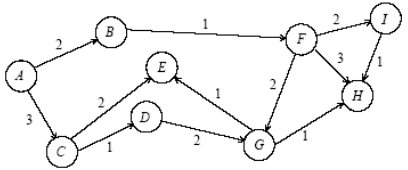
Matrix-Multiplication Based Algorithm

- Consider the multiplication of the weighted adjacency matrix with itself - except, in this case, we replace the multiplication operation in matrix multiplication by addition, and the addition operation by minimization
- Notice that the product of weighted adjacency matrix with itself returns a matrix that contains shortest paths of length 2 between any pair of nodes
- It follows from this argument that A^n contains all shortest paths
- A^n is computed by doubling powers - i.e., as A, A^2, A^4, A^8, ...



$$A^1 = \begin{pmatrix} 0 & 2 & 3 & \infty & \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty & 1 & \infty & \infty & \infty \\ \infty & \infty & 0 & 1 & 2 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & 0 & \infty & \infty & 2 & \infty & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix} \quad A^2 = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 3 & \infty & \infty & \infty \\ \infty & 0 & \infty & \infty & \infty & 1 & 3 & 4 & 3 \\ \infty & \infty & 0 & 1 & 2 & \infty & 3 & \infty & \infty \\ \infty & \infty & \infty & 0 & 3 & \infty & 2 & 3 & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 3 & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$

$$A^4 = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 3 & 5 & 6 & 5 \\ \infty & 0 & \infty & \infty & 4 & 1 & 3 & 4 & 3 \\ \infty & \infty & 0 & 1 & 2 & \infty & 3 & 4 & \infty \\ \infty & \infty & \infty & 0 & 3 & \infty & 2 & 3 & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 3 & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix} \quad A^8 = \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 3 & 5 & 6 & 5 \\ \infty & 0 & \infty & \infty & 4 & 1 & 3 & 4 & 3 \\ \infty & \infty & 0 & 1 & 2 & \infty & 3 & 4 & \infty \\ \infty & \infty & \infty & 0 & 3 & \infty & 2 & 3 & \infty \\ \infty & \infty & \infty & \infty & 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & 3 & 0 & 2 & 3 & 2 \\ \infty & \infty & \infty & \infty & 1 & \infty & 0 & 1 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 0 & \infty \\ \infty & \infty & \infty & \infty & \infty & \infty & \infty & 1 & 0 \end{pmatrix}$$

- We need $\log n$ matrix multiplications, each taking time $O(n^3)$.
- The serial complexity of this procedure is $O(n^3 \log n)$.
- This algorithm is not optimal, since the best known algorithms have complexity $O(n^3)$.

Parallel formulation

- Each of the $\log n$ matrix multiplications can be performed in parallel.
- We can use $n^3 / \log n$ processors to compute each matrix-matrix product in time $\log n$.
- The entire process takes $O(\log^2 n)$ time.

Dijkstra's Algorithm

- Execute n instances of the single-source shortest path problem, one for each of the n source vertices.
- Complexity is $O(n^3)$.

Parallel formulation

Two parallelization strategies - execute each of the n shortest path problems on a different processor (***source partitioned***), or use a parallel formulation of the shortest path problem to increase concurrency (***source parallel***).

Dijkstra's Algorithm: Source Partitioned Formulation

- Use n processors, each processor P_i finds the shortest paths from vertex v_i to all other vertices by executing Dijkstra's sequential single-source shortest paths algorithm.
- It requires no interprocess communication (provided that the adjacency matrix is replicated at all processes).

- The parallel run time of this formulation is: $\Theta(n^2)$.
- While the algorithm is cost optimal, it can only use n processors. Therefore, the isoefficiency due to concurrency is $\Theta(p^3)$.

Dijkstra's Algorithm: Source Parallel Formulation

- In this case, each of the shortest path problems is further executed in parallel. We can therefore use up to n^2 processors.
- Given p processors ($p > n$), each single source shortest path problem is executed by p/n processors.
- Using previous results, this takes time:

$$T_p = \overbrace{\Theta\left(\frac{n^3}{p}\right)}^{\text{computation}} + \overbrace{\Theta\left(n \log p\right)}^{\text{communication}}$$

- For cost optimality, we have $p = O(n^2/\log n)$ and the isoefficiency is $\Theta((p\log p)^{1.5})$.

Floyd's Algorithm

- Let $G = (V, E, w)$ be the weighted graph with vertices $V = \{v_1, v_2, ..., v_n\}$.
- For any pair of vertices $v_i, v_j \in V$, consider all paths from v_i to v_j whose intermediate vertices belong to the subset $\{v_1, v_2, \ldots, v_k\}$ ($k \leq n$). Let $p_{i,j}^{(k)}$ (of weight $d_{i,j}^{(k)}$) be the minimum-weight path among them.
- If vertex v_k is not in the shortest path from v_i to v_j, then $p_{i,j}^{(k)}$ is the same as $p_{i,j}^{(k-1)}$.
- If v_k is in $p_{i,j}^{(k)}$, then we can break $p_{i,j}^{(k)}$ into two paths - one from v_i to v_k and one from v_k to v_j. Each of these paths uses vertices from $\{v_1, v_2, \ldots, v_{k-1}\}$.

From our observations, the following recurrence relation follows:

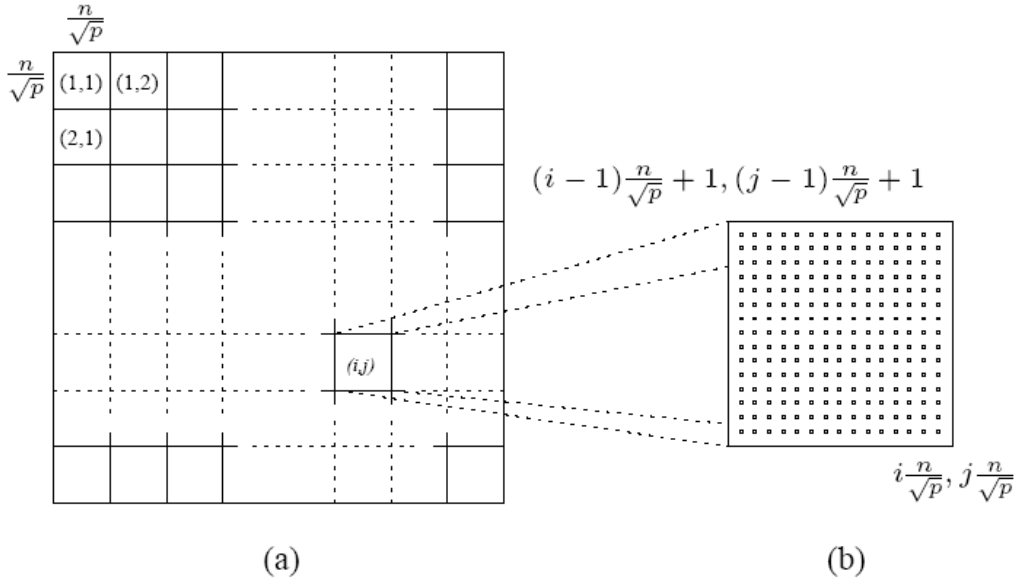
$$d_{i,j}^{(k)} = \begin{cases} w(v_i, v_j) & \text{if } k = 0 \\ \min\left\{d_{i,j}^{(k-1)}, d_{i,k}^{(k-1)} + d_{k,j}^{(k-1)}\right\} & \text{if } k \geq 1 \end{cases}$$

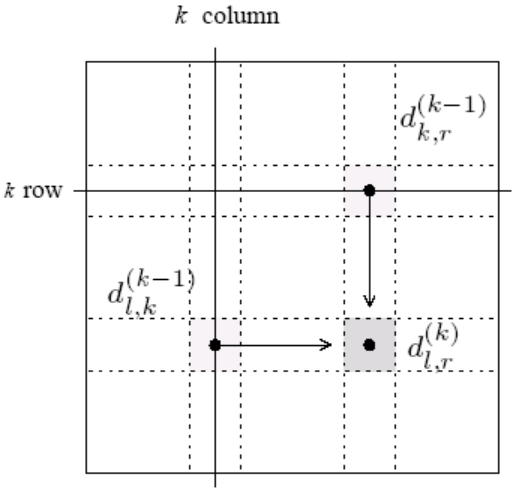
This equation must be computed for each pair of nodes and for $k = 1, n$. The serial complexity is $O(n^3)$.

```
procedure FLOYD_ALL_PAIRS_SP(A)
begin
  D^0 = A;
  for k := 1 to n do
    for i := 1 to n do
      for j := 1 to n do
        d_{i,j}^{(k)} := min(d_{i,j}^{(k-1)}, d_{i,k}^{(k-1)} + d_{k,j}^{(k-1)});
end FLOYD_ALL_PAIRS_SP
```

Parallel formulation: 2D Block Mapping

- Matrix $D^{(k)}$ is divided into p blocks of size $(n/\sqrt{p}) \times (n/\sqrt{p})$.
- Each processor updates its part of the matrix during each iteration.
- To compute $d_{l,r}^{(k-1)}$ processor $P_{i,j}$ must get $d_{l,k}^{(k-1)}$ and $d_{k,r}^{(k-1)}$.
- In general, during the k^{th} iteration, each of the $\sqrt{p})$ processes containing part of the k^{th} row send it to the $\sqrt{p}-1$ processes in the same column.
- Similarly, each of the $\sqrt{p}$ processes containing part of the k^{th} column sends it to the $\sqrt{p}-1$ processes in the same row.


$\frac{n}{\sqrt{p}}$
$\frac{n}{\sqrt{p}}$
(1,1)
(1,2)
(2,1)
(i,j)
$(i-1)\frac{n}{\sqrt{p}}+1, (j-1)\frac{n}{\sqrt{p}}+1$
$i\frac{n}{\sqrt{p}}, j\frac{n}{\sqrt{p}}$
(a)
(b)


k column
k row
$d_{k,r}^{(k-1)}$
$d_{l,k}^{(k-1)}$
$d_{l,r}^{(k)}$
(a)
k column
(b)

```
procedure FLOYD_2DBLOCK(D^(0))
begin
  for k := 1 to n do
    begin
      each process P_{i,j} that has a segment of the k^th row of D^(k-1)
        broadcasts it to the P_{*,j} processes;
      each process P_{i,j} that has a segment of the k^th column of D^(k-1)
        broadcasts it to the P_{i,*} processes;
      each process waits to receive the needed segments;
      each process P_{i,j} computes its part of the D^(k) matrix;
    end
end FLOYD_2DBLOCK
```

- During each iteration of the algorithm, the k^{th} row and k^{th} column of processors perform a one-to-all broadcast along their rows/columns.
- The size of this broadcast is $n/\sqrt{p}$ elements, taking time $\Theta((n \log p)/\sqrt{p})$.
- The synchronization step takes time $\Theta(\log p)$.

- The computation time is $\Theta(n^2/p)$.
- The parallel run time of the 2-D block mapping formulation of Floyd's algorithm is

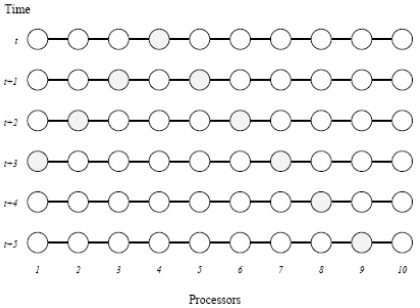
$$T_p = \overbrace{\Theta\left(\frac{n^3}{p}\right)}^{\text{computation}} + \overbrace{\Theta\left(\frac{n^2}{\sqrt{p}}\log p\right)}^{\text{communication}}$$

- The above formulation can use $O(n^2/\log^2 n)$ processors cost-optimally.
- The isoefficiency of this formulation is $\Theta(p^{1.5}\log^3 p)$.
- This algorithm can be further improved by relaxing the strict synchronization after each iteration.

Speeding things up by pipelining

- The synchronization step in parallel Floyd's algorithm can be removed without affecting the correctness of the algorithm.

- A process starts working on the k^{th} iteration as soon as it has computed the $k-1^{\text{th}}$ iteration and has the relevant parts of the $D^{(k-1)}$ matrix.



Communication protocol followed in the pipelined 2-D block mapping formulation of Floyd's algorithm. Assume that process 4 at time t has just computed a segment of the k^{th} column of the $D^{(k-1)}$ matrix. It sends the segment to processes 3 and 5. These processes receive the segment at time $t+1$ (where the time unit is the time it takes for a matrix segment to travel over the communication link between adjacent processes). Similarly, processes farther away from process 4 receive the segment later. Process 1 (at the boundary) does not forward the segment after receiving it.

- In each step, $n/\sqrt{p}$ elements of the first row are sent from process $P_{i,j}$ to $P_{i+1,j}$.

- Similarly, elements of the first column are sent from process $P_{i,j}$ to process $P_{i,j+1}$.
- Each such step takes time $\Theta(n/\sqrt{p})$.
- After $\Theta(\sqrt{p})$ steps, process $P_{\sqrt{p},\sqrt{p}}$ gets the relevant elements of the first row and first column in time $\Theta(n)$.
- The values of successive rows and columns follow after time $\Theta(n^2/p)$ in a pipelined mode.
- Process $P_{\sqrt{p},\sqrt{p}}$ finishes its share of the shortest path computation in time $\Theta(n^3/p) + \Theta(n)$.
- When process $P_{\sqrt{p},\sqrt{p}}$ has finished the $(n-1)^{\text{th}}$ iteration, it sends the relevant values of the n^{th} row and column to the other processes.
- The overall parallel run time of this formulation is

$$T_p = \overbrace{\Theta\left(\frac{n^3}{p}\right)}^{\text{computation}} + \overbrace{\Theta(n)}^{\text{communication}}$$

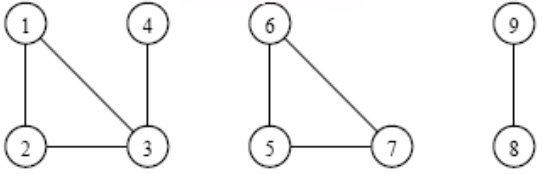
- The pipelined formulation of Floyd's algorithm uses up to $O(n^2)$ processes efficiently.
- The corresponding isoefficiency is $\Theta(p^{1.5})$.

All-pairs Shortest Path: Comparison

	Maximum Number of Processes for $E = \Theta(1)$	Corresponding Parallel Run Time	Isoefficiency Function
Dijkstra source-partitioned	$\Theta(n)$	$\Theta(n^2)$	$\Theta(p^3)$
Dijkstra source-parallel	$\Theta(n^2/\log n)$	$\Theta(n \log n)$	$\Theta((p \log p)^{1.5})$
Floyd 1-D block	$\Theta(n/\log n)$	$\Theta(n^2 \log n)$	$\Theta((p \log p)^3)$
Floyd 2-D block	$\Theta(n^2/\log^2 n)$	$\Theta(n \log^2 n)$	$\Theta(p^{1.5} \log^3 p)$
Floyd pipelined 2-D block	$\Theta(n^2)$	$\Theta(n)$	$\Theta(p^{1.5})$

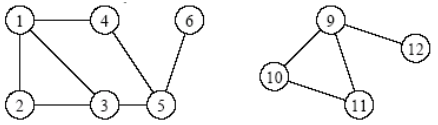
12.5 Connected Components

- The connected components of an undirected graph are the equivalence classes of vertices under the "is reachable from" relation
- A graph with three connected components: {1,2,3,4}, {5,6,7}, and {8,9}:

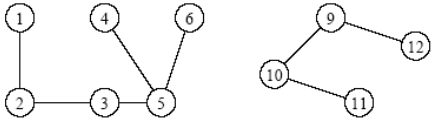


Depth-First Search (DFS) Based Algorithm

- Perform DFS on the graph to get a forest - each tree in the forest corresponds to a separate connected component
- Part (b) is a depth-first forest obtained from depth-first traversal of the graph in part (a). Each of these trees is a connected component of the graph in part (a):



(a)

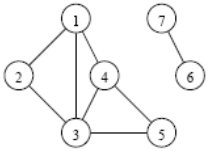


(b)

Parallel Formulation

- Partition the graph across processors and run independent connected component algorithms on each processor. At this point, we have p spanning forests.

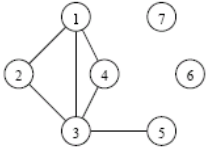
- In the second step, spanning forests are merged pairwise until only one spanning forest remains.



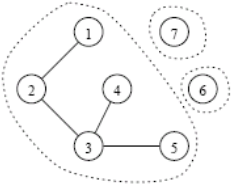
(a)

	1	2	3	4	5	6	7	
1	0	1	1	1	0	0	0	
2	1	0	1	0	0	0	0	Processor 1
3	1	1	0	1	1	0	0	
4	1	0	1	0	1	0	0	
5	0	0	1	1	0	0	0	
6	0	0	0	0	0	0	1	Processor 2
7	0	0	0	0	0	1	0	

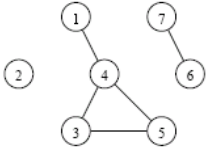
(b)



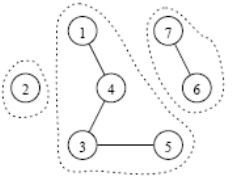
(c)



(d)



(e)



(f)

Computing connected components in parallel:

The adjacency matrix of the graph G in (a) is partitioned into two parts (b).

Each process gets a subgraph of G ((c) and (e)).

Each process then computes the spanning forest of the subgraph ((d) and (f)).

Finally, the two spanning trees are merged to form the solution.

- To merge pairs of spanning forests efficiently, the algorithm uses disjoint sets of edges.
- We define the following operations on the disjoint sets:
- **find(x)**
  - returns a pointer to the representative element of the set containing x . Each set has its own unique representative.
- **union(x, y)**
  - unites the sets containing the elements x and y. The two sets are assumed to be disjoint prior to the operation.
- For merging forest A into forest B, for each edge (u, v) of A, a find operation is performed to determine if the vertices are in the same tree of B.
- If not, then the two trees (sets) of B containing u and v are united by a union operation.

- Otherwise, no union operation is necessary.
- Hence, merging A and B requires at most $2(n-1)$ find operations and $(n-1)$ union operations.

Parallel 1-D Block Mapping

- The $n \times n$ adjacency matrix is partitioned into p blocks.
- Each processor can compute its local spanning forest in time $\Theta(n^2/p)$.
- Merging is done by embedding a logical tree into the topology. There are $\log p$ merging stages, and each takes time $\Theta(n)$. Thus, the cost due to merging is $\Theta(n \log p)$.
- During each merging stage, spanning forests are sent between nearest neighbors. Recall that $\Theta(n)$ edges of the spanning forest are transmitted.

- The parallel run time of the connected-component algorithm is

$$T_p = \overbrace{\Theta\left(\frac{n^2}{p}\right)}^{\text{local computation}} + \overbrace{\Theta\left(n \log p\right)}^{\text{forest merging}}$$

- For a cost-optimal formulation $p = O(n/\log n)$. The corresponding isoefficiency is $\Theta(p^2 \log^2 p)$.