Viscoelastic Dynamics for a Scalar Two Dimensional Continuum Model for Microstructure creation at an Austenite-Martensite Interface

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1. Examples of microstructure predicted by static energy minimization

Kohn and Müller. 1992

KEY COMMENT: Not clear if reasonable dynamics can lead to refining microstructure

2. Dynamical equations

\[
\begin{align*}
\rho \frac{\partial u}{\partial t} &= -\beta u_{xx} + \gamma (u_{xx} - u_{yy}) + \alpha u_{xxx} \\
\rho \frac{\partial u}{\partial t} &= -\beta u_{xx} + \gamma (u_{xx} - u_{yy}) + \alpha u_{xxx}
\end{align*}
\]

acoustic velocity due to internal strain energy dissipation

Dissipative systems with Lyapunov function

\[
L[u] = \frac{\beta}{4} u_{xx}^2 + \frac{\gamma}{4} (u_{xx} - u_{yy})^2 + \frac{\alpha}{4} u_{xxx}^2
\]

3. Chebyshev Spectral Integration vs. Spectral Differentiation

Linear boundary value problem: Differentiation Formulation

\[
Au_{xx} + Bu_{xx} + Cu = f(x)
\]

Linear boundary value problem: Integration Formulation

\[
Au + Bu + Cu + f(x) = 0
\]

Sparsity pattern for Chebyshev Integration matrix

Example showing stable integration and unstable differentiation

\[
\begin{align*}
u_{xx} + 2u_{xx} + u &= \cos(x) \\
\alpha(-1) &= \alpha(1) = 0 \\
\alpha(i) &= \alpha(i+1) = 0
\end{align*}
\]

KEY COMMENT: Need to solve the highest order linear term implicitly to use integral formulation

4. Time stepping and temporal convergence

Second order accurate scheme with discretized Lyapunov function

Backward differentiation for linear terms and extrapolation for nonlinear terms

Time stepping scheme for eq. (1)

\[
\begin{align*}
u_{xx}^{n+1} &= \frac{-\beta u_{xx}^n + \gamma (u_{xx}^n - u_{yy}^n) + \alpha u_{xxx}^n}{\rho} \\
\frac{1}{2} \beta u_{xx}^{n+1} &= \frac{4}{3} \nu_{xx}^{n+1} + \nu_{xx}^{n+1} - \nu_{xx}^{n+1} + \frac{1}{3} \nu_{xx}^{n+1}
\end{align*}
\]

Relation to linear boundary value problem

\[
Au_{xx} + Bu_{xx} + Cu = f(x)
\]

Convergence to a traveling wave solution

KEY COMMENT: Final steady state is dependent upon initial iterate

5. Microstructure formation in one dimension

Boundary Conditions for simulation of eq. (1)

\[
\begin{align*}
u(x, y = 1) &= 0, \quad \nu(x, y = -1) = 0 \\
\alpha_{xx}(x, y = 1) &= 0, \quad \alpha_{xx}(x, y = -1) = 0
\end{align*}
\]

Parameters for simulation of eq. (1)

\[
\begin{align*}
\rho &= 1, & \beta &= 1, & \epsilon &= 10^{-4}, & \alpha &= 100
\end{align*}
\]

Zero initial velocity, 1024 modes, timestep of \((10^{-4})

6. Microstructure formation in two dimensions

Boundary Conditions for simulation of eq. (2)

\[
\begin{align*}
u(x, y = 1) &= 0, \quad \nu(x, y = -1) = 0 \\
\alpha_{xx}(x, y = 1) &= 0, \quad \alpha_{xx}(x, y = -1) = 0
\end{align*}
\]

Parameters for simulation of eq. (2)

\[
\begin{align*}
\rho &= 1, & \beta &= 0.5, & \epsilon &= 2 \times 10^{-6}
\end{align*}
\]

256 modes in x, 128 modes in y, timestep of 0.01/25 Zero initial velocity

7. Conclusions

• A method to obtain high resolution numerical solutions to semi-linear initial boundary value problems
• By using integration formulation can obtain high order derivatives without numerical instabilities
• See beginnings of self-similar refinement at boundary in two-dimensions
• Can apply a wide variety of boundary conditions

8. Further Work

• Implement deferred correction time stepping to get high order solutions in time
• See if hysteresis occurs during cyclic loading
• Extend to 2D vectorial models for nonlinear elastodynamics

9. Acknowledgments

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References