R})$ which is either bounded or of a polynomial growth as $| x | \to \infty$, and converse the formula of the Wiener interpolant starting from (8.1), (8.6), (8.9)):

$$(Q_{h,m}f)(x) = \sum_{k\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} a_{k-j} f((j+\frac{m}{2})h) \right) B_m(h^{-1}x - k)$$

$$= \sum_{j\in\mathbb{Z}} f((j+\frac{m}{2})h) \left(\sum_{k\in\mathbb{Z}} a_{k-j} B_m(h^{-1}x - k) \right)$$

$$= \sum_{j\in\mathbb{Z}} f((j+\frac{m}{2})h) \left(\sum_{k\in\mathbb{Z}} a_k B_m(h^{-1}x - k - j) \right).$$

Denoting

$$F_m(x) := \sum_{k\in\mathbb{Z}} a_k B_m(x-k),$$

we obtain the representation

$$(8.27) \qquad (Q_{h,m}f)(x) = \sum_{j\in\mathbb{Z}} f((j+\frac{m}{2})h)F_m(h^{-1}x-j).$$

The function F_m is called the *fundamental spline* of order m, since it satisfies the condition $F_m(i+\frac{m}{2}) = \delta_{i,0}$, $i\in\mathbb{Z}$, where $\delta_{i,j}$ is the Kronecker symbol. Indeed,

$$F_m(i+\frac{m}{2}) = \sum_{k\in\mathbb{Z}} a_k B_m(i+\frac{m}{2}-k) = \sum_{k\in\mathbb{Z}} a_k b_{i-k} = \delta_{i,0}, \;\; i\in\mathbb{Z}$$

(see Section 8.3). Note that $\operatorname{supp} F_m = \mathbb{R}$ and this restricts the utility of representation (8.27) in the practice although F_m decays exponentially as $|\, x \,| \to \infty$.

Denote $\varphi_m(x) := \sum_{j\in\mathbb{Z}} |\, F_m(x-j) \,|$. This is an 1-periodic continuous function, and $\varphi_m(\frac{m}{2}-x) = \varphi_m(\frac{m}{2}+x)$, $x\in\mathbb{R}$. For any $f\in BC(\mathbb{R})$, (8.27) implies that

$$\| Q_{h,m}f \|_\infty \le \| f \|_\infty \sup_{x\in\mathbb{R}} \sum_{j\in\mathbb{Z}} |\, F_m(h^{-1}x-j) \,| = \| f \|_\infty \sup_{x\in\mathbb{R}} \sum_{j\in\mathbb{Z}} |\, F_m(x-j \,|$$

$$= \| f \|_\infty \sup_{x\in\mathbb{R}} \varphi_m(x) \; = \| f \|_\infty \max_{x\in[\frac{m}{2},\frac{m+1}{2}]} \varphi_m(x).$$

Take $x_0\in[0,1]$ such that $\varphi_m(x_0) = \max_{\frac{m}{2}\le x\le\frac{m+1}{2}} \varphi_m(x)$. For a function $f\in BC(\mathbb{R})$ such that $\| f \|_\infty = 1$ and $f((j+\frac{m}{2})h) = \operatorname{sign} F_m(x_0-j)$, $j\in\mathbb{Z}$, we have

$$(Q_{h,m}f)(hx_0) = \sum_{j\in\mathbb{Z}} f((j+\frac{m}{2})h)F_m(x_0-j) = \sum_{j\in\mathbb{Z}} |\, F_m(x_0-j) \,| = \max_{\frac{m}{2}\le x\le\frac{m+1}{2}} \varphi_m(x),$$

$$\| Q_{h,m}f \|_\infty \ge (Q_{h,m}f)(hx_0) = \| f \|_\infty \max_{\frac{m}{2}\le x\le\frac{m+1}{2}} \varphi_m(x).$$

Hence

$$q_m := \| Q_{h,m} \|_{BC(\mathbb{R})\to BC(\mathbb{R})} = \max_{x\in[\frac{m}{2},\frac{m+1}{2}]} \sum_{j\in\mathbb{Z}} |\, F_m(x-j) \,| \,.$$

Numerical values of q_m for $3\le m\le 10$ are presented in the following table; we recall also the values of $\sum_k |\, a_{k,m} \,|$.

m	3	4	5	6	7	8	9	10	20
q_m	1.414	1.549	1.706	1.816	1.916	2.000	2.075	2.142	2.583
$\sum_k \mid a_{k,m} \mid$	2.000	3.000	4.800	7.500	11.80	18.53	29.11	45.73	4182

For $4\le m\le 20$, q_m fits into a provisional model $q_m \le \frac{e}{4}+\frac{2}{\pi}\log m$, and it seems that $q_m - (\frac{e}{4}+\frac{2}{\pi}\log m) \to 0$ as $m\to\infty$; for $m=20$ this difference is of order 0.003. It is challenging to examine q_m analytically.

The quantity q_m characterises the stability of the interpolation process: to errors ε_j of $f((j+\frac{m}{2})h))$, $j\in\mathbb{Z}$, there corresponds the error $\varepsilon(x)$ of $(Q_{h,m}f)(x)$ having the bound $\sup_{x\in\mathbb{R}} |\, \varepsilon(x) \,| \le q_m \sup_{j\in\mathbb{Z}} |\, \varepsilon_j \,|$. We see that the interpolation process is rather stable with respect to the errors in the function f.

Similarly, $\sum_k |\, a_{k,m} \,|$ characterises the stability of the coefficients $d_k = \sum_{j\in\mathbb{Z}} a_{k-j} f((j+\frac{m}{2})h))$, $k\in\mathbb{Z}$, in the B-spline representation (8.1) of the interpolant $Q_{h,m}f$. A qualitative consequence of the table above is that the uniform error of $Q_{h,m}f$ may be significantly smaller than the supremum over k of the absolute values of the errors in d_k.

8.8. Expressions for the coefficients of the Wiener interpolant. As we know, for $m \geq 3$ the Wiener interpolant $Q_{h,m}f$ has the representation

$$(Q_{h,m}f)(x) = \sum_{k\in\mathbb{Z}} d_k B_m(h^{-1}x - k), \;\; d_k = \sum_{j\in\mathbb{Z}} a_j f_{k-j}, \;\; f_k = f((k + \frac{m}{2})h), \;\; k \in \mathbb{Z}.$$

In this Section we give new expressions for the coefficients d_k, $k \in \mathbb{Z}$.

Introduce the vector space $\mathfrak{s}(\mathbb{Z})$ of bisequences $\underline{a} = (a_j)_{j\in\mathbb{Z}}$ such that

$$\forall r \geq 0 \; \exists c_r < \infty : \;\; | \, a_j \, | \leq c_r \, | \, j \, |^{-r}, \;\; 0 \neq j \in \mathbb{Z},$$

and the subspace

$$\mathfrak{s}_{\text{sym}}(\mathbb{Z}) = \{\underline{a} \in \mathfrak{s}(\mathbb{Z}) : \; a_{-j} = a_j, \; j \in \mathbb{Z}\} \subset \mathfrak{s}(\mathbb{Z}).$$

Consider the difference operators

$$D^+ : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z}), \;\; (D^+\underline{a})_j = a_{j+1} - a_j, \; j \in \mathbb{Z} \;\; \text{(forward difference)},$$

$$D^- : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z}), \;\; (D^-\underline{a})_j = a_j - a_{j-1}, \; j \in \mathbb{Z} \;\; \text{(backward difference)}$$

and their one side inverses $J^\pm : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z})$,

$$(J^+\underline{a})_k = \left\{ \begin{array}{ll} \sum_{j=-\infty}^{k-1} a_j, & k \leq 0 \\ -\sum_{j=k}^{\infty} a_j, & k > 0 \end{array} \right\}, \; (J^-\underline{a})_k = \left\{ \begin{array}{ll} \sum_{j=-\infty}^{k} a_j, & k < 0 \\ -\sum_{j=k+1}^{\infty} a_j, & k \geq 0 \end{array} \right\}, \; k \in \mathbb{Z}.$$

Namely, denoting

$$\underline{e} = (\delta_{j,0})_{j\in\mathbb{Z}} = (..., 0, 0, 1, 0, 0...),$$

it is easy to check that

$$J^\pm D^\pm \underline{a} = \underline{a}, \;\; D^\pm J^\pm \underline{a} = \underline{a} - \left(\sum_{j\in\mathbb{Z}} a_j \right) \underline{e} \;\; \text{for} \;\; \underline{a} \in \mathfrak{s}(\mathbb{Z}). \tag{8.28}$$

Our main tool will be the second order central difference operator

$$D = D^+D^- = D^-D^+ : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z}), \;\; (D\underline{a})_j = a_{j-1} - 2a_j + a_{j+1}, \; j \in \mathbb{Z},$$

with its one side inverse

$$J = J^+J^- : \mathfrak{s}(\mathbb{Z}) \to \mathfrak{s}(\mathbb{Z}).$$

Formulae (8.28) imply for $\underline{a} \in \mathfrak{s}(\mathbb{Z})$

$$JD\underline{a} = (J^+J^-)(D^+D^-)\underline{a} = J^+(J^-D^-)D^+\underline{a} = J^+D^+\underline{a} = \underline{a},$$

$$DJ\underline{a} = (D^+D^-)(J^+J^-)\underline{a} = D^-(D^+J^+)(J^-\underline{a}) = D^- \left(J^-\underline{a} - \left(\sum_{j\in\mathbb{Z}} (J^-\underline{a})_j \right) \underline{e} \right)$$

$$= \underline{a} - \left(\sum_{j\in\mathbb{Z}} a_j \right) \underline{e} - \left(\sum_{j\in\mathbb{Z}} (J^-\underline{a})_j \right) D^-\underline{e}, \;\; \underline{a} \in \mathfrak{s}(\mathbb{Z});$$

note that $\sum_{j\in\mathbb{Z}}(J^{\pm}\underline{a})_j = 0$ for $\underline{a} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z})$, and the expression of $DJ\underline{a}$ simplifies to

$$DJ\underline{a} = \underline{a} - \left(\sum_{j\in\mathbb{Z}} a_j\right) \underline{e} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z}) \text{ for } \underline{a} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z}). \tag{8.29}$$

It is also easy to check that $J\underline{a} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z})$ for $\underline{a} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z})$. By induction over $p \in \mathbb{N}$ we next prove that

$$D^p J^p \underline{a} = \underline{a} - \sum_{q=0}^{p-1} \gamma_q D^q \underline{e} \text{ with } \gamma_q = \sum_{j\in\mathbb{Z}} (J^q \underline{a})_j \text{ for } \underline{a} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z}). \tag{8.30}$$

For $p = 1$, (8.30) coincides with (8.29). Assuming that for a $p \in \mathbb{N}$, (8.30) holds, we check that (8.30) holds also for $p+1$. Indeed, due to the induction assumption and (8.29),

$$D^{p+1}J^{p+1}\underline{a} = D(D^pJ^p)(J\underline{a}) = D\left(J\underline{a} - \sum_{q=0}^{p-1}\left(\sum_{j\in\mathbb{Z}}(J^qJ\underline{a})_j\right)D^q\underline{e}\right)$$

$$= \underline{a} - \left(\sum_{j\in\mathbb{Z}} a_j\right)\underline{e} - \sum_{q=0}^{p-1}\left(\sum_{j\in\mathbb{Z}}(J^{q+1}\underline{a})_j\right)D^{q+1}\underline{e} = \underline{a} - \sum_{q=0}^{p}\left(\sum_{j\in\mathbb{Z}}(J^q\underline{a})_j\right)D^q\underline{e}$$

as expected.

Formula (8.30) can be interpreted as follows: $\underline{a} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z})$ *has the representations*

$$\underline{a} = \sum_{q=0}^{p-1}\gamma_q D^q \underline{e} + D^pJ^p\underline{a}, \;\; \gamma_q = \sum_{j\in\mathbb{Z}}(J^q\underline{a})_j, \;\; p \in \mathbb{N}. \tag{8.31}$$

Applying this result to the Wiener sequence $\underline{a} = (a_k)_{k\in\mathbb{Z}} \in \mathfrak{s}_{\mathrm{sym}}(\mathbb{Z})$ defined in (8.9) we get the following result.

Theorem 8.6. *For $\underline{a} = (a_k)_{k\in\mathbb{Z}}$ defined in* (8.9), *formula* (8.31) *holds with*

$$\gamma_0 = 1, \;\; \gamma_q = \gamma_{q,m} = \sum_{\nu=1}^{\mu}\frac{(1+z_\nu)z_\nu^{\mu+q-1}}{(1-z_\nu)^{2q+1}P'_{2\mu}(z_\nu)}, \;\; q \geq 1. \tag{8.32}$$

The coefficients $d_k = \sum_{j\in\mathbb{Z}} a_j f_{k-j}$ of the interpolant $(Q_{h,m}f)(x) = \sum_{k\in\mathbb{Z}} d_k B_m(h^{-1}x - k)$ can be expressed in the form

$$d_k = f_k + \sum_{q=1}^{p-1}\gamma_q D^q f_k + \delta_k^{(p)} = \sum_{|j|\leq p-1} a_j^{(p)} f_{k-j} + \delta_k^{(p)}, \quad k \in \mathbb{Z}, \tag{8.33}$$

where

$$\delta_k^{(p)} = \sum_{j\in\mathbb{Z}}(J^p\underline{a})_{k-j}D^p f_j, \;\; k \in \mathbb{Z}, \tag{8.34}$$

$$a_j^{(p)} = a_{j,m}^{(p)} = \sum_{q=|j|}^{p-1}(-1)^{j+q}\binom{2q}{j+q}\gamma_{q,m}, \;\; |\, j \,| \leq p-1. \tag{8.35}$$

Proof. According to (8.9) and (8.10),

$$a_j = \sum_{\nu=1}^{\mu}\frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)}z_\nu^{|j|}, \;\; j \in \mathbb{Z}, \;\; \gamma_0 = \sum_{j\in\mathbb{Z}} a_j = 1.$$

Let us establish (8.32) for $q \geq 1$. For the sequence $\underline{z}^{(\nu)} := (z_\nu^{|j|})_{j\in\mathbb{Z}}$ we have

$$(J^- \underline{z}^{(\nu)})_k = \frac{1}{1-z_\nu} \left\{ \begin{array}{ll} z_\nu^{-k}, & k < 0 \\ -z_\nu^{k+1}, & k \geq 0 \end{array} \right\},$$

$$(J\underline{z}^{(\nu)})_k = (J^+ J^- \underline{z}^{(\nu)})_k = \frac{1}{(1-z_\nu)^2} \left\{ \begin{array}{ll} z_\nu^{-k+1}, & k \leq 0 \\ z_\nu^{k+1}, & k > 0 \end{array} \right\} = \frac{z_\nu}{(1-z_\nu)^2} z_\nu^{|k|}, \;\; k \in \mathbb{Z}.$$

Thus

$$(J\underline{a})_k = \sum_{\nu=1}^{\mu} \frac{z_\nu}{(1-z_\nu)^2} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} z_\nu^{|k|}, \;\; k \in \mathbb{Z}.$$

By repeated application of this formula we find that

$$(J^q \underline{a})_k = \sum_{\nu=1}^{\mu} \frac{z_\nu^q}{(1-z_\nu)^{2q}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} z_\nu^{|k|}, \;\; k \in \mathbb{Z}, \;\; q \in \mathbb{N}. \tag{8.36}$$

Since $\sum_{k\in\mathbb{Z}} z_\nu^{|k|} = \frac{1+z_\nu}{1-z_\nu}$, (8.32) follows.

To establish (8.33), we need some formulae of summation by parts. For $\underline{a} \in \mathfrak{s}(\mathbb{Z})$ and a bounded or polynomially growing sequence $\underline{f}$, an elementary check confirms that

$$\sum_{j\in\mathbb{Z}} f_j D^+ a_j = -\sum_{j\in\mathbb{Z}} (D^- f_j) a_j, \;\; \sum_{j\in\mathbb{Z}} f_j D^- a_j = -\sum_{j\in\mathbb{Z}} (D^+ f_j) a_j.$$

For $D = D^+ D^-$ these formulae imply

$$\sum_{j\in\mathbb{Z}} f_j D a_j = \sum_{j\in\mathbb{Z}} (D f_j) a_j, \;\; \sum_{j\in\mathbb{Z}} f_j D^p a_j = \sum_{j\in\mathbb{Z}} (D^p f_j) a_j, \;\; p \in \mathbb{N}. \tag{8.37}$$

Recalling that $\underline{e} = (e_j) = (\delta_{j,0})$, we obtain with the help of (8.31) and (8.37)

$$d_k = \sum_{j\in\mathbb{Z}} a_{k-j} f_j = \sum_{j\in\mathbb{Z}} f_{k-j} a_j = \sum_{j\in\mathbb{Z}} f_{k-j} (\sum_{q=0}^{p-1} \gamma_q D^q \underline{e} + D^p J^p \underline{a})_j$$

$$= \sum_{q=0}^{p-1} \gamma_q \sum_{j\in\mathbb{Z}} (D^q f_{k-j}) e_j + \sum_{j\in\mathbb{Z}} (D^p f_{k-j}) (J^p \underline{a})_j = \sum_{q=0}^{p-1} \gamma_q D^q f_k + \sum_{j\in\mathbb{Z}} (J^p \underline{a})_{k-j} D^p f_j.$$

We took into account that $D_j f_{k-j} = D_k f_{k-j}$ where the designations $D_j f_{k-j}$ and $D_k f_{k-j}$ mean that the second central difference $D f_{k-j}$ is taken with respect to j or k, respectively; due to the equality of these differences, we may omit the indexes j or k by D. Thus the first expression form (8.33) for d_k is established. To obtain the second representation form (8.33) for d_k, it remains to observe that

$$f_k + \sum_{q=1}^{p-1} \gamma_q D^q f_k = \sum_{|j|\leq p-1} a_j^{(p)} f_{k-j}.$$

This proof detail is left as an exercise. □

8.9. Quasi-interpolation. Let $m \geq 3$. For the coefficients d_k of the Wiener interpolant $(Q_{h,m}f)(x) = \sum_{k\in\mathbb{Z}} d_k B_m(h^{-1}x - k)$ we derived the expressions (8.33). Dropping the "rest" term $\delta_k^{(p)} = \sum_{j\in\mathbb{Z}} (J^p\underline{a})_{k-j} D^p f_j$ in (8.33), we arrive at the *quasi-interpolants*

$$(Q_{h,m}^{(p)}f)(x) = \sum_{k\in\mathbb{Z}} d_k^{(p)} B_m(h^{-1}x - k), \ d_k^{(p)} = \sum_{|j|\leq p-1} a_j^{(p)} f_{k-j}, \ p \in \mathbb{N}, \tag{8.38}$$

with $a_j^{(p)} = a_{j,m}^{(p)}$ defined in (4.35) and $f_k = f((k+\frac{m}{2})h)$, $k \in \mathbb{Z}$.

Lemma 8.4. *Assume that $f \in C(\mathbb{R})$ is bounded or polynomially growing as $|x| \to \infty$. Then for $i \in \mathbb{Z}$, $p \in \mathbb{N}$, there hold the representations*

$$f((i+\frac{m}{2})h) - (Q_{h,m}^{(p)}f)((i+\frac{m}{2})h) \tag{8.39}$$

$$= \sum_{j=-\mu+1}^{\mu-1} \left(\sum_{\nu=1}^{\mu} \frac{z_\nu^p}{(1-z_\nu)^{2p}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{|k|\leq\mu} b_k\, z_\nu^{|k-j|} \right) D^p f_{i+j}$$

$$= \sum_{j=-\mu}^{\mu} \left(\sum_{\nu=1}^{\mu} \frac{z_\nu^p}{(1-z_\nu)^{2p}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{|k|\leq\mu} b_k\, D_j z_\nu^{|k-j|} \right) D^{p-1} f_{i+j}$$

$$= \sum_{j=-\mu+1}^{\mu} \left(\sum_{\nu=1}^{\mu} \frac{z_\nu^p}{(1-z_\nu)^{2p}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{k=-\mu}^{\mu} b_k (z_\nu^{|k-j|} - z_\nu^{|k-j-1|}) \right) D^{p-1} D^+ f_{i+j}$$

where $\mu = \operatorname{int}((m-1)/2)$, b_k are defined in (8.4), and z_ν, $\nu = 1, ..., \mu$, are the characteristic roots (the roots of the characteristic polynomial $P_{2\mu}$) in $(-1,0)$ whereas the index j in $D_j z_\nu^{|k-j|}$ *indicates that the difference $D = D^+D^-$ is applied to $z_\nu^{|k-j|}$ with respect to j.*

Proof. Due to (8.33)–(8.34),

$$(Q_{h,m}f)(x) - (Q_{h,m}^{(p)}f)(x) = \sum_{k\in\mathbb{Z}} \delta_k^{(p)} B_m(h^{-1}x - k) \text{ with } \delta_k^{(p)} = \sum_{j\in\mathbb{Z}} (J^p\underline{a})_{k-j} D^p f_j;$$

due to (8.4) and (8.36),

$$f((i+\frac{m}{2})h) - (Q_{h,m}^{(p)}f)((i+\frac{m}{2})h) = (Q_{h,m}f)((i+\frac{m}{2})h) - (Q_{h,m}^{(p)}f)((i+\frac{m}{2})h)$$

$$= \sum_{k\in\mathbb{Z}} \delta_k^{(p)} B_m(i+\frac{m}{2}-k) = \sum_{k\in\mathbb{Z}} b_{i-k}\delta_k^{(p)} = \sum_{k\in\mathbb{Z}} b_{-k}\delta_{k-i}^{(p)} = \sum_{k\in\mathbb{Z}} b_k \delta_{k-i}^{(p)}$$

$$= \sum_{|k|\leq\mu} b_k \left(\sum_{j\in\mathbb{Z}} (J^p\underline{a})_{k-i-j} D^p f_j \right) = \sum_{|k|\leq\mu} b_k \left(\sum_{j\in\mathbb{Z}} (J^p\underline{a})_{k-j} D^p f_{i+j} \right)$$

$$= \sum_{j\in\mathbb{Z}} \left(\sum_{|k|\leq\mu} b_k (J^p\underline{a})_{k-j} \right) D^p f_{i+j}$$

$$= \sum_{j\in\mathbb{Z}} \left(\sum_{\nu=1}^{\mu} \frac{z_\nu^p}{(1-z_\nu)^{2p}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{|k|\leq\mu} b_k\, z_\nu^{|k-j|} \right) D^p f_{i+j}.$$

Representing $D^p f_{i+j} = D \cdot D^{p-1} f_{i+j} = D^- D^{p-1} D^+ f_{i+j}$ and using the summation formulae, in particular, $\sum_{j\in\mathbb{Z}} a_j D^- f_j = -\sum_{j\in\mathbb{Z}} (D^+ a_j) f_j = \sum_{j\in\mathbb{Z}} (a_j - a_{j+1}) f_j$ we obtain also

$$f((i+\frac{m}{2})h) - (Q_{h,m}^{(p)} f)((i+\frac{m}{2})h)$$

$$= \sum_{j\in\mathbb{Z}} \left(\sum_{\nu=1}^{\mu} \frac{z_\nu^p}{(1-z_\nu)^{2p}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{|k|\le\mu} b_k D_j z_\nu^{|k-j|} \right) D^{p-1} f_{i+j}$$

$$= \sum_{j\in\mathbb{Z}} \left(\sum_{\nu=1}^{\mu} \frac{z_\nu^p}{(1-z_\nu)^{2p}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{|k|\le\mu} b_k (z_\nu^{|k-j|} - z_\nu^{|k-j-1|}) \right) D^{p-1} D^+ f_{i+j}.$$

These formulae take the form (8.39) since for the characteristic values z_ν we have $\sum_{|k|\le\mu} b_k\, z_\nu^{|k-j|} = 0$ for $|\, j \,| \ge \mu$:

$$\text{for } j \le -\mu, \quad \sum_{|k|\le\mu} b_k\, z_\nu^{|k-j|} = \sum_{|k|\le\mu} b_k\, z_\nu^{k-j} = z_\nu^{-j} \sum_{|k|\le\mu} b_k\, z_\nu^{k} = 0,$$

$$\text{for } j \ge \mu, \quad \sum_{|k|\le\mu} b_k\, z_\nu^{|k-j|} = \sum_{|k|\le\mu} b_k\, z_\nu^{j-k} = z_\nu^{j} \sum_{|k|\le\mu} b_k\, z_\nu^{-k} = 0.$$

Recall that together with z_ν, also z_ν^{-1} is a characteristic value. □

It holds $Q_{h,m} g = g$ for $g \in S_{h,m}$, and $Q_{h,m} f - Q_{h,m}^{(p)} f = Q_{h,m}(f - Q_{h,m}^{(p)} f)$. Together with the first one of representations (8.39) we obtain for $f \in V^{2p,\infty}(\mathbb{R})$

$$(8.40) \qquad \| Q_{h,m} f - Q_{h,m}^{(p)} f \|_\infty \le q_m \sup_{i\in\mathbb{Z}} |\, f((i+\frac{m}{2})h) - (Q_{h,m}^{(p)} f)((i+\frac{m}{2})h) \,|$$

$$\le q_m c_m^{(p)} h^{2p} \| f^{(2p)} \|_\infty$$

where $q_m = \| Q_{h,m} \|_{BC(\mathbb{R})\to BC(\mathbb{R})}$ (see Section 8.7 for its numerical values) and

$$c_m^{(p)} = \sum_{j=-\mu+1}^{\mu-1} \left| \sum_{\nu=1}^{\mu} \frac{z_\nu^p}{(1-z_\nu)^{2p}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{|k|\le\mu} b_k\, z_\nu^{|k-j|} \right|.$$

Differently from $\| f - Q_{h,m} f \|_\infty$ which is saturated at the accuracy $O(h^m)$, there is no saturation in the error $\| Q_{h,m} f - Q_{h,m}^{(p)} f \|_\infty$ – according to (8.40), the accuracy order $O(h^{2p})$ grows with p if f is sufficiently regular. It is reasonable to quasi-interpolate with the smallest $p \in \mathbb{N}$ for which $2p > m$; denote it by p', i.e.,

$$p' = \text{int}\left(\frac{m+2}{2}\right) = \left\{ \begin{array}{ll} \frac{m}{2}+1, & m \text{ even} \\ \frac{m+1}{2}, & m \text{ odd} \end{array} \right\}.$$

Denote also $Q'_{h,m} := Q_{h,m}^{(p')}$, $a'_{j,m} := a_{j,m}^{(p')}$. As we see from Theorem 8.7 below, $\| f - Q'_{h,m} f \|_\infty$ asymptotically maintains the accuracy of $\| f - Q_{h,m} f \|_\infty$ for C^m-smooth f.

Note that a quasi-interpolant can be determined from local values of f since for $x \in [ih, (i+1)h]$, $i \in \mathbb{Z}$, the sum in (8.38) is restricted to the following terms:

$$(Q_{h,m}^{(p)} f)(x) = \sum_{k=i-m+1}^{i} \left(\sum_{|j|\le p-1} a_j^{(p)} f_{k-j} \right) B_m(h^{-1}x - k).$$

In this expression, index $k-j$ varies between $(i-m+1)-(p-1)$ and $i+(p-1)$, and $f_{k-j} = f((k-j+\frac{m}{2})h)$ exploits values of f from the interval $[(i-\frac{m}{2}-p+2)h,(i+\frac{m}{2}+p-1)h]$. Thus $(Q_{h,m}^{(p)}f)(x)$ is well defined for $x \in [ih,(i+1)h]$ if f is given on the interval $[(i-\frac{m}{2}-p+2)h,(i+\frac{m}{2}+p-1)h]$. Also the total error $f(x)-(Q_{h,m}^{(p)}f)(x)$ can be estimated locally for any $p \in \mathbb{N}$. Taking into account our later interests we restrict ourselves to the case $p = p'$ and $x \in [0,1]$. The quasi-interpolant

$$(Q'_{h,m}f)(x) = \sum_{k=-m+1}^{n-1} \left(\sum_{|j|\le p'-1} a'_{j,m} f_{k-j} \right) B_m(h^{-1}x-k), \;\; 0 \le x \le 1,$$

is well defined for $f \in C(-mh, 1+mh)$.

Theorem 8.7. *For $f \in W^{m,\infty}(-\delta,1+\delta)$, $\delta \ge mh$, it holds*

$$\max_{0\le x\le 1} | f(x) - (Q'_{h,m}f)(x) | \le (\Phi_{m+1}\pi^{-m} + q_m c'_m) h^m \sup_{-\delta<x<1+\delta} | f^{(m)}(x) |, \tag{8.41}$$

where $q_m = \| Q_{h,m} \|_{BC(\mathbb{R})\to BC(\mathbb{R})}$ and the constant c'_m is defined by

$$c'_m = \sum_{j=-\mu}^{\mu} \left| \sum_{\nu=1}^{\mu} \frac{z_\nu^{p'}}{(1-z_\nu)^{2p'}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{k=-\mu}^{\mu} b_k D_j z_\nu^{|k-j|} \right| \text{ for even } m,$$

$$c'_m = \sum_{j=-\mu+1}^{\mu} \left| \sum_{\nu=1}^{\mu} \frac{z_\nu^{p'}}{(1-z_\nu)^{2p'}} \frac{z_\nu^{\mu-1}}{P'_{2\mu}(z_\nu)} \sum_{k=-\mu}^{\mu} b_k (z_\nu^{|k-j|} - z_\nu^{|k-j-1|}) \right| \text{ for odd } m.$$

Moreover, for any compact subset M of $C^m[-\delta,1+\delta]$, *it holds*

$$\sup_{f\in M} \max_{0\le x\le 1} | f(x) - (Q'_{h,m}f)(x) | \tag{8.42}$$

$$\le \Phi_{m+1}\pi^{-m}h^m \max_{-\delta\le x\le 1+\delta} | f^{(m)}(x) | + h^m \varepsilon_{h,m,M}, \;\; \varepsilon_{h,m,M} \to 0 \text{ as } h \to 0.$$

Proof. Let us extend $f \in W^{m,\infty}(-\delta,1+\delta) \subset C^{m-1}[-\delta,1+\delta]$ up to function $\overline{f} \in V^{m,\infty}(\mathbb{R})$ setting

$$\overline{f}(x) = \left\{ \begin{array}{cc} f_-(x), & x < -\delta \\ f(x), & -\delta \le x \le 1+\delta \\ f_+(x), & x > 1+\delta \end{array} \right\}$$

where $f_\mp$ are the Taylor polynomials of f of degree $m-1$ with expansion centers $-\delta$ and $1+\delta$, respectively. For $0 \le x \le 1$ we have $f(x) - (Q'_{h,m}f)(x) = \overline{f}(x) - (Q'_{h,m}\overline{f})(x)$, and together with the equality

$$\overline{f} - Q'_{h,m}\overline{f} = \overline{f} - Q_{h,m}\overline{f} + Q_{h,m}(Q_{h,m}\overline{f} - Q'_{h,m}\overline{f})$$

we obtain

$$\max_{0\le x\le 1} | f(x) - (Q'_{h,m}f)(x) |$$

$$\le \| \overline{f} - Q_{h,m}\overline{f} \|_\infty + q_m \sup_{l\in\mathbb{Z}} | \overline{f}((l+\frac{m}{2})h) - (Q_{h,m}^{(p')}\overline{f})((l+\frac{m}{2}) | .$$

By Theorem 8.2, $\| \overline{f} - Q_{h,m}\overline{f} \|_\infty \leq \Phi_{m+1}\pi^{-m}h^m \| \overline{f}^{(m)} \|_\infty$. Using the second and third representation (8.39) respectively for even and odd m, we get

$$\sup_{l\in\mathbb{Z}} | \overline{f}((l+\frac{m}{2})h) - (Q^{(p')}_{h,m}\overline{f})((l+\frac{m}{2})h) | \leq c'_m \left\{ \begin{array}{l} \sup_{j\in\mathbb{Z}} | D^{p'-1}\overline{f}_j |, \ m \text{ even} \\ \sup_{j\in\mathbb{Z}} | D^{p'-1}D^+\overline{f}_j |, \ m \text{ odd} \end{array} \right\}.$$

Further,

$$| D^{p'-1}\overline{f}_j | = | D^{m/2}\overline{f}_j | \leq h^m \| \overline{f}^{(m)} \|_\infty = h^m \| f^{(m)} \|_\infty \ \text{ for even } m,$$

$$| D^{p'-1}D^+\overline{f}_j | = | (D^{(m-1)/2}D^+)\overline{f}_j | \leq h^m \| \overline{f}^{(m)} \|_\infty = h^m \| f^{(m)} \|_\infty \ \text{ for odd } m$$

where $\| f^{(m)} \|_\infty := \sup_{-\delta<x<1+\delta} | f^{(m)}(x) |$. Thus

$$\| Q_{h,m}(Q_{h,m}\overline{f} - Q^{(p')}_{h,m}\overline{f}) \|_\infty \leq q_m c'_m h^m \sup_{-\delta<x<1+\delta} | f^{(m)}(x) | \tag{8.43}$$

and (8.41) follows.

To prove (8.42), introduce the operator

$$A_{h,m} : C^m[-\delta, 1+\delta] \to C[0,1], \ \ A_{h,m}f = h^{-m}Q_{h,m}(Q_{h,m}\overline{f} - Q'_{h,m}\overline{f})$$

where the extension $\overline{f}$ of f now is built using the Taylor polynomials of f of degree m. For $f \in C^{m+1}[-\delta, 1+\delta]$, we then have $\overline{f} \in V^{m+1,\infty}(\mathbb{R})$, and similarly as (8.43) we obtain (cf. also (8.40)) an estimate of order $\| Q_{h,m}(Q_{h,m}\overline{f} - Q^{(p')}_{h,m}\overline{f}) \|_\infty = O(h^{m+1})$. Thus $\| A_{h,m}f \|_{C[0,1]} \to 0$ as $h \to 0$ for f from $C^{m+1}[-\delta, 1+\delta]$ which is a dense set in $C^m[-\delta, 1+\delta]$. According to (8.43), $\| A_{h,m} \|_{C^m[-\delta,1+\delta]\to C[0,1]} \leq q_m c'_m$ for all sufficiently small h (for $h \leq \delta/m$). By Banach–Steinhaus theorem, the convergence $\| A_{h,m}f \|_{C[0,1]} \to 0$ as $h \to 0$ takes place for all $f \in C^m[-\delta, 1+\delta]$; the convergence is uniform with respect to $f \in M$ where $M \subset C^m[-\delta, 1+\delta]$ is a compact set. This proves (8.42) with $\varepsilon_{h,m,M} = \sup_{f\in M} \| A_{h,m}f \|_{C[0,1]} \to 0$ as $h \to 0$. □

The weights $a'_{j,m} := a^{(p')}_{j,m} = a^{(p')}_{-j,m} = a'_{-j,m}$, $j = 0, ..., p'-1$, of the quasi-interpolant

$$(Q'_{h,m}f)(x) = \sum_{k\in\mathbb{Z}} \left(\sum_{|j|\leq p'-1} a'_{j,m} f((k-j+\frac{m}{2})h) \right) B_m(h^{-1}x - k)$$

can be computed by (8.35) once for ever. For $m = 3, ..., 10$ they are as follows:

$a'_{j,m}$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$m=3$	1.2500000	−0.1250000				
$m=4$	1.5000000	−0.2777778	0.0277778			
$m=5$	1.6614583	−0.3715278	0.0407986			
$m=6$	2.0541667	−0.6385417	0.1229167	−0.0114583		
$m=7$	2.3113137	−0.8030165	0.1629774	−0.0156178		
$m=8$	2.9285825	−1.2534083	0.3430732	−0.0587258	0.0047696	
$m=9$	3.3532232	−1.5474118	0.4418932	−0.0774754	0.0063823	
$m=10$	4.3468295	−2.3113639	0.8030947	−0.1918579	0.0287522	−0.0020398

The value of $q'_m := \| Q'_{h,m} \|_{BC(\mathbb{R})\to BC(\mathbb{R})}$ can be computed according to the formula (cf. Section 8.7)

$$q^{(p)}_m := \| Q^{(p)}_{h,m} \|_{BC(\mathbb{R})\to BC(\mathbb{R})} = \max_{x\in[\frac{m}{2},\frac{m+1}{2}]} \sum_{j\in\mathbb{Z}} | F_{m,p}(x-j) |, \tag{8.44}$$

$$F_{m,p}(x) := \sum_{|k|\leq p-1} a^{(p)}_{k,m} B_m(x-k), \quad p \in \mathbb{N}.$$

For $m = 3, ..., 10$, the numerical values of c'_m (see (8.41)) and q'_m are presented in the following table; we recall also the values of $q_m = \| Q_{h,m} \|_{BC(\mathbb{R})\to BC(\mathbb{R})}$.

m	3	4	5	6	7	8	9	10	20
c'_m	0.016	0.019	0.015	0.0085	0.0060	0.0030	0.0022	0.0010	$6.5 \cdot 10^{-6}$
q'_m	1.250	1.354	1.329	1.403	1.356	1.413	1.378	1.419	1.514
q_m	1.414	1.549	1.706	1.816	1.916	2.000	2.075	2.142	2.583

We see that the quasi-interpolation is even more stable than the interpolation. Nevertheless, it can be guessed that also q'_m grows logarithmically as $m \to \infty$.

The interpolation operator $Q_{h,m}$ is a projector ($Q^2_{h,m} = Q_{h,m}$) with the range $S_{h,m}$. The range of a quasi-interpolation operator $Q^{(p)}_{h,m}$ is also in $S_{h,m}$ but $Q^{(p)}_{h,m}$ is not a projector. For instance, for $m \geq 3$ the support of $Q'_{h,m}B_m(nx-k)$ is wider than $[kh, (k+m)h]$, the support of $B_m(nx-k)$, hence functions $Q'_{h,m}B_m(nx-k)$ and $B_m(nx-k)$ are different.

8.10. Approximation of periodic functions. Introduce the space $C_{\text{per}}(\mathbb{R})$ of continuous 1-periodic functions equipped with the norm

$$\| f \|_\infty = \max_{0\leq x\leq 1} | f(x) | = \sup_{x\in\mathbb{R}} | f(x) | .$$

Denote also

$$C^m_{\text{per}}(\mathbb{R}) = C^m(\mathbb{R}) \cap C_{\text{per}}(\mathbb{R}), \;\; W^{m,\infty}_{\text{per}}(\mathbb{R}) = W^{m,\infty}(\mathbb{R}) \cap C_{\text{per}}(\mathbb{R}), \;\; m \in \mathbb{N}.$$

In that follows we specify the algorithms to find the spline interpolant of a function $f \in C_{\text{per}}(\mathbb{R})$. We also compare the accuracy-complexity intercourse for interpolation and quasi-interpolation.

Set $h = 1/n$, $n \in \mathbb{N}$. Then for $f \in C_{\text{per}}(\mathbb{R})$, the Wiener interpolant

$$(Q_{h,m}f)(x) = \sum_{k\in\mathbb{Z}} d_k B_m(nx-k)$$

is 1-periodic and the bisequence of its coefficients $d_k = \sum_{j\in\mathbb{Z}} a_{k-j} f((j+\frac{m}{2})h)$ is n-periodic (i.e., $d_{k+n} = d_k$, $k \in \mathbb{Z}$), so it is sufficient to compute $d_0, ..., d_{n-1}$. Representing $j \in \mathbb{Z}$ as $j = i + ln$, $0 \leq i \leq n-1$, $l \in \mathbb{Z}$, we have

$$d_k = \sum_{i=0}^{n-1} \sum_{l\in\mathbb{Z}} a_{k-i-ln} f((i+\frac{m}{2})h) = \sum_{i=0}^{n-1} \widetilde{a}_{k-i} f((i+\frac{m}{2})h), \;\; k = 0, ..., n-1,$$

where

$$\widetilde{a}_k = \sum_{l\in\mathbb{Z}} a_{k-ln}, \;\; | k | \leq n-1.$$

Thus the computation of $d_0, ...d_{n-1}$ is reduced to the application of the Toeplitz $n \times n$-matrix $(\widetilde{a}_{k-i})^{n-1}_{k,i=0}$ to the n-vector $(f(\frac{m}{2}h), ..., f((n-1+\frac{m}{2})h))$. The complexity of this application is n^2 flops in the usual matrix algebra and $O(n\log n)$ flops if the fast Fourier transform (FFT) is involved. A further reduction of the complexity can be achieved using the periodised version of B-splines as discussed below.

Introduce the dilated-shifted B-splines $B_{n,m,k}(x) = B_m(nx-k)$, $k \in \mathbb{Z}$, and their 1-periodised version

$$\widetilde{B}_{n,m.k}(x) = \sum_{l\in\mathbb{Z}} B_{n,m,k}(x+l) = \sum_{l\in\mathbb{Z}} B_m(nx+ln-k), \;\; k \in \mathbb{Z}.$$

Clearly, $\widetilde{B}_{n,m.k} = \widetilde{B}_{n,m.k+m}$, $k \in \mathbb{Z}$, and $\widetilde{B}_{n,m.k}$, $k = 0,...n-1$, constitute a basis of the n-dimensional space $\widetilde{S}_{h,m} = S_{h,m} \cap C_{\text{per}}(\mathbb{R})$ of 1-periodic splines. The interpolation operator $Q_{h,m}$ defined in Section 8.2 maps $C_{\text{per}}(\mathbb{R})$ into $\widetilde{S}_{h,m}$; representing $j \in \mathbb{Z}$ as $j = k + ln$, $0 \le k \le n-1$, $l \in \mathbb{Z}$, we have for $f \in C_{\text{per}}(\mathbb{R})$

$$Q_{h,m}f = \sum_{j\in\mathbb{Z}} d_j B_{n,m,j} = \sum_{k=0}^{n-1}\sum_{l\in\mathbb{Z}} d_{k+ln} B_{n,m,k+ln} = \sum_{k=0}^{n-1} d_k \widetilde{B}_{n,m.k}.$$

Due to periodicity, it is sufficient to pose the interpolation conditions (8.2) only for $i = 0, ..., n-1$. Thus, for $f \in C_{\text{per}}(\mathbb{R})$, the interpolant $Q_{h,m}f = \sum_{k=0}^{n-1} d_k \widetilde{B}_{n,m.k}$ is determined by conditions

$$(Q_{h,m}f)((i + \frac{m}{2})h) = f((i + \frac{m}{2})h), \;\; i = 0, ..., n-1,$$

that lead to the $n \times n$-system of linear algebraic equations

$$\text{(8.45)} \qquad \sum_{k=0}^{n-1} \widetilde{B}_{n,m,k}((i + \frac{m}{2})h) d_k = f((i + \frac{m}{2})h), \;\; i = 0, ..., n-1,$$

with respect to the expansion coefficients d_k, $k = 0, ..., n-1$. From the presented considerations we can conclude that $(\widetilde{a}_{k-i})_{k,i=0}^{n-1}$ is the inverse matrix to the matrix of system (8.45). There is not much use of this fact for a fast solving of system (8.45). For $n > m$, the matrix of system (8.45) preserves the band structure with a band width $2\mu+1$ but, in addition, μ nonzero diagonals appear in the left lower and right upper corners of the matrix. The solving of system (8.45) by the Gauss elimination method without pivoting costs approximately $N \approx \frac{1}{2}m^2 n$ flops and $N \approx \frac{3}{2}m^2 n$ flops in case of pivoting along columns of the lower triangle part of the matrix. A method of complexity mn or perhaps cmn flops were of great interest but, unfortunately, the construction and justification of so fast stable (possibly iterative) methods is complicated if possible at all. Note that already one application of the matrix of system (8.45) to an n-vector costs $(2\mu+1)n \approx mn$ flops. Actually there is a method of complexity $2mn$ flops but it is hopelessly unstable: theoretically, in infinite precise arithmetics, we could find $d_0, ..., d_{2\mu-1}$ by the application of the matrix $(\widetilde{a}_{k-i})_{k,i=0}^{n-1}$ to the r.h.s. of (8.45) and after that determine $d_{2\mu}, ..., d_{n-1}$ recursively from (8.45).

According to Theorem 8.2, for $f \in W_{\text{per}}^{m,\infty}(\mathbb{R})$,

$$\text{(8.46)} \qquad \| f - Q_{h,m}f \|_\infty \le \Phi_{m+1}\pi^{-m}h^m \| f^{(m)} \|_\infty .$$

This is a remarkable estimate also in the sense that (see [8]) the Kolmogorov n-width of the set $\{f \in W_{\text{per}}^{m,\infty}(\mathbb{R}) : \| f^{(m)} \|_\infty \le 1\}$ in $C_{\text{per}}(\mathbb{R})$ equals to $\Phi_{m+1}\pi^{-m}n^{-m}$ for even n. Hence, for arbitrary n-dimensional subspace $E_n \subset C_{\text{per}}(\mathbb{R})$ with even n, no mapping $M_n : W_{\text{per}}^{m,\infty}(\mathbb{R}) \to E_n$ (linear or nonlinear!) and no $\varepsilon_n < \Phi_{m+1}\pi^{-m}n^{-m}$ exist such that $\| f - M_n f \|_\infty \le \varepsilon_n \| f^{(m)} \|_\infty$ for all $f \in W_{\text{per}}^{m,\infty}(\mathbb{R})$.

With respect to the complexity $N \approx \frac{1}{2}m^2 n$ corresponding to the Gauss elimination to solve system (8.45), estimate (8.46) takes the form

$$\text{(8.47)} \qquad \| f - Q_{h,m}f \|_\infty \le \Phi_{m+1}(2\pi)^{-m} m^{2m} N^{-m} \| f^{(m)} \|_\infty .$$

Let us turn to quasi-interpolation. For $f \in C_{\text{per}}(\mathbb{R})$, the quasi-interpolant $Q'_{h,m}f$ is still defined by

$$(Q'_{h,m}f)(x) = \sum_{k\in\mathbb{Z}} d'_k B_m(nx - k), \;\; d'_k = \sum_{|j|\le p'-1} a'_{j,m} f((k - j + \frac{m}{2})h), \;\; k \in \mathbb{Z},$$

but now $d'_k = d'_{k+n}$, $k \in \mathbb{Z}$, so we have to compute only d'_k for $k = 0, ..., n-1$ that costs approximately $N = mn$ flops. By Theorem 8.7,

$$\| f - Q'_{h,m}f \|_\infty \le (\Phi_{m+1}\pi^{-m} + q_m c'_m)h^m \| f^{(m)} \|_\infty, \;\; f \in C_{\text{per}}^m(\mathbb{R}).$$

With respect to the complexity $N = mn$, this estimate takes the form

(8.48) $$\| f - Q'_{h,m} f \|_\infty \le (\Phi_{m+1}\pi^{-m} + q_m c'_m) m^m N^{-m} \| f^{(m)} \|_\infty$$

that is better than (8.47): the factors m^{2m} and m^m dominate in (8.47) and (8.48), respectively; for great m, the constant c'_m seems to be of the order $o(2^{-m/2})$. For an individual $f \in C^m_{\text{per}}(\mathbb{R})$, the asymptotic estimate (8.42) even more strongly inclines us to prefer the quasi-interpolation.

9. Spline collocation and quasi-collocation for weakly singular integral equations

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9.1. Operator form of the quasi-collocation method. Let us return to the weakly singular integral equation (5.1) , $u = Tu + f$, with $K \in \mathcal{S}^{m,\nu}$, $f \in C^{m,\nu}(0,1)$, $m \ge 2$, $0 < \nu < 1$. Using the smoothing change of variables we rewrite (5.1) in the form (5.15), $v = T_\varphi v + f_\varphi$, in the inteval $0 \le t \le 1$. Introduce the extension operator (already expoloited in Section 7)

$$E_\delta : C[0,1] \to C[-\delta, 1+\delta], \;\; (E_\delta f)(t) = \left\{ \begin{array}{ll} f(0), & -\delta \le t \le 0 \\ f(t), & 0 \le t \le 1 \\ f(1), & 1 \le t \le 1+\delta \end{array} \right\}.$$

For $h = 1/n$, $n \in \mathbb{N}$, introduce the spline quasi-interpolation operator (see Section 4.9) $Q'_{h,m}$: $C[-\delta, 1+\delta] \to C[0,1]$,

$$(Q'_{h,m} v)(t) = \sum_{i=-m+1}^{n-1} \left(\sum_{|j| \le p'-1} a'_{j,m} v((i - j + \frac{m}{2})h) \right) B_m(nt - i), \;\; 0 \le t \le 1.$$

We approximate equation (5.15) by the finite dimensional equation

(9.1) $$v_h = Q'_{h,m} E_\delta T_\varphi v_h + Q'_{h,m} E_\delta f_\varphi.$$

In analogy to the collocation method, we call this method *spline quasicollocation method.* Note that $Q'_{h,m} E_\delta : C[0,1] \to C[0,1]$ is not a projection operator but this is no obstacle to obtain an effective method.

Theorem 9.1. *Let $K \in \mathcal{S}^{m,\nu}$, $f \in C^{m,\nu}(0,1)$, $m \ge 3$, $\nu < 1$, and let $\varphi : [0,1] \to [0,1]$ satisfy the conditions of Theorem* 5.3. *Further, assume that $\mathcal{N}(I - T) = \{\mathbf{0}\}$ (or equivalently, $\mathcal{N}(I - T_\varphi) = \{\mathbf{0}\}$). Then there exists an n_0, such that for $n \ge n_0$ the quasicollocation equation* (9.1) *has a unique solution v_h. The error of v_h can be estimated by*

(9.2) $$\| v - v_h \|_\infty \le c h^m \| v^{(m)} \|_\infty$$

where $v(t) = u_\varphi(t) = u(\varphi(t))$ is the solution of (5.15), *$u(x)$ is the solution of* (5.1). *The constant* c in (9.2) *is independent of n and f (it depends on K, m and φ).*

Proof. This formulation is almost identical to that of Theorem 7.1 but now the claims concern the spline quasicollocation method (9.1). The proof of the theorem repeats the argument in the proof of Theorem 7.1. There is no need to reproduce all the details of the proof again. We comment only on details that are different from those in the proof of Theorem 7.1.

First of all, we have to justify the pointwise convergence of $Q'_{h,m} E_\delta$ to I in $C[0,1]$. This follows by Banach–Steinhaus theorem (Theorem 2.2): (i) clearly

$$\| Q'_{h,m} E_\delta \|_{C[0,1] \to C[0,1]} \le \| Q'_{h,m} \|_{C[-\delta,1+\delta] \to C[0,1]} \le \text{const}, \;\; n \in \mathbb{N};$$

(ii) the set

$$V^{(m)} := \{ v \in C^m[0,1] : v^{(j)}(0) = v^{(j)}(1) = 0, \; j = 1, ..., m \}$$

is dense in $C[0,1]$, $E_\delta V^{(m)} \subset C^m[-\delta, 1+\delta]$ and by (8.41) $Q'_{h,m}E_\delta v \to v$ for $v \in V^{(m)}$ as $n \to \infty$ with the estimate

$$\| v - Q'_{h,m}E_\delta v \|_\infty \le ch^m \| v^{(m)} \|_\infty . \tag{9.3}$$

Now similarly as estimate (7.19) in the proof of Theorem 7.1, we obtain for the solution v of equation (5.15) and the solution v_h of equation (9.1)

$$\| v - v_h \|_\infty \le \kappa_h \| v - Q'_{h,m}E_\delta v \|_\infty \tag{9.4}$$

where

$$\kappa_h := \| (I - Q'_{h,m}E_\delta T_\varphi)^{-1} \|_{C[0,1]\to C[0,1]} \le \frac{\kappa}{1-\kappa\varepsilon_h} \to \kappa \ \text{ as } n \to \infty,$$

$$\varepsilon_h := \| T_\varphi - Q'_{h,m}E_\delta T_\varphi \|_{C[0,1]\to C[0,1]} \to 0 \ \text{ as } n \to \infty.$$

By Corollary 5.1, $v \in V^{(m)}$ for the solution of (5.15), thus estimate (9.4), (9.3) holds for the solution of (5.15) and (9.1) implying (9.2). □

9.2. Matrix form of the spline quasicollocation method. The solution v_h of the quasicollocation equation (9.1) has the form

$$v_h(t) = \sum_{i=-m+1}^{n-1} c_i B_{m-1}(nt-i) \tag{9.5}$$

in which we have to determine the $n+m-1$ unknown parameters c_i, $i = -m+1, ..., n-1$. The two terms in the r.h.s. of (9.1) are

$$(Q'_{h,m}E_\delta f_\varphi)(t) = \sum_{i=-m+1}^{n-1} \left(\sum_{|j|\le p'-1} a'_{j,m}(E_\delta f_\varphi)((i-j+\frac{m}{2})h) \right) B_m(nt-i),$$

and

$$(Q'_{h,m}E_\delta T_\varphi v_h)(t) = \sum_{i=-m+1}^{n-1} \left(\sum_{|j|\le p'-1} a'_{j,m}(E_\delta T_\varphi v_h)((i-j+\frac{m}{2})h) \right) B_m(nt-i).$$

Here

$$(E_\delta T_\varphi v_h)(t) = \left\{ \begin{array}{ll} \int_0^1 K_\varphi(0,s)v_h(s)ds, & t<0 \\ \int_0^1 K_\varphi(t,s)v_h(s)ds, & 0\le t\le 1 \\ \int_0^1 K_\varphi(1,s)v_h(s)ds, & t>1 \end{array} \right\}$$

$$= \sum_{k=-m+1}^{n-1} \left\{ \begin{array}{ll} \int_0^1 K_\varphi(0,s)B_m(ns-k)ds, & t<0 \\ \int_0^1 K_\varphi(t,s)B_m(ns-jk)ds, & 0\le t\le 1 \\ \int_0^1 K_\varphi(1,s)B_m(ns-jk)ds, & t>1 \end{array} \right\} c_k,$$

thus

$$(E_\delta T_\varphi v_h)((i-j+\frac{m}{2})h)$$

$$= \sum_{k=-m+1}^{n-1} \left\{ \begin{array}{ll} \int_0^1 K_\varphi(0,s)B_m(ns-k)ds, & (i-j+\frac{m}{2})h<0 \\ \int_0^1 K_\varphi((i+\frac{m}{2})h,s)B_m(ns-k)ds, & 0\le (i-j+\frac{m}{2})h\le 1 \\ \int_0^1 K_\varphi(1,s)B_{m-1}(ns-k)ds, & (i-j+\frac{m}{2})h>1 \end{array} \right\} c_k.$$

From equality of coefficients by $B_{m-1}(nt-i)$, $i=-m+1,...,n-1$, in the l.h.s and r.h.s. of equation (9.1) we obtain the following system of linear equations for the determining of the parameters c_i, $i=-m+1,...,n-1$, of v_h:

$$c_i = \sum_{k=-m+1}^{n-1} \alpha_{i,k} c_k + \beta_i, \quad i=-m+1,...,n-1, \tag{9.6}$$

where

$$\beta_i = \sum_{|j|\le p'-1} a'_{j,m} \sigma_{i-j}, \quad \alpha_{i,k} = \sum_{|j|\le p'-1} a'_{j,m} \tau_{i-j,k}, \quad i,k=-m+1,...,n-1,$$

$$\sigma_i = \left\{ \begin{array}{cc} f_\varphi(0), & (i+\frac{m}{2})h<0 \\ f_\varphi((i+\frac{m}{2})h), & 0\le (i+\frac{m}{2})h \le 1 \\ f_\varphi(1), & (i+\frac{m}{2})h>1 \end{array} \right\},$$

$$\tau_{i,k} = \left\{ \begin{array}{cc} \int_0^1 K_\varphi(0,s) B_{m-1}(ns-k)ds, & (i+\frac{m}{2})h<0 \\ \int_0^1 K_\varphi((i+\frac{m}{2})h,s) B_{m-1}(ns-k)ds, & 0\le (i+\frac{m}{2})h \le 1 \\ \int_0^1 K_\varphi(1,s) B_{m-1}(ns-k)ds, & (i+\frac{m}{2})h>1 \end{array} \right\}.$$

Having found c_i, $i=-m+1,...,n-1$, by solving the system (9.6), the quasi-collocation solution v_h is given by (9.5).

9.3. Periodization of the integral equation and the spline collocation method. Introduce the one dimensional projection operator

$$\Pi: C[0,1]\to C[0,1], \quad (\Pi v)(x) = [v(1)-v(0)]x$$

(clearly $\Pi^2=\Pi$). Equation (5.15), $v=T_\varphi v + f_\varphi$, is equivalent to the system of two equations

$$\Pi v = \Pi T_\varphi \Pi v + \Pi T_\varphi (I-\Pi)v + \Pi f_\varphi,$$

$$(I-\Pi)v = (I-\Pi)T_\varphi(\Pi v) + (I-\Pi)T_\varphi(I-\Pi)v + (I-\Pi)f_\varphi$$

with unknowns Πv and $(I-\Pi)v =: \widetilde{v}$. With respect to the unknowns $\alpha := v(1)-v(0)\in\mathbb{R}$ and $\widetilde{v}\in C[0,1]$, this system can be written as

$$\alpha = \theta\alpha + \int_0^1 \sigma(s)\widetilde{v}(s)ds + \beta, \quad \widetilde{v}(t) = \alpha\widetilde{\tau}(t) + \int_0^1 \widetilde{K}_\varphi(t,s)\widetilde{v}(s)ds + \widetilde{f}_\varphi(t) \tag{9.7}$$

where

$$\beta = f_\varphi(1) - f_\varphi(0), \quad \widetilde{f}_\varphi(t) = f_\varphi(t) - \beta t,$$

$$\sigma(s) = K_\varphi(1,s) - K_\varphi(0,s), \quad \theta = \int_0^1 \sigma(s)sds,$$

$$\widetilde{\tau}(t) = \int_0^1 K_\varphi(t,s)sds - \theta t, \quad \widetilde{K}_\varphi(t,s) = K_\varphi(t,s) - t\sigma(s).$$

If $(\alpha,\widetilde{v})$ is a solution to system (9.7) then $v(t)=\alpha t+\widetilde{v}(t)$ is a solution of equation (5.15). Observe that

$$\widetilde{\tau}(0)=\widetilde{\tau}(1), \quad \widetilde{K}_\varphi(0,s) = \widetilde{K}_\varphi(1,s);$$

we extend $\widetilde{\tau}(t)$ and $\widetilde{K}_\varphi(t,s)$ into 1-periodic functions of t maintaining the same designations for the extensions. For $\widetilde{v} = (I - \Pi)v$, $v \in C[0,1]$, we also have $\widetilde{v}(0) = \widetilde{v}(1)$, and the may treat $\widetilde{v}$ and $\widetilde{f}_\varphi$ as 1-periodic functions. So we can consider system (9.7) as an equation in the space $X = \mathbb{R} \times C_{\text{per}}(\mathbb{R})$,

$$\begin{pmatrix} \alpha \\ \widetilde{v} \end{pmatrix} = \begin{pmatrix} \theta & \Sigma \\ \widetilde{\tau} & \widetilde{T}_\varphi \end{pmatrix} \begin{pmatrix} \alpha \\ \widetilde{v} \end{pmatrix} + \begin{pmatrix} \beta \\ \widetilde{f}_\varphi \end{pmatrix} \tag{9.8}$$

where

$$\Sigma : C_{\text{per}}(\mathbb{R}) \to \mathbb{R}, \quad \Sigma\widetilde{v} = \int_0^1 \sigma(s)\widetilde{v}(s)ds,$$

$$\widetilde{T}_\varphi : C_{\text{per}}(\mathbb{R}) \to C_{\text{per}}(\mathbb{R}), \quad (\widetilde{T}_\varphi\widetilde{v})(t) = \int_0^1 \widetilde{K}_\varphi(t,s)\widetilde{v}(s)ds.$$

We build the collocation solution of (9.8) with the help of the interpolation projection operator $Q_{h,m}$ which maps $C_{\text{per}}(\mathbb{R})$ into $\widetilde{S}_{h,m} := S_{h,m} \cap C_{\text{per}}(\mathbb{R})$, see Section 8.10:

$$\begin{pmatrix} \alpha_h \\ \widetilde{v}_h \end{pmatrix} = \begin{pmatrix} \theta & \Sigma \\ Q_{h,m}\widetilde{\tau} & Q_{h,m}\widetilde{T}_\varphi \end{pmatrix} \begin{pmatrix} \alpha_h \\ \widetilde{v}_{nh} \end{pmatrix} + \begin{pmatrix} \beta \\ Q_{h,m}\widetilde{f}_\varphi \end{pmatrix}. \tag{9.9}$$

This is a system with repect to $\begin{pmatrix} \alpha_h \\ \widetilde{v}_h \end{pmatrix} \in \mathbb{R} \times \widetilde{S}_{h,m}$; the approximate solution v_h to equation (5.15) is given by

$$v_h(t) = \alpha_h t + \widetilde{v}_h(t), \quad 0 \le t \le 1. \tag{9.10}$$

Recall from Section 8.10 the designations $B_{n,m,k}(x) = B_m(nx - k)$,

$$\widetilde{B}_{n,m,k}(x) = \sum_{l\in\mathbb{Z}} B_{n,m,k}(x + l) = \sum_{l\in\mathbb{Z}} B_m(nx + ln - k), \quad k \in \mathbb{Z}.$$

Recall also that $\widetilde{B}_{n,m,k}$, $k = 0, ...n-1$, constitute a basis of the n-dimensional space $\widetilde{S}_{h,m} = S_{h,m} \cap C_{\text{per}}(\mathbb{R})$ of 1-periodic splines. Representing $\widetilde{v}_h = \sum_{k=0}^{n-1} c_k\widetilde{B}_{n,m,k}$, (9.9) yields with respect to $n+1$ unknowns α_h and $c_0, ..., c_{n-1}$ the system of $n+1$ linear algebraic equations

$$\alpha_h = \theta\alpha_h + \sum_{k=0}^{n-1} \int_0^1 \sigma(s)\widetilde{B}_{n,m,k}(s)ds\, c_k + \beta, \tag{9.11}$$

$$\sum_{k=0}^{n-1} \widetilde{B}_{n,m,k}((i + \frac{m}{2})h)c_k = \alpha_h\widetilde{\tau}((i + \frac{m}{2})h) + \sum_{k=0}^{n-1} \int_0^1 [\widetilde{K}_\varphi((i + \frac{m}{2})h, s)\widetilde{B}_{n,m,k}(s)ds\, c_k$$

$$+\widetilde{f}_\varphi((i + \frac{m}{2})h), \quad i = 0, ..., n-1.$$

Here $\widetilde{\tau}(t)$, $\widetilde{K}_\varphi(t,s)$ and $\widetilde{f}_\varphi(t)$ are understood as 1-periodic functions of t.

Theorem 9.2. *Assume the conditions of Theorem* 9.1. *Then there exists an* n_0, *such that for* $n \ge n_0$ *the collocation system* (9.11) *has a unique solution* α_h, $c_0, ..., c_{n-1}$. *The accuracy of* v_h *defined by* (9.10) *can be estimated by*

$$\| v - v_h \|_\infty \le ch^m \| v^{(m)} \|_\infty \tag{9.12}$$

where the constant c is independent of n (of $h = 1/n$).

Proof. $X := \mathbb{R} \times C_{\text{per}}(\mathbb{R})$ is a Banach space with the norm $\| (\alpha, \widetilde{v}) \| = | \alpha | + \| \widetilde{v} \|_\infty$. The identity operator in X is given by $I_X = \begin{pmatrix} 1 & 0 \\ 0 & I \end{pmatrix}$ where I is the identity operator in $C_{\text{per}}(\mathbb{R})$. Denote

$$\mathcal{T} = \begin{pmatrix} \theta & \Sigma \\ \widetilde{\tau} & \widetilde{T}_\varphi \end{pmatrix} : X \to X, \ \ \mathcal{T}_h = \begin{pmatrix} \theta & \Sigma \\ Q_{h,m}\widetilde{\tau} & Q_{h,m}\widetilde{T}_\varphi \end{pmatrix} : X \to X.$$

The operator $\mathcal{T} : X \to X$ is compact that easily follows from the compactness of the operator $\widetilde{T}_\varphi : C_{\text{per}}(\mathbb{R}) \to C_{\text{per}}(\mathbb{R})$. If

$$\begin{pmatrix} \alpha \\ \widetilde{v} \end{pmatrix} = \begin{pmatrix} \theta & \Sigma \\ \widetilde{\tau} & \widetilde{T}_\varphi \end{pmatrix} \begin{pmatrix} \alpha \\ \widetilde{v} \end{pmatrix}$$

for some $(\alpha, \widetilde{v}) \in X$ then $v := \alpha t + \widetilde{v}$ is the solution of the homogenous equation $v = T_\varphi v$, and by the assumption of the Theorem, $v = 0$ that implies $\alpha = 0$, $\widetilde{v} = 0$. Hence $I_X - \mathcal{T} : X \to X$ has a bounded inverse $(I_X - \mathcal{T})^{-1} : X \to X$. Further, since $\widetilde{T}_\varphi : C_{\text{per}}(\mathbb{R}) \to C_{\text{per}}(\mathbb{R})$ is compact and $\| \widetilde{v} - Q_{h,m}\widetilde{v} \|_\infty \to 0$ for every $\widetilde{v} \in C_{\text{per}}(\mathbb{R})$, we have by Theorem 2.6 $\| (I - Q_{h,m})\widetilde{T}_\varphi \|_{C_{\text{per}}(\mathbb{R}) \to C_{\text{per}}(\mathbb{R})} \to 0$ that implies $\| \mathcal{T} - \mathcal{T}_h \|_{X \to X} \to 0$ as $n \to \infty$. For n such that $\| (I_X - \mathcal{T})^{-1} \|_{X \to X} \| \mathcal{T} - \mathcal{T}_h \|_{X \to X} < 1$, also $I_X - \mathcal{T}_h$ is invertible and

$$\zeta_h := \| (I_X - \mathcal{T}_h)^{-1} \|_{X \to X}$$

$$\leq \frac{\| (I_X - \mathcal{T})^{-1} \|_{X \to X}}{1 - \| (I_X - \mathcal{T})^{-1} \|_{X \to X} \| \mathcal{T} - \mathcal{T}_h \|_{X \to X}} \to \| (I_X - \mathcal{T})^{-1} \|_{X \to X} =: \zeta$$

as $n \to \infty$. In particular, collocation system (9.11) is uniquely solvable for all sufficiently large n. For the solution $(\alpha, \widetilde{v})$ of (9.8) and the solution $(\alpha_h, \widetilde{v}_h)$ of (9.9) we have

$$(I_X - \mathcal{T}_h)\{(\alpha, \widetilde{v}) - (\alpha_h, \widetilde{v}_h)\} = (I_X - \mathcal{T}_h)(\alpha, \widetilde{v}) - (\beta, Q_{h,m}\widetilde{f}_\varphi)$$

$$= (I_X - \mathcal{T})(\alpha, \widetilde{v}) + (\mathcal{T} - \mathcal{T}_h)(\alpha, \widetilde{v}) - (\beta, Q_{h,m}\widetilde{f}_\varphi)$$

$$= (\beta, \widetilde{f}_\varphi) + (0, \, \alpha(I - Q_{h,m})\widetilde{\tau} + (I - Q_{h,m})\widetilde{T}_\varphi \widetilde{v}) - (\beta, Q_{h,m}\widetilde{f}_\varphi)$$

where for simplicity we wrote $(\alpha, \widetilde{v})$ instead of $\begin{pmatrix} \alpha \\ \widetilde{v} \end{pmatrix}$. This implies

$$| \alpha - \alpha_h | + \| \widetilde{v} - \widetilde{v}_h \|_\infty = \| (\alpha, \widetilde{v}) - (\alpha_h, \widetilde{v}_h) \|_X \leq \zeta_h \| (I - Q_{h,m})\widetilde{v} \|_\infty .$$

By (8.46), $\| (I - Q_{h,m})\widetilde{v} \|_\infty \leq \Phi_{m+1}\pi^{-m}h^m \| \widetilde{v}^{(m)} \|_\infty$. For the solution $v(t) = \alpha t + \widetilde{v}(t)$ of equation (5.15) and the collocation solution $v_h(t) = \alpha_h t + \widetilde{v}_h(t)$, these estimate yield (9.12) completing the proof of the Theorem. □

Exercises and problems.

1. Prove the compactness in $C[0,1]$ of the Fredholm and Volterra integral operators with a continuous kernel, see Section 2.4.
2. Prove the Faa di Bruno differentiation formula (2.1). *Hint*: induction.
3. Establish the Leibnitz rule for $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^l$:

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^l [a(x,y)b(x,y)] = \sum_{j=0}^{l} \begin{pmatrix} l \\ j \end{pmatrix} \left[\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^j a(x,y)\right] \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^{l-j} b(x,y).$$

4. Show that the kernel (3.1) with $0 < \nu < 1$ belongs to $\mathcal{S}^{m,\nu}$ if $a \in C^m([0,1] \times [0,1])$ or, more generally, if $a \in \mathcal{S}^{m,-\delta}$, $\delta > 0$.

5. Show that the kernel $K(x,y) = a(x,y) \log \mid x - y \mid$ with $a \in C^m([0,1] \times [0,1])$ belongs to $\mathcal{S}^{m,0}$ or, more generally, if $a \in \mathcal{S}^{m,-\delta}$, $\delta > 0$.

6. Prove the claims of Lemma 3.1.

7. Prove Lemma 4.1.

8. Present a detailed proof of Theorem 4.3.

9. Prove the compactness of the imbedding $C^{m,\nu}(0,1) \subset C[0,1]$, $m \geq 1$, $\nu < 1$.

10. Prove (5.4) and the compactness of the imbedding operator.

11. Prove that the spaces $C^{m,\nu}(0,1)$ and $C^{m,\nu}(0,1]$ are complete.

12. Prove that $uv \in C^{m,\nu}(0,1)$ for $u, v \in C^{m,\nu}(0,1)$ and

$$\| uv \|_{C^{m,\nu}(0,1)} \leq c \| u \|_{C^{m,\nu}(0,1)} \| v \|_{C^{m,\nu}(0,1)}$$

with a constant c that is independent of u and v.

13. Prove that $\| u' \|_{C^{m-1,\nu+1}(0,1)} \leq \| u \|_{C^{m,\nu}(0,1)}$ for $u \in C^{m,\nu}(0,1)$, $m \geq 1$, $\nu < 0$.

14. Prove that equation (6.1) with $K \in \mathcal{S}^{m,\nu}(\triangle)$, $m \geq 0$, $\nu < 1$, $f \in C[0,1]$ has a unique solution $u \in C[0,1]$.

15. Present a detailed proof of Lemma 7.1.

16. Assume the conditions of Theorem 7.1 but purely $f_\varphi \in C[0,1]$. Prove that $\| v - v_h \|_\infty \to 0$ as $n \to \infty$. Relax the condition also for K assuming that $K \in \mathcal{S}^{0,\nu}$, $\nu < 1$.

17*. Present and prove a counterpart of Theorem 7.1 in case $m = 1$ using piecewise constant interpolants with the central dislocation of the interpolation points, cf. Section 7.3. Examine the superconvergence of the collocation solution at the collocation points, i.e., the convergence with a speed exceeding the global convergence speed $\| v - v_n \|_\infty \leq ch \| v' \|_\infty$. Examine full discretizations of the method and two grid iteration schemes of complexity $O(n^2)$ flops to solve the discretized collocation system . Solve numerical examples and comment on them.

18. Present the proof of the equivalence of Definitions 8.1 and 8.2.

19. Prove the properties of B_m listed in Section 8.1.

20. Show that for $m \geq 2$, B_m is strictly increasing on $[0, \frac{m}{2}]$ and strictly decreasing on $[\frac{m}{2}, m]$.

21. Prove that, with B_1 given in Definition 8.2, B_m satisfy the recursion

$$B_m(x) = \frac{1}{m-1}[xB_{m-1}(x) + (m-x)B_{m-1}(x-1)], \quad m = 2, 3, \dots .$$

22. Establish for $b_{k,m} = B_m(k + \frac{m}{2})$, $\mid k \mid \leq \mu = \text{int}((m-1/2)$, $m \geq 3$, the recursion formula

$$b_{k,m} = \frac{1}{4(m-1)(m-2)} \left((m-2k)^2 b_{k-1,m-1} + 2(m^2 - 2m - 4k^2) b_{k.m-1} + (m+2k)^2 b_{k+1,m-1} \right).$$

23. Prove the inequality $\| Q_{h,m} \|_{BC(\mathbb{R}) \to BC(\mathbb{R})} \leq \sum_{k \in \mathbb{Z}} \mid a_{k,m} \mid$.

24. Show that for $m \geq 3$, the null space $\mathcal{N}(\mathfrak{B})$ of the matrix $\mathfrak{B} = (b_{k-j,m})_{k,j \in \mathbb{Z}}$ in the vector space X of all bi-infinite vectors $(d_j)_{j \in \mathbb{Z}}$ is of the dimension 2μ being spanned by $(z_\nu^j)_{j \in \mathbb{Z}}$, $\nu = 1, ..., 2\mu$, where z_ν, $\nu = 1, ..., 2\mu$, are the characteristic roots.

25. Present a detailed proof of (8.11).

26. Prove estimate (8.16) for $m = 1$ and for $m = 2$.

27. Prove that the inclusion $W^{m,\infty}(\mathbb{R}) + \mathcal{P}_m \subset V^{m,\infty}(\mathbb{R})$ is strict for $m = 2$.

28. Let f be periodic with a period $p = nh$, $n \in \mathbb{N}$. Prove that then $Q_{h,m} f$ is also p-periodic. Further, prove that $Q_{h,m} f = f$ for $f(x) \equiv 1$; this claim was exploited in the proof of estimate (8.16), so use the formula for $Q_{h,m} f$ and do not use (8.16) or (8.17).

29. Prove claims 1 and 4 formulated in the part (ii) of the proof of Theorem 8.2.

30. Prove that condition (8.18) implies (8.19).

31. Present a detailed proof of Remark 8.5.

32. Let $h = 1/n$ where $n \in \mathbb{N}$ is even. Determine $\| E_{h,m} \|_{L^2(0,1)}$ and use (8.16) in order to estimate $\| f - Q_{h,m} f \|_{L^2(0,1)}$ for $f \in W_{\text{per}}^{\infty,m}(\mathbb{R})$.

33. By Definition 8.1, the functions $B_{j,m}(x) := B_m(x-j)$, $j = 0,...,m-1$, are polynomials of degree $m-1$ on $[m-1,m]$. Prove that $\text{span}\{B_{j,m} : j = 0,...,m-1\} = \mathcal{P}_{m-1}$ and hence $\{B_{j,m} : j = 0,...,m-1\}$ are linearly independent on $[m-1,m]$. Observe a consequence: if $\sum_{j\in\mathbb{Z}} d_j B_m(h^{-1}x-j) \equiv 0$ on $[ih,(i+1)h]$ then $d_j = 0$ for $j = i-m+1,...,i$. *Hint*: Remark 8.1.

34. Let $h = 1/n$, $n \in \mathbb{N}$. Prove that the functions $B_m(nx-i)$, $i = -m+1,...,n-1$, are linearly independent on the interval $[0,1]$. Nevertheless, already for moderate and especially for great m, one must be careful using this basis: compare the maximums of $B_m(nx-i)$ on $[0,1]$ for extreme $i = -m+1$ and $i = n-1$ with the maximum of $B_m(nx-i)$ for central $i = 0,...,n-m-1$.

35. Prove that

$$(1/\alpha_m)\sup_{j\in\mathbb{Z}} \mid c_j \mid \leq \sup_{x\in\mathbb{R}} \mid \sum_{j\in\mathbb{Z}} c_j B(h^{-1}x-j) \mid \leq \sup_{j\in\mathbb{Z}} \mid c_j \mid,$$

$$\beta_m \left(h \sum_{j\in\mathbb{Z}} \mid c_j \mid^2 \right)^{\frac{1}{2}} \leq \left(\int_{\mathbb{R}} \mid \sum_{j\in\mathbb{Z}} c_j B(h^{-1}x-j) \mid^2 dx \right)^{\frac{1}{2}} \leq \left(h \sum_{j\in\mathbb{Z}} \mid c_j \mid^2 \right)^{\frac{1}{2}}$$

where $\alpha_m = \sum_{k\in\mathbb{Z}} \mid a_{k,m} \mid$ and β_m is a positive constant depending only on m; estimate it.

36. Prove that B_m satisfies the relation $B_m(x) = \sum_{j=0}^{m} \beta_{j,m} B_m(2x-j)$, $x \in \mathbb{R}$, where $\beta_{j,m} = \sum_k a_{j-k,m} B_m(\frac{1}{2}(k+\frac{m}{2}))$. *Hints*: $B_m = Q_{\frac{1}{2},m} B_m$, Exercise 33.

37. Present a detailed proof of (8.28).

38. Prove that $\sum_{j\in\mathbb{Z}} (J^{\pm}\underline{a})_j = 0$ and $J\underline{a} \in \mathfrak{s}_{\text{sym}}(\mathbb{Z})$ for $\underline{a} \in \mathfrak{s}_{\text{sym}}(\mathbb{Z})$.

39. Prove that $f_k + \sum_{q=1}^{p-1} \gamma_q D^q f_k = \sum_{|j|\leq p-1} a_j^{(p)} f_{k-j}$, see (8.32), (8.35).

40.Establish for $m = 3$ (quadratic splines), $x \in [ih,(i+1)h]$, $i \in \mathbb{Z}$, the formula

$$(Q_{h,3}^{(1)} f)(x) = \sum_{k=i-2}^{i} \left(-\frac{1}{8} f((k+\frac{1}{2}))h + \frac{5}{4} f((k+\frac{3}{2})h) - \frac{1}{8} f((k+\frac{5}{2})h \right) B_3(nx-k).$$

41. Establish for $m = 4$ (cubic splines), $ih \leq x \leq (i+1)h$, $i \in \mathbb{Z}$, the formula

$$(Q'_{h,4} f)(x)$$

$$= \sum_{j=i-3}^{i} \left\{ \frac{3}{2} f((j+2)h) - \frac{5}{18}[f((j+1)h) + f((j+3)h)] + \frac{1}{36}[f(jh) + f((j+4)h)] \right\}$$

$$\cdot B_4(h^{-1}x-j) \cdot$$

42. Prove (8.44).

43*. Establish the counterpart of Theorem 9.1 for the collocation method

$$v_n = Q_{h,m} E_\delta T_\varphi v_n + Q_{h,m} E_\delta f_\varphi.$$

Present the matrix form of the method. Examine suitable full discretizations of the collocation and quasi-collocation methods and present two grid iterations to solve the systems trying to restrict all the computations to $O(n^2)$ flops; of course, the accuracy $O(h^m)$ should be maintained by the approximate solution. Solve numerical examples an comment on them.

44*. Establish a counterpart of Theorem 9.2 for the the quasi-collocation method and present the matrix form of the method. Examine suitable full discretizations of the collocation and quasi-collocation methods for the periodised problem and present two grid iterations to solve the systems trying to restrict all the computations to $O(n^2)$ flops; the accuracy $O(h^m)$ should be maintained by the approximate solution. Solve numerical examples and comment on them.

Exercises labelled by $\star$ propose possible topics for master or doctor theses.

Comments and bibliographical remarks.

With proofs, the results of Sections 2.1–2.4, except Theorem 2.8, can be found in any text book on functional analysis, see e.g. in [7], [10] or [13]. The proof of Theorem 2.8, in its full extent, is based on the Fredholm theory for compact operators, see [37] for details. Strange enought, the Faa di Bruno's differentiation formula (Theorem 2.9) is not included into standard text books on calculus although its proof by induction is instructive and simple.

The study of the singularities of a solution of weakly singular integral equations has a long history, see e.g. [1–3], [16], [17], [28] [36,37] and the literature quoted there; the results of Section 5 can be extended to multidimensional weakly singular integral equations, see [17], [28]. In the last time, the smoothness/singularity results have been extended to integral equations of the type

$$u(x) = \int_0^1 K(x,y) y^{-\lambda}(1-y)^{-\mu} u(y) dy + f(x)$$

where $K \in \mathcal{S}^{m,\nu}$, $m \geq 1$, $\nu < 1$, $\nu + \lambda < 1$, $\nu + \mu < 1$. It occurs that the boundary singularities $y^{-\lambda}(1-y)^{-\mu}$ by the kernel shift the solutions from $C^{m,\nu}(0,1)$ into the space $C^{m,\nu+\lambda,\nu+\mu}(0,1)$ of functions that have the singularities of the type $C^{m,\nu+\lambda}$ in a vicinity of 0 and of the type $C^{m,\nu+\mu}$ in a vicinity of 1. See [16] for precise (and more general) formulations.

There is a varity of literature on the numerical solution of integral equations, including weakly singular ones, see in particular, [1–3], [6], [7], [9], [19], [23], [26–28]. In the practice, large discretised problems can be effectively solved by iteration methods such as two grid iterations, GMRES and conjugate gradients. For the algorithms and justificatin of those, see, in addition to previous items, also [12], [18], [25], [29] .

Piecewise polynomial collocation method can be applied to integral equation (5.1) directly, without a smoothing transformation. The optimal convergence order $O(n^{-m})$ can be achieved by using a suitable graded grid of the type

$$x_i = \frac{1}{2}\left(\frac{i}{n}\right)^r, \quad i = 0, ..., n, \quad x_{n+i} = 1 - x_{n-i}\,, \quad i = 1, ..., n,$$

where $r \geq 1$ is the grading parameter. For $r = 1$ the grid is uniform; for greater r the grid points x_i are more densely located near the end points of the interval $[0,1]$. On every subinterval $[x_i, x_{i+1}]$, $i = 0, ..., 2n-1$, take m interpolation points $\xi_{i,l} = x_i + b_l(x_{i+1} - x_i)$, $l = 1, ..., m$, where $0 \leq b_1 < ... < b_m \leq 1$ are parameters that are independent of i and n. Using these interpolation points we can build a polynomial interpolant of degree $m-1$ of a given function $f \in C[0,1]$ on every interval $[x_i, x_{i+1}]$, $i = 0, ..., 2n-1$, independently and compose from those partial interpolants a piecewise polynomial function on $[0,1]$ that we denote by $Q_n f$. It occurs that for $f \in C^{m,\nu}(0,1)$ and sufficiently large $r = r(m,\nu)$ described in [28],[37], $\| f - Q_n f \|_\infty \leq cn^{-m} \| w_{m+\nu-1} f^{(m)} \|_\infty$. Assuming that $f \in C^{m,\nu}(0,1)$, $K \in \mathcal{S}^{m,\nu}$ and $\mathcal{N}(I-T) = \{0\}$, the collocation method

$$u_n = Q_n T u_n + Q_n f$$

applied to equation (5.1) converges with the optimal accuracy order

$$\| u - u_n \|_\infty \leq cn^{-m} \| w_{m+\nu-1} u^{(m)} \|_\infty .$$

In [14,15], this method is combined with the smoothing change of variables to reduce the restriction on the grading parameter r. In particular, the uniform grid ($r = 1$) can be used setting suitable conditions on φ. The collocation method introdoced and examined in Section 7 of the present lecture notes is different. This method similarly as the two methods of Section 9 seem to be new. The periodization of the problem allows to use not only periodic splines (as in Section 9.3) but also trigonometric or wavelet trial functions, cf. [19].

The spline interpolation problem has been found much attention in the literature, see, in particular, the monographs [5], [8], [21–23], [39]. Usually the interpolation problem is formulated for an interval, say, for $[0,1]$, but we followed [23] where the treatment of interpolation starts from $\mathbb{R}$. For us the interpolation on $\mathbb{R}$ is comfortable since, due to boundary conditions (5.20) satisfied by the solution of the transformed integral equation (5.15) on $[0, 1]$, we have a simple way to extend the solution onto $\mathbb{R}$ maintaining the C^m-smoothness. Our idea to use the Wiener theorem for the construction of the (Wiener) interpolant is equivalent to the idea of constructing the interpolant with a bounded derivative of order $m-1$ exploited in [23]. The central result of Section 8 – Theorem 8.2 – has been established for 1-periodic functions and $h = 1/n$ with even n by Korneychuk [8] in 1984 and recently extended by Vainikko [31,33] to the full extent in the process of lecturing in Helsinki and Tartu; consequences for the quasi-interpolation are published in [11]. In the literature, the spline quasi-interpolants have been usually introduced through the condition that they reproduce the polynomials of degree $\leq m-1$, without any connection to the real interpolant, see [5], [22], [39]. In [22], the quasi-interpolants are systematically exploited to estimate, in a varity of norms, the distance of a given function from the subspace of splines. This approach leads to optimal convergence orders but the constants in estimates remain undetermined or are rather coarse, for instance, the estimate $\| Q^\star_{n,m} \|_{C\to C} \leq (2m)^m$ is established for the quasi-interpolation operators described in [22]. Our treatment of quasi-interpolants based on the difference representation of the Wiener solution is different and has the advantage that we obtain simple closed formulae for the quasi-interpolants of any approximation degree, and at least for $m \leq 20$ the norms of the interpolation and quasi-interpolation operators are quite acceptable to be sure that the numerical schemes are stable with respect to truncation and rounding errors.

The problem of a full discretization of the collocation schemes and of a fast solution of the collocation systems remained untouched in these lectures. In the section Exercises and Problems we formulated some problems to construnt fully discrete schemes of the optimal accuracy order and complexity $O(n^2)$ flops for their implementation. Similarly as in [30,32,34,35,38] for the case of smooth kernels without singularities, it is a **challenging problem** how to reduce the arithmetical work to $O(n)$ flops maintaining the optimal accuracy $\| v - v_n \|_\infty \leq ch^m \| v^{(m)} \|_\infty$ of the approximate solution under the assumptions that $f \in C^{m,\nu}(0,1)$, $K \in \mathcal{S}^{2m,\nu}$, $m \geq 1$, $\nu < 1$ and $\mathcal{N}(I - T) = \{0\}$. Actually a radical **open complexity problem** is whether (or under which additional conditions on the kernel) this is possible at all; in analogy to the case of smooth kernels, we already strengthened the condition $K \in \mathcal{S}^{m,\nu}$ up to $K \in \mathcal{S}^{2m,\nu}$ but it is not clear whether this is enought.

Fast solution is supremely important in the case of **multidimensinal** integral equations.

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