A SPLINE PRODUCT QUASI-INTERPOLATION METHOD FOR WEAKLY SINGULAR FREDHOLM INTEGRAL EQUATIONS*

EERO VAINIKKO AND GENNADI VAINIKKO†

Abstract. A discrete method of accuracy $O(h^m)$ is constructed and justified for a class of Fredholm integral equations of the second kind with kernels that may have weak diagonal and boundary singularities. The method is based on improving the boundary behaviour of the kernel with the help of a change of variables, and on the product integration using quasi-interpolation by smooth splines of order m.

Key words. Weakly singular integral equations, boundary singularities, spline quasi-interpolation, spline interpolation, product integration methods, Nyström type methods

AMS subject classifications. 65R20, 65D07, 65D30

1. Introduction. In the present paper we call attention to some high order fully discrete methods for a class of Fredholm integral equations of the second kind with different weak singularities. The idea of the method is close to those in [2], [4], [9]. Consider the weakly singular integral equation

$$u(x) = \int_0^1 \left(a(x,y)|x-y|^{-\nu} + b(x,y)\right) u(y)dy + f(x), \ \ 0 \le x \le 1, \tag{1.1}$$

depending on the parameter ν, $0 < \nu < 1$. A traditional product integration method (see [2], [4] and the literature quoted there) is based on the approximation of (1.1) by the equation

$$u(x) = \int_0^1 \left(|x-y|^{-\nu} P_{n,m}(a(x,y)u(y)) + P_{n,m}(b(x,y)u(y))\right) dy + f(x), \ \ 0 \le x \le 1,$$

where $P_{n,m}$ is an interpolation projector onto the space of piecewise polynomial functions of degree $m-1$ associated with a subdivision of $[0,1]$ into n subintervals by some knots $0 = x_{0,n} < x_{1,n} < ... < x_{n,n} = 1$; $P_{n,m}$ is applied to the products $a(x,y)u(y)$ and $b(x,y)u(y)$ as functions of y treating x as a parameter. This results to some Nyström type methods of accuracy $O(n^{-m})$ provided that the grid $\{x_{0,n}, x_{1,n}, ..., x_{n,n}\}$ is properly graded to compensate the generic boundary singularities of the derivatives of the exact solution to (1.1); even the accuracy $O(n^{-m-1+\nu})$ can be achieved for a skilled choice of the interpolation points, see [13]. The method of work [9] is different: the authors first perform a change of variables which improves the boundary behaviour of the solution and of the coefficient functions $a(x,y)$, $b(x,y)$, and after that, using a polynomial approximation of the solution, they apply a Gauss type quadrature formula for the integral in the transformed equation.

In our method we follow [9], [10] performing a smoothing change of variables and then, instead of discontinuous piecewise polynomial interpolation by $P_{n,m}$, we use a product quasi-interpolation (or even the real interpolation) by smooth splines of degree $m-1$ on the uniform grid of the step size $h = 1/n$. This leads to some reduction of degrees of freedom compared with the use of piecewise polynomials. Our algorithm still has features of the Nyström method. In Nyström type methods for weakly singular integral equations, the computation of the quadrature coefficients

*This work was supported by the Estonian Information Technology Foundation, grant No 07-03-01-01, and by the Estonian Science Foundation, grant No 7353.

†Faculty of Mathematics and Computer Science, University of Tartu, Liivi 2, Tartu 50409, Estonia (eero.vainikko@ut.ee and gennadi.vainikko@ut.ee)

is a laborious and delicate part of the algorithms. The use of quasi-interpolation (or real interpolation) by smooth splines on uniform grids enables us to obtain simple but still somewhat delicate formulae for the quadrature coefficients, and so we obtain a fully discrete method of accuracy $O(h^m)$. This is the main message of the present paper.

In our considerations, the coefficient functions $a(x,y)$ and $b(x,y)$ may have boundary singularities with respect to y. Denote by T the integral operator of equation (1.1),

$$(Tu)(x) = \int_0^1 \left(a(x,y)|x-y|^{-\nu} + b(x,y)\right) u(y)dy.$$

LEMMA 1.1. *Assume that $a, b \in C([0,1]\times(0,1))$ satisfy the inequalities*

$$|a(x,y)| \le cy^{-\lambda_0}(1-y)^{-\lambda_1},\ |b(x,y)| \le cy^{-\mu_0}(1-y)^{-\mu_1},\ (x,y)\in[0,1]\times(0,1),$$

where $0<\nu<1$, $\lambda_0,\lambda_1,\mu_0,\mu_1\in\mathbb{R}$, $\lambda_0+\nu<1$, $\lambda_1+\nu<1$, $\mu_0<1$, $\mu_1<1$. Then T maps $C[0,1]$ into $C[0,1]$, and $T: C[0,1]\to C[0,1]$ is compact.

The proof is standard, cf. [5], [7]; a detailed argument can be found in [11].

For $m\in\mathbb{N}$, $\theta_0,\theta_1\in\mathbb{R}$, $\theta_0,\theta_1<1$, denote by $C^m_\star(0,1)$ and $C^{m,\theta_0,\theta_1}(0,1)$ the weighted spaces of functions $u\in C[0,1]\cap C^m(0,1)$ such that, respectively,

$$\| u \|_{C^m_\star(0,1)} := \sum_{k=0}^m \sup_{0<x<1} x^k(1-x)^k|u^{(k)}(x)| < \infty,$$

$$\| u \|_{C^{m,\theta_0,\theta_1}(0,1)} := \sum_{k=0}^m \sup_{0<x<1} w_{k-1+\theta_0}(x)w_{k-1+\theta_1}(1-x)|u^{(k)}(x)| < \infty$$

where

$$w_\rho(r) = \begin{cases} 1, & \rho<0, \\ 1/(1+|\log r|), & \rho=0, \\ r^\rho, & \rho>0, \end{cases} \quad r,\rho\in\mathbb{R},\ r>0.$$

Clearly, $C^m[0,1]\subset C^{m,\theta_0,\theta_1}(0,1)\subset C^m_\star(0,1)$.

LEMMA 1.2. ([11]). *Let $a,b\in C^m([0,1]\times(0,1))$ and*

$$(1.2)\ |\partial_x^k\partial_y^l a(x,y)| \le cy^{-\lambda_0-l}(1-y)^{-\lambda_1-l},\ |\partial_x^k\partial_y^l b(x,y)| \le cy^{-\mu_0-l}(1-y)^{-\mu_1-l},$$

$$(x,y)\in[0,1]\times(0,1),\ \ k+l\le m,$$

where $m\in\mathbb{N}$, $0<\nu<1$, $\lambda_0+\nu<1$, $\lambda_1+\nu<1$, $\mu_0<1$, $\mu_1<1$. Then T maps $C^m_\star(0,1)$ into $C^m_\star(0,1)$, $T: C^m_\star(0,1)\to C^m_\star(0,1)$ is bounded and $T^2: C^m_\star(0,1)\to C^m_\star(0,1)$ is compact. Further, T maps $C^{m,\theta_0,\theta_1}(0,1)$ with $\theta_0=\max\{\lambda_0+\nu,\mu_0\}$, $\theta_1=\max\{\lambda_1+\nu,\mu_1\}$ into $C^{m,\theta_0,\theta_1}(0,1)$ and $T: C^{m,\theta_0,\theta_1}(0,1)\to C^{m,\theta_0,\theta_1}(0,1)$ is compact.

Here $\partial_x^k\partial_y^l = (\partial/\partial x)^k(\partial/\partial y)^l$. An example of a satisfying (1.2) is given by $a(x,y)=y^{-\lambda_0}(1-y)^{-\lambda_1}\overline{a}(x,y)$ where $\overline{a}\in C^m([0,1]\times[0,1])$.

Denote $\mathcal{N}(I-T)=\{u\in C[0,1]: u=Tu\}$. The following theorem is a consequence of Lemmas 1.1 and 1.2.

THEOREM 1.3. *Assume the conditions of Lemma 1.2 and $\mathcal{N}(I-T)=\{0\}$. Then for $f\in C^m_\star(0,1)$, equation (1.1) has a solution $u\in C^m_\star(0,1)$ which is unique in $C[0,1]$, and $\| u \|_{C^m_\star(0,1)} \le c \| f \|_{C^m_\star(0,1)}$ where the constant c is independent of f. Further, if $f\in C^{m,\theta_0,\theta_1}(0,1)$, $\theta_0=\max\{\lambda_0+\nu,\mu_0\}$, $\theta_1=\max\{\lambda_1+\nu,\mu_1\}$*

then $u \in C^{m,\theta_0,\theta_1}(0,1)$, *and* $\| u \|_{C^{m,\theta_0,\theta_1}(0,1)} \leq c_{m,\theta_0,\theta_1} \| f \|_{C^{m,\theta_0,\theta_1}(0,1)}$ *where the constant* c_{m,θ_0,θ_1} *is independent of* f.

The main results of the paper are established under assumptions of Theorem 1.3. Using the approach of [12], our treatment can be extended to the case where f has a singularity at a point $x = \xi$, $\xi \in (0,1)$, and/or a and b have certain singularities on the lines $x = \xi$ and $y = \xi$ or on a system of such inner points/lines – we can build the approximate solution on $[0,\xi]$ and $[\xi,1]$ in a similar manner as this is done on $[0,1]$ in the Section 4.

The rest of the paper is organised as follows. In Section 2 we recall some results about the interpolation and quasi-interpolation of functions by splines with a uniform knot set and, in particular, about the optimality of such approximations. In Section 3 we transform equation (1.1) into a new form by the change of variables. Section 4 is central in the paper – we introduce the product quasi-interpolation method for the transformed equation, examine its accuracy and present the matrix form of the method. In Section 5 we discuss different computational details of the method and in Section 6 we test the algorithms on a numerical example.

Actually the method can be applied to somewhat more general class of equations rather than (1.1). The case of integral equation with logarithmic diagonal singularity of the kernel will be treated in a separate work with more attention to the computational environment. In [17] the method is modified for the Volterra integral equation corresponding to (1.1).

2. Approximation tools.

2.1. The father B-spline. The *father* B-*spline* B_m *of order* m in the terminology of [3], [15], or of degree $m-1$ in the terminology of [6], [16], can be defined by the formula

$$B_m(x) = \frac{1}{(m-1)!}\sum_{i=0}^{m}(-1)^i \mathcal{C}_i^{(m)}(x-i)_+^{m-1}, \quad x \in \mathbb{R}, \quad m \in \mathbb{N}. \tag{2.1}$$

Here, as usual, $0! = 1$, $0^0 := \lim_{x\downarrow 0} x^x = 1$,

$$\mathcal{C}_i^{(m)} = \begin{pmatrix} m \\ i \end{pmatrix} = \frac{m!}{i!\,(m-i)!}, \quad (x-i)_+^{m-1} = \begin{cases} (x-i)^{m-1}, & x-i \geq 0, \\ 0, & x-i < 0. \end{cases}$$

Let us recall some properties of B_m: $B_m \in C^{(m-2)}(\mathbb{R})$,

$$\operatorname{supp} B_m = [0,m], \quad B_m(x) = B_m(m-x) > 0 \text{ for } 0 < x < m, \tag{2.2}$$

$$B_m^{(m-1)}(x) = (-1)^l \mathcal{C}_l^{(m-1)} \text{ for } l < x < l+1, \quad l = 0,...,m-1, \tag{2.3}$$

$$\sum_{j\in\mathbb{Z}} B_m(x-j) = 1 \text{ for } x \in \mathbb{R}, \quad \int_{\mathbb{R}} B_m(x)dx = 1. \tag{2.4}$$

2.2. Spline interpolation on the uniform grid in $\mathbb{R}$. Introduce in $\mathbb{R}$ the uniform grid $h\mathbb{Z} = \{ih : i \in \mathbb{Z}\}$ of the step size $h > 0$. Denote by $S_{h,m}$, $m \in \mathbb{N}$, the space of splines of order m (of degree $m-1$) and defect 1 with the knot set $h\mathbb{Z}$. The family B-spline $B_m(h^{-1}x - j)$, $j \in \mathbb{Z}$, belongs to $S_{h,m}$, and the same is true for $\sum_{j\in\mathbb{Z}} d_j B_m(h^{-1}x - j)$ with arbitrary coefficients d_j; there are no problems with the convergence of the series since it is locally finite: it follows from (2.2) that

$$\sum_{j\in\mathbb{Z}} d_j B_m(h^{-1}x-j) = \sum_{j=i-m+1}^{i} d_j B_m(h^{-1}x-j) \ \text{ for } \ x\in[ih,(i+1)h),\ i\in\mathbb{Z}.$$

Given a function $f \in C(\mathbb{R})$, bounded or of at most polynomial growth as $|x|\to\infty$, we determine the interpolant $Q_{h,m}f\in S_{h,m}$ by the conditions

$$(Q_{h,m}f)(x) = \sum_{j\in\mathbb{Z}} d_j B_m(h^{-1}x-j),\ \ x\in\mathbb{R}, \tag{2.5}$$

$$(Q_{h,m}f)((k+\frac{m}{2})h) = f((k+\frac{m}{2})h),\ \ \ k\in\mathbb{Z}. \tag{2.6}$$

For $m=1$ and $m=2$, $Q_{h,m}f$ is the usual piecewise constant, respectively, piecewise linear interpolant which can be determined on every subinterval $[ih,(i+1)h)$, $i\in\mathbb{Z}$, independently of other subintervals. For $m\ge 3$, the value of $Q_{h,m}f$ at a given point $x\in\mathbb{R}$ depends on the values of f at all interpolation knots $(k+\frac{m}{2})h$, $k\in\mathbb{Z}$. It occurs (see [14], [16]) that for $m\ge 3$ conditions (2.5)–(2.6) really determine d_j, $j\in\mathbb{Z}$, uniquely in the space of bounded or polynomially growing bisequences (d_j), namely,

$$d_j = \sum_{k\in\mathbb{Z}} \alpha_{j-k,m} f((k+\frac{m}{2})h) = \sum_{k\in\mathbb{Z}} \alpha_{k,m} f((j-k+\frac{m}{2})h),\ \ j\in\mathbb{Z}, \tag{2.7}$$

where

$$\alpha_{k,m} = \sum_{l=1}^{m_0} \frac{z_{l,m}^{m_0-1}}{P_m'(z_{l,m})} z_{l,m}^{|k|},\ \ k\in\mathbb{Z},\ \ \ m_0 = \begin{cases} (m-2)/2 & \text{if } m \text{ is even,} \\ (m-1)/2 & \text{if } m \text{ is odd,}\end{cases} \tag{2.8}$$

and $z_{l,m}\in(-1,0)$, $l=1,...,m_0$, are the roots of the characteristic polynomial

$$P_m(z) = \sum_{|k|\le m_0} B_m(k+\frac{m}{2}) z^{k+m_0}$$

(it is a polynomial of degree $2m_0$); it occurs that P_m has exactly m_0 simple roots $z_{l,m}$, $l=1,...,m_0$, in the interval $(-1,0)$, and the remaining m_0 roots are of the form $z_{l+m_0,m} = 1/z_{l,m}\in(-\infty,-1)$, $l=1,...,m_0$.

Denote by $BC(\mathbb{R})$ the space of bounded continuous functions on $\mathbb{R}$ equipped with the norm $\| f \|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$, by $V^{m,\infty}(\mathbb{R})$ the space of functions having bounded mth (distributional) derivative in $\mathbb{R}$, by $W^{m,\infty}(\mathbb{R})$ the standard Sobolev space of functions on $\mathbb{R}$ having bounded derivatives of order $\le m$, and by $W^{m,\infty}_{(0,1)}(\mathbb{R})$ the space of functions $f\in W^{m,\infty}(\mathbb{R})$ with supports in $(0,1)$.

LEMMA 2.1. ([6,21,22]). *For $f\in V^{m,\infty}(\mathbb{R})$, it holds $f-Q_{h,m}f\in BC(\mathbb{R})$ and*

$$\| f - Q_{h,m}f \|_\infty \le \ \Phi_{m+1}\pi^{-m}h^m \| f^{(m)} \|_\infty \tag{2.9}$$

where $\Phi_m = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^{km}}{(2k+1)^m}$, $m\in\mathbb{N}$, is the Favard constant,

$$\Phi_1 < \Phi_3 < \Phi_5 < ... < 4/\pi < ... < \Phi_6 < \Phi_4 < \Phi_2,\ \ \lim_{m\to\infty} \Phi_m = 4/\pi.$$

The following lemma tells that, in some sense, the spline interpolation yields the best approximation of the function classes $W^{m,\infty}(\mathbb{R})$ and $V^{m,\infty}(\mathbb{R})$, asymptotically also of $W^{m,\infty}_{(0,1)}(\mathbb{R})$, compared with other methods that use the same information as

$Q_{h,m}f$ – the values $f|_{\mathbb{Z}_{h,m}}$ of f on the grid $\mathbb{Z}_{h,m} = \{(j+\frac{m}{2})h : j \in \mathbb{Z}\}$. Denote by $C(\mathbb{Z}_{h,m})$ the vector space of all (grid) functions defined on $\mathbb{Z}_{h,m}$.

LEMMA 2.2. ([21,22]). *For given $\gamma > 0$, we have by Lemma 2.1*

$$\sup_{f\in V^{m,\infty}(\mathbb{R}),\ \|f^{(m)}\|_\infty\le\gamma} \| f - Q_{h,m}f \|_\infty \le \Phi_{m+1}\pi^{-m}h^m\gamma,$$

whereas for any mapping $M_h : C(\mathbb{Z}_{h,m}) \to C(\mathbb{R})$ (linear or nonlinear, continuous or discontinuous), it holds

$$\sup_{f\in W^{m,\infty}(\mathbb{R}),\ \|f^{(m)}\|_\infty\le\gamma} \| f - M_h(f|_{\mathbb{Z}_{h,m}}) \|_\infty \ge \Phi_{m+1}\pi^{-m}h^m\gamma,$$

$$\liminf_{h\to 0} \sup_{f\in W^{m,\infty}_{(0,1)}(\mathbb{R}),\ \|f^{(m)}\|_\infty\le\gamma} \| f - M_h(f|_{\mathbb{Z}_{h,m}}) \|_\infty / (\Phi_{m+1}\pi^{-m}h^m\gamma) \ge 1.$$

2.3. Quasi-interpolation by splines. Let $m \ge 3$. In a quasi-interpolant $Q^{(p)}_{h,m}f$, the infinite sum (2.7) defining the coefficients d_j of the spline interpolant (2.5)–(2.6) is replaced by a finite sum:

$$(Q^{(p)}_{h,m}f)(x) = \sum_{j\in\mathbb{Z}} d^{(p)}_j B_m(h^{-1}x - j),\ \ x \in \mathbb{R}, \tag{2.10}$$

$$d^{(p)}_j = \sum_{|k|\le p-1} \alpha^{(p)}_{k,m} f((j-k+\frac{m}{2})h),\ \ p \in \mathbb{N}. \tag{2.11}$$

A simple truncation of the series in (2.7) does not give acceptable results. Using a special difference calculus for fast decaying bisequences, the following formulae for $\alpha^{(p)}_{k,m}$ have been proposed in [8]:

$$\alpha^{(p)}_{k,m} = \sum_{q=|k|}^{p-1} (-1)^{k+q} \begin{pmatrix} 2q \\ k+q \end{pmatrix} \gamma_{q,m},\ \ |k| \le p-1, \tag{2.12}$$

$$\gamma_{0,m} = 1,\ \ \gamma_{q,m} = \sum_{l=1}^{m_0} \frac{(1+z_{l,m})z_{l,m}^{m_0+q-1}}{(1-z_{l,m})^{2q+1}P'_m(z_{l,m})},\ \ q \ge 1. \tag{2.13}$$

Then it occurs that

$$\| Q_{h,m}f - Q^{(p)}_{h,m}f \|_\infty \le c_{m,p}h^{2p} \| f^{(2p)} \|_\infty \quad \text{for } f \in V^{2p,\infty}(\mathbb{R}) \tag{2.14}$$

with a constant $c_{m,p}$ that can be described. A consequence of (2.14) is that for $f \in V^{m,\infty}(\mathbb{R})$ with uniformly continuous $f^{(m)}$ and $2p > m$, it holds $\| Q_{h,m}f - Q^{(p)}_{h,m}f \|_\infty h^{-m} \to 0$ as $h \to 0$, i.e., the quasi-interpolant $Q^{(p)}_{h,m}f$ is asymptotically of the same accuracy as the interpolant $Q_{h,m}f$. It is reasonable to take the smallest $p \in \mathbb{N}$ for which $2p > m$; denote it by m_1,

$$m_1 = \begin{Bmatrix} \frac{m}{2}+1, & m \text{ even} \\ \frac{m+1}{2}, & m \text{ odd} \end{Bmatrix} = m - m_0. \tag{2.15}$$

Denote also $Q'_{h,m} := Q^{(m_1)}_{h,m}$, $\alpha'_{k,m} := \alpha^{(m_1)}_{k,m}$, $|k| < m_1$. Note that $(Q'_{h,m}f)(x)$ is well defined for $x \in [ih,(i+1)h]$ with an $i \in \mathbb{Z}$ if f is given on $[(i-m+1)h,(i+m)h] \cap \mathbb{Z}_{h,m}$.

LEMMA 2.3. ([8]). *For $i \in \mathbb{Z}$, $f \in C^m[(i-m+1)h, (i+m)h]$, it holds*

$$\max_{ih \le x \le (i+1)h} |f(x) - (Q'_{h,m}f)(x))| \tag{2.16}$$

$$\le (\Phi_{m+1}\pi^{-m} + q_m c'_m)h^m \sup_{(i-m+1)h \le x \le (i+m)h} |f^{(m)}(x)|.$$

For a relatively compact set $\mathcal{M}$ in $C[-\delta, 1+\delta]$, $\delta > 0$, it holds

$$\sup_{f \in \mathcal{M}} \max_{0 \le x \le 1} |f(x) - (Q'_{h,m}f)(x))| \to 0 \text{ as } h \to 0. \tag{2.17}$$

Formulae for $q_m := \| Q_{h,m} \|_{BC(\mathbb{R}) \to BC(\mathbb{R})}$, $q'_m := \| Q'_{h,m} \|_{BC(\mathbb{R}) \to BC(\mathbb{R})}$ and c'_m from (2.16) are presented in [8] with the following numerical values:

m	3	4	5	6	7	8	9	10	20
c'_m	0.016	0.019	0.015	0.0085	0.0060	0.0030	0.0022	0.0010	6.5e-6
q_m	1.414	1.549	1.706	1.816	1.916	2.000	2.075	2.142	2.583
$q_m c'_m$	0.023	0.029	0.026	0.015	0.012	0.006	0.005	0.002	1.7e-5
q'_m	1.250	1.354	1.329	1.403	1.356	1.413	1.378	1.419	1.514

Relatively small values of the norms q_m and q'_m tell us about good stability properties of interpolation and quasi-interpolation processes.

We assumed that $m \ge 3$. For $m = 1$ and $m = 2$, we may put $Q'_{h,m} = Q_{h,m}$.

3. A smoothing change of variables. In the integral equation (1), we perform the change of variables

$$x = \varphi(t), \quad y = \varphi(s), \quad 0 \le t \le 1, \quad 0 \le s \le 1, \tag{3.1}$$

where $\varphi : [0,1] \to [0,1]$ is defined by the formula

$$\varphi(t) = \frac{1}{c_\star} \int_0^t \sigma^{r_0 - 1}(1-\sigma)^{r_1 - 1} d\sigma, \tag{3.2}$$

$$c_\star = \int_0^1 \sigma^{r_0 - 1}(1-\sigma)^{r_1 - 1} d\sigma = \frac{\Gamma(r_0)\Gamma(r_1)}{\Gamma(r_0 + r_1)},$$

Γ is the Euler gamma function. We assume that $r_0, r_1 \in \mathbb{R}$, $r_0 \ge 1$, $r_1 \ge 1$, but we keep in mind that practicable algorithms correspond to the case where at least one of parameters r_0 and r_1 is natural. If $r_1 \in \mathbb{N}$, (3.2) takes the form

$$\varphi(t) = \frac{1}{c_\star} t^{r_0} \sum_{k=0}^{r_1 - 1} \frac{(-1)^k}{r_0 + k} \mathcal{C}_k^{(r_1 - 1)} t^k, \ 0 \le t \le 1, \ c_\star = \frac{(r_1 - 1)!}{r_0(r_0+1)...(r_0 + r_1 - 1)},$$

but in our computations, the following expansion of φ with positive terms occurred to be preferable (essentially more stable numerically):

$$\varphi(t) = t^{r_0} \left[1 + \sum_{k=1}^{r_1 - 1} \frac{r_0(r_0+1)...(r_0+k-1)}{k!} (1-t)^k \right], \ 0 \le t \le 1. \tag{3.3}$$

If $r_0, r_1 \in \mathbb{N}$, the integral in (3.2) can be alternatively evaluated [9] in a stable way by an exact Gauss rule, since the integrand is a polynomial of degree $r_0 + r_1 - 2$.

Clearly, $\varphi(0) = 0$, $\varphi(1) = 1$ and $\varphi(t)$ is strictly increasing in $[0,1]$. Hence $\frac{\varphi(t)-\varphi(s)}{t-s} > 0$, $|\varphi(t)-\varphi(s)| = \frac{\varphi(t)-\varphi(s)}{t-s}|t-s|$ for $s \neq t$, and equation (1.1) takes with respect to $v(t) = u(\varphi(t))$ the form

$$v(t) = \int_0^1 \left(\mathcal{A}(t,s)|t-s|^{-\nu} + \mathcal{B}(t,s)\right) v(s)ds + g(t), \quad 0 \le t \le 1, \tag{3.4}$$

where

$$\mathcal{A}(t,s) = a(\varphi(t),\varphi(s))\Phi(t,s)^{-\nu}\varphi'(s), \quad \Phi(t,s) = \begin{cases} \frac{\varphi(t)-\varphi(s)}{t-s}, & t \neq s, \\ \varphi'(s), & t = s, \end{cases} \tag{3.5}$$

$$\mathcal{B}(t,s) = b(\varphi(t),\varphi(s))\varphi'(s), \quad g(t) = f(\varphi(t)). \tag{3.6}$$

Let us characterise the boundary behaviour of functions in equation (3.4). Clearly,

$$\begin{aligned} &0 \le \varphi(t) \le ct^{r_0}, \quad 0 \le 1-\varphi(t) \le c(1-t)^{r_1}, \\ &|\varphi^{(k)}(t)| \le ct^{r_0-k}(1-t)^{r_1-k}, \quad 0 < t < 1, \quad k = 1,...m. \end{aligned} \tag{3.7}$$

Lemma 3.1. *For $u \in C^m_\star(0,1)$, $v(t) = u(\varphi(t))$, it holds $v \in C^m_\star(0,1)$ and*

$$\| v \|_{C^m_\star(0,1)} \le c \| u \|_{C^m_\star(0,1)} \tag{3.8}$$

where the constant c is independent of u. Further, if $u \in C^{m,\theta_0,\theta_1}(0,1)$, $\theta_0 < 1$, $\theta_1 < 1$, then for $j = 1,...,m$, $0 < t < 1$,

$$|v^{(j)}(t)| \le c \| u \|_{C^{m,\theta_0,\theta_1}(0,1)} \times \tag{3.9}$$

$$\left\{ \begin{array}{rl} t^{r_0-j}, & \theta_0 < 0 \\ t^{r_0-j}(1+|\log t|), & \theta_0 = 0 \\ t^{(1-\theta_0)r_0-j}, & \theta_0 > 0 \end{array} \right\} \left\{ \begin{array}{rl} (1-t)^{r_1-j}, & \theta_1 < 0 \\ (1-t)^{r_1-j}(1+|\log(1-t)|), & \theta_1 = 0 \\ (1-t)^{(1-\theta_1)r_1-j}, & \theta_1 > 0 \end{array} \right\}.$$

Proof. Clearly $\| v \|_\infty = \| u \|_\infty$. The proof of (3.8) and (3.9) is based on (3.7) and the formula of Faà di Bruno

$$\left(\frac{d}{dt}\right)^j u(\varphi(t)) = \sum_{k_1+...+jk_j=j} \frac{j!}{k_1!..k_j!} u^{(k_1+...+k_j)}(\varphi(t)) \left(\frac{\varphi'(t)}{1!}\right)^{k_1} ... \left(\frac{\varphi^{(j)}(t)}{j!}\right)^{k_j}$$

where the sum is taken over all $k_1 \ge 0, ..., k_j \ge 0$ such that $k_1 + 2k_2 + ... + jk_j = j$. Here are the details for (3.9). For $0 < t \le \frac{1}{2}$, $1 \le j \le m$, we have

$$|v^{(j)}(t)| \le c \sum_{k_1+2k_2...+jk_j=j} \left\{ \begin{array}{cl} 1, & k_1+...+k_j < 1-\theta_0 \\ 1+|\log t|, & k_1+...+k_j = 1-\theta_0 \\ t^{(1-\theta_0-k_1-...-k_j)r_0}, & k_1+...+k_j > 1-\theta_0 \end{array} \right\} \times$$

$$\times \| u \|_{C^{m,\theta_0,\theta_1}} t^{k_1(r_0-1)} t^{k_2(r_0-2)} ... t^{k_j(r_0-j)}$$

$$= c \sum_{k_1+...+jk_j=j} \left\{ \begin{array}{ll} t^{(k_1+...+k_j)r_0-j}, & k_1+...+k_j < 1-\theta_0 \\ t^{(1-\theta_0)r_0-j}(1+|\log t|), & k_1+...+k_j = 1-\theta_0 \\ t^{(1-\theta_0)r_0-j}, & k_1+...+k_j > 1-\theta_0 \end{array} \right\} \| u \|_{C^{m,\theta_0,\theta_1}} .$$

The condition $k_1+2k_2...+jk_j = j$ implies the following. (i) The relation $k_1+...+k_j < 1-\theta_0$ is possible only if $\theta_0 < 0$, and then t^{r_0-j} is the leading term as $t \to 0$ in the last sum (it corresponds to $k_1 = ... = k_{j-1} = 0$, $k_j = 1$). (ii) The relation $k_1 + ... + k_j = 1 - \theta_0$ is possible only if $\theta_0 \le 0$, $\theta_0 \in \mathbb{Z}$; in case $\theta_0 = 0$, the leading term is $t^{r_0-j}(1 + |\log t|)$ whereas in case $\theta_0 \le -1$, similarly as in (i), the leading term is t^{r_0-j} . (iii) If $0 < \theta_0 < 1$ then all terms in the last sum are $t^{(1-\theta_0)r_0-j}$. These considerations prove (3.9) for $0 < t \le \frac{1}{2}$. By a symmetry argument we obtain (3.9) also for $\frac{1}{2} \le t < 1$. □

LEMMA 3.2. *Let a and b satisfy the conditions of Lemma 1.2. Then for $j = 0, ..., m$, $0 \le t \le 1$, $0 < s < 1$, it holds*

$$|\partial_s^j a(\varphi(t), \varphi(s))| \le c s^{-r_0\lambda_0 - j}(1-s)^{-r_1\lambda_1 - j}, \tag{3.10}$$

$$|\partial_s^j b(\varphi(t), \varphi(s))| \le c s^{-r_0\mu_0 - j}(1-s)^{-r_1\mu_1 - j}. \tag{3.11}$$

These inequalities elementarily follow by the formula of Faà di Bruno.

The function $\Phi(t,s)^{-\nu}$ has singularities at $(0,0)$ and $(1,1)$, the only zeroes of $\Phi(t,s)$ in $[0,1] \times [0,1]$. It is easy to see that

$$\partial_s^k \Phi(t,s) \asymp (t+s)^{r_0-k-1}((1-t)+(1-s))^{r_1-k-1} \text{ as } t,s \to 0 \text{ or as } t,s \to 1$$

that together with the formula of Faà di Bruno implies the following result.

LEMMA 3.3. *For $j = 0, ..., m$, $0 \le t \le 1$, $0 < s < 1$, it holds*

$$\left|\partial_s^j \left((\Phi(t,s))^{-\nu}\right)\right| \le c(t+s)^{-\nu(r_0-1)-j}((1-t)+(1-s))^{-\nu(r_1-1)-j}. \tag{3.12}$$

Next we present estimates for functions $\mathcal{A}(t,s)$, $\mathcal{B}(t,s)$ and $\partial_s^m[\mathcal{A}(t,s)v(s)]$, $\partial_s^m[\mathcal{B}(t,s)v(s)]$ in somewhat specific form for the needs of Section 4.

COROLLARY 3.4. *Let a and b satisfy the conditions of Lemma 1.1. Then the following holds true: (i) if*

$$r_0, r_1 \ge 1, \ \ r_0 > (1-\nu)/(1-\nu-\lambda_0), \ \ r_1 > (1-\nu)/(1-\nu-\lambda_1), \tag{3.13}$$

then with $\delta_0 := (1-\nu-\lambda_0)r_0 - (1-\nu) > 0$, $\delta_1 := (1-\nu-\lambda_1)r_1 - (1-\nu) > 0$, it holds

$$|\mathcal{A}(t,s)| \le c s^{\delta_0}(1-s)^{\delta_1}, \ \ (t,s) \in [0,1] \times (0,1); \tag{3.14}$$

(ii) if

$$r_0, r_1 \ge 1, \ \ r_0 > 1/(1-\mu_0), \ \ r_1 > 1/(1-\mu_1), \tag{3.15}$$

then with $\delta_0 := (1-\mu_0)r_0 - 1 > 0$, $\delta_1 := (1-\mu_1)r_1 - 1 > 0$, it holds

$$|\mathcal{B}(t,s)| \le c s^{\delta_0}(1-s)^{\delta_1}, \ \ (t,s) \in [0,1] \times (0,1). \tag{3.16}$$

Proof. These estimates are direct consequences of Lemmas 3.2 and 3.3. By (3.7), (3.10) and (3.12) we have

$$|\mathcal{A}(t,s)| \le c s^{-r_0\lambda_0 + r_0 - 1}(t+s)^{-\nu(r_0-1)}(1-s)^{-r_1\lambda_1 + r_1 - 1}(2-t-s)^{-\nu(r_1-1)}$$

that for r_0, r_1 satisfying (3.13) yields (3.14). Similarly, by (3.7), (3.11) and (3.12),

$$|\mathcal{B}(t,s)| \le c s^{-r_0\mu_0 + r_0 - 1}(1-s)^{-r_1\mu_1 + r_1 - 1}$$

that for r_0, r_1 satisfying (3.15) yields (3.16). □

COROLLARY 3.5. *Let a and b satisfy the conditions of Lemma 1.2, and let $u \in C^m_\star(0,1)$, $v(t) = u(\varphi(t))$. Then the following estimates hold true for $(t,s) \in [0,1]\times(0,1)$: (i) if*

$$r_0, r_1 \geq 1, \;\; r_0 > m/(1-\nu-\lambda_0), \;\; r_1 > m/(1-\nu-\lambda_1), \tag{3.17}$$

then with $\delta_0 := (1-\nu-\lambda_0)r_0 - m > 0$, $\delta_1 := (1-\nu-\lambda_1)r_1 - m > 0$,

$$|\mathcal{A}(t,s)| \leq c s^{m-(1-\nu)+\delta_0}(1-s)^{m-(1-\nu)+\delta_1}, \tag{3.18}$$

$$|\partial_s^m[\mathcal{A}(t,s)v(s)]| \leq c s^{-(1-\nu)+\delta_0}(1-s)^{-(1-\nu)+\delta_1} \parallel u \parallel_{C^m_\star(0,1)}; \tag{3.19}$$

(ii) if

$$r_0, r_1 \geq 1, \;\; r_0 > m/(1-\mu_0), \;\; r_1 > m/(1-\mu_1), \tag{3.20}$$

then with $\delta_0 := (1-\mu_0)r_0 - m > 0$, $\delta_1 := (1-\mu_1)r_1 - m > 0$,

$$|\mathcal{B}(t,s)| \leq c s^{m-1+\delta_0}(1-s)^{m-1+\delta_1}, \tag{3.21}$$

$$|\partial_s^m[\mathcal{B}(t,s)v(s)]| \leq c s^{-1+\delta_0}(1-s)^{-1+\delta_1} \parallel u \parallel_{C^m_\star(0,1)}; \tag{3.22}$$

(iii) if

$$r_0, r_1 \geq 1, \;\; r_0 > (m+\nu)/(1-\lambda_0), \;\; r_1 > (m+\nu)/(1-\lambda_1) \tag{3.23}$$

then with $\delta_0 := (1-\lambda_0)r_0 - m - \nu > 0$, $\delta_1 := (1-\lambda_1)r_1 - m - \nu > 0$,

$$\varphi'(t)^\nu |\mathcal{A}(t,s)| \leq c s^{m-(1-\nu)+\delta_0}(1-s)^{m-(1-\nu)+\delta_1}, \tag{3.24}$$

$$\varphi'(t)^\nu |\partial_s^m[\mathcal{A}(t,s)v(s)]| \leq c s^{-(1-\nu)+\delta_0}(1-s)^{-(1-\nu)+\delta_1} \parallel u \parallel_{C^m_\star(0,1)} . \tag{3.25}$$

Proof. These estimates are direct consequences of Lemmas 3.1–3.3. Observe that $|\partial_s^k[\varphi'(t)^\nu \Phi(t,s)^{-\nu}]| \leq c(t+s)^{-k}((1-t)+(1-s))^{-k}$. □

Let us extend $\mathcal{A}(t,s)$ and $\mathcal{B}(t,s)$ with respect to s outside $(0,1)$ by the zero value. Under conditions (3.13) and (3.15) we obtain continuous functions on $[0,1]\times\mathbb{R}$, see (3.14) and (3.16).

4. The product quasi-interpolation method.

4.1. The operator form of the method. Let $h = 1/n$, $n \in \mathbb{N}$, $n \geq m$. For $0 \leq s \leq 1$, the quasi-interpolant $Q'_{h,m}w$ of a function $w \in C[-\delta, 1+\delta]$ has the expansion

$$(Q'_{h,m}w)(s) = \sum_{j=-m+1}^{n-1} \sum_{|p|<m_1} \alpha'_{p,m} w((j-p+\frac{m}{2})h) B_m(ns-j). \tag{4.1}$$

We approximate equation (3.4) by its discretised version: for $0 \leq t \leq 1$,

$$v_n(t) = \int_0^1 [|t-s|^{-\nu} Q'_{h,m}(\mathcal{A}(t,s)v_n(s)) + Q'_{h,m}(\mathcal{B}(t,s)v_n(s))]ds + g(t) \tag{4.2}$$

where $Q'_{h,m}$ is applied to the products $\mathcal{A}(t,s)v(s)$ and $\mathcal{B}(t,s)v(s)$ as functions of s. This can be done for v given on $[0,1]$ since $\mathcal{A}(t,s) = \mathcal{B}(t,s) = 0$ for $s \leq 0$ and for $s \geq 1$; recall that under condition (3.13) and (3.15), $\mathcal{A}, \mathcal{B} \in C([0,1]\times\mathbb{R})$.

THEOREM 4.1. *(i) Let a and b satisfy the conditions of Lemma 1.1, $\mathcal{N}(I-T)=\{0\}$, $f\in C[0,1]$, and let r_0 and r_1 satisfy condition* (3.13) *and* (3.15). *Then for sufficiently large n, equation* (4.2) *has a unique solution v_n, and*

$$\| v-v_n \|_\infty := \max_{0\le t\le 1} |v(t)-v_n(t)| \to 0 \ \text{ as } n\to\infty \tag{4.3}$$

where v is the solution of equation (3.4).

(ii) Under assumptions of Theorem 1.3, $f\in C_\star^m(0,1)$ and conditions (3.17) *and* (3.20) *on r_0 and r_1, it holds*

$$\| v-v_n \|_\infty \le ch^m \| f \|_{C_\star^m(0,1)} . \tag{4.4}$$

(iii) Under assumptions of Theorem 1.3, $f\in C_\star^m(0,1)$ and conditions (3.20) *and* (3.23) *on r_0 and r_1, it holds*

$$\max_{0\le t\le 1} \varphi'(t)^\nu |v(t)-v_n(t)| \le ch^m \| f \|_{C_\star^m(0,1)} . \tag{4.5}$$

The constant c in (4.4) *and* (4.5) *is independent of n and f.*

Proof. Accept the assumptions formulated in (i). Denote by $\mathcal{T}$ and $\mathcal{T}_n$ the integral operators of equations (3.4) and (4.2),

$$(\mathcal{T}v)(t)=\int_0^1 [|t-s|^{-\nu}\mathcal{A}(t,s)+\mathcal{B}(t,s)]v(s)ds,$$

$$(\mathcal{T}_n v)(t)=\int_0^1 [|t-s|^{-\nu}Q'_{h,m}(\mathcal{A}(t,s)v(s))+Q'_{h,m}(\mathcal{B}(t,s)v(s))]ds.$$

We claim that $\mathcal{T}_n\to\mathcal{T}$ compactly in $C[0,1]$ as $n\to\infty$, i.e.,

$$\| \mathcal{T}_n v-\mathcal{T}v \|_\infty \to 0 \text{ for every } v\in C[0,1], \tag{4.6}$$

$$(v_n)\subset C[0,1],\ \| v_n \|_\infty \le 1 \ \Rightarrow\ (\mathcal{T}_n v_n) \text{ is relatively compact in } C[0,1]. \tag{4.7}$$

Indeed, the sets $\{\mathcal{A}(t,\cdot):\ 0\le t\le 1\}$ and $\{\mathcal{B}(t,\cdot):\ 0\le t\le 1\}$ are relatively compact in $C[-\delta,1+\delta]$, and by Lemma 2.3, for a fixed $v\in C[0,1]$ extended by $v(t)=v(0)$ for $-\delta\le s\le 0$, $v(t)=v(1)$ for $1\le s\le 1+\delta$, it holds

$$\sup_{0\le t\le 1}\max_{0\le s\le 1} |\mathcal{A}(t,s)v(s)-Q'_{h,m}(\mathcal{A}(t,s)v(s))| \to 0 \text{ for } n\to\infty,$$

$$\sup_{0\le t\le 1}\max_{0\le s\le 1} |\mathcal{B}(t,s)v(s)-Q'_{h,m}(\mathcal{B}(t,s)v(s))| \to 0 \text{ for } n\to\infty.$$

This together with the equality $\| Q'_{h,m} \| = q'_m$ implies (4.6). The proof of (4.7) can be built using the Arzela theorem.

Due to the condition $\mathcal{N}(I-T)=\{0\}$, also $\mathcal{N}(I-\mathcal{T})=\{0\}$. As well known (see [1], [2], [7] or [18]), relations (4.6), (4.7) and $\mathcal{N}(I-\mathcal{T})=\{0\}$ imply that, for sufficiently large n, the operators $I-\mathcal{T}_n$ are invertible and the inverses are uniformly bounded:

$$\| (I-\mathcal{T}_n)^{-1} \|_{C[0,1]\to C[0,1]} \le c,\ \ n\ge n_0. \tag{4.8}$$

Let v and v_n be the solutions of equations (3.4) and (4.2), respectively. Then

$$v-v_n=(I-\mathcal{T}_n)^{-1}(\mathcal{T}v-\mathcal{T}_n v), \tag{4.9}$$

and due to (4.8) and (4.6), the convergence (4.3) follows.

Next we establish the uniform estimate (4.4). Assume the conditions of Theorem 1.3, $f \in C_\star^m(0,1)$ and (3.17), (3.20). For the solutions u and v of (1.1) and (3.4) we have $v(t) = u(\varphi(t))$ and $u \in C_\star^m(0,1)$ by Theorem 1.3. On the basis of (2.16) and (3.19),

$$\int_{2mh}^{1-2mh} |t-s|^{-\nu} |\mathcal{A}(t,s)v(s) - Q'_{h,m}(\mathcal{A}(t,s)v(s))| ds$$

$$\leq ch^m \int_{mh}^{1-mh} |t-s|^{-\nu} s^{-(1-\nu)+\delta_0}(1-s)^{-(1-\nu)+\delta_1} ds \parallel u \parallel_{C_\star^m(0,1)} \leq c' h^m \parallel u \parallel_{C_\star^m(0,1)}$$

where we took into account that for $2mh \leq s \leq 1-2mh$, it holds

$$\max_{|\sigma - s| \leq mh} \sigma^{-(1-\nu)+\delta_0}(1-\sigma)^{-(1-\nu)+\delta_1} \leq cs^{-(1-\nu)+\delta_0}(1-s)^{-(1-\nu)+\delta_1}.$$

Further, due to (3.18)

$$\max_{0\leq t \leq 1,\, s \in [0,2mh]\cup[1-2mh,1]} |\mathcal{A}(t,s)| \leq ch^{m-(1-\nu)+\delta'}, \;\; \delta' = \min\{\delta_0, \delta_1)\} > 0,$$

that implies

$$\left(\int_0^{2mh} + \int_{1-2mh}^1 \right) |t-s|^{-\nu} |\mathcal{A}(t,s)v(s) - Q'_{h,m}(\mathcal{A}(t,s)v(s))| ds$$

$$\leq (1+q'_m) ch^{m-(1-\nu)+\delta'} \left(\int_0^{2mh} + \int_{1-2mh}^1 \right) |t-s|^{-\nu} ds \parallel v \parallel_\infty \leq c' h^{m+\delta'} \parallel u \parallel_\infty$$

and together with the previous estimate,

$$\int_0^1 |t-s|^{-\nu} |\mathcal{A}(t,s)v(s) - Q'_{h,m}(\mathcal{A}(t,s)v(s))| ds \leq ch^m \parallel u \parallel_{C_\star^m(0,1)}, \;\; 0 \leq t \leq 1.$$

With the help of (3.21) and (3.22) we obtain in a similar way that

$$\int_0^1 |\mathcal{B}(t,s)v(s) - Q'_{h,m}(\mathcal{B}(t,s)v(s))| ds \leq ch^m \parallel u \parallel_{C_\star^m(0,1)}, \;\; 0 \leq t \leq 1.$$

Exploiting also Theorem 1.3 we get

$$\parallel \mathcal{T}v - \mathcal{T}_n v \parallel_\infty \leq ch^m \parallel u \parallel_{C_\star^m(0,1)} \leq c' h^m \parallel f \parallel_{C_\star^m(0,1)} .$$

Together with (4.8) and (4.9) this proves (4.4).

Finally, let us prove the weighted error estimate (4.5) under conditions (3.20), (3.23) on r_0 and r_1. Let v and v_n be the solutions of equations (3.4) and (4.2), respectively; then $w(t) = \varphi'(t)^\nu v(t)$ and $w_n(t) = \varphi'(t)^\nu v_n(t)$ are the solutions of equations

$$w(t) = \int_0^1 [|t-s|^{-\nu} \overline{\mathcal{A}}(t,s) w(s) + \overline{\mathcal{B}}(t,s)) w(s)] ds + \varphi'(t)^\nu g(t), \tag{4.10}$$

$$w_n(t) = \int_0^1 [|t-s|^{-\nu} Q'_{h,m}(\overline{\mathcal{A}}(t,s) w_n(s)) + Q'_{h,m}(\overline{\mathcal{B}}(t,s)) w_n(s))] ds \tag{4.11}$$

$$+\varphi'(t)^\nu g(t), \quad 0 \le t \le 1,$$

respectively, where

$$\overline{\mathcal{A}}(t,s) = \varphi'(t)^\nu \mathcal{A}(t,s)\varphi'(s)^{-\nu} = \varphi'(t)^\nu a(\varphi(t),\varphi(s))\Phi(t,s)^{-\nu}\varphi'(s)^{1-\nu},$$

$$\overline{\mathcal{B}}(t,s) = \varphi'(t)^\nu \mathcal{B}(t,s)\varphi'(s)^{-\nu} = \varphi'(t)^\nu b(\varphi(t),\varphi(s))\varphi'(s)^{1-\nu}.$$

Denote by $\overline{\mathcal{T}}$ and $\overline{\mathcal{T}}_n$ the integral operators of equations (4.10) and (4.11). Arguing similarly as above it is easily seen that $\overline{\mathcal{T}}_n \to \overline{\mathcal{T}}$ compactly in $C[0,1]$; by the way, this holds even under the conditions of the theorem part (i), in particular, (3.13) and (3.15) guarantee the continuity of $\overline{\mathcal{A}}(t,s)$ and $\overline{\mathcal{B}}(t,s)$ on $[0,1]\times\mathbb{R}$. Hence

$$\| (I - \overline{\mathcal{T}}_n)^{-1} \|_{C[0,1]\to C[0,1]} \le c, \quad n \ge n_0, \tag{4.12}$$

$$\| w - w_n \|_\infty \le c \| \overline{\mathcal{T}} w - \overline{\mathcal{T}}_n w \|_\infty = c \max_{0\le t\le 1} \varphi'(t)^\nu |(\mathcal{T}v)(t) - (\mathcal{T}_n v)(t)|.$$

Using (3.24), (3.25), (3.21), (3.22) and Theorem 1.3, we obtain

$$\max_{0\le t\le 1} \varphi'(t)^\nu |(\mathcal{T}v)(t) - (\mathcal{T}_n v)(t)| \le ch^m \| u \|_{C^m_\star(0,1)} \le c'h^m \| f \|_{C^m_\star(0,1)}$$

that completes the proof of estimate (4.5). □

Remark 4.1. Under conditions of Theorem 4.1(i), we have

$$\| \mathcal{T}_n \|_{C[0,1]\to C[0,1]} \le c, \quad \| \overline{\mathcal{T}}_n \|_{C[0,1]\to C[0,1]} \le c, \quad n \ge m. \tag{4.13}$$

Remark 4.2. Under conditions of Theorem 4.1(ii) but slack inequalities instead of strict ones in (3.17), (3.20), a slight modification in the proof yields

$$\| v - v_n \|_\infty \le ch^m |\log h| \| f \|_{C^m_\star(0,1)} . \tag{4.14}$$

Similar remark concerns Theorem 4.1(iii).

The numerical example in Section 6 does not confirm the presence of the factor $|\log h|$ in (4.14). If it really can be dropped, the proof needs new ideas.

Remark 4.3. Assume that m is odd, f$\in C^{m+1}_\star(0,1)$, $a, b \in C^{m+1}([0,1]\times(0,1))$ satisfy (1.2) for $k+l \le m+1$, and $\mathcal{N}(I-T) = \{0\}$. Then under conditions

$$\begin{aligned} r_0, r_1 \ge 1, \ \ r_0 \ge (m+1-\nu)/(1-\nu-\lambda_0), \ \ r_1 \ge (m+1-\nu)/(1-\nu-\lambda_1), \\ r_0 \ge (m+1-\nu)/(1-\mu_0), \qquad r_1 \ge (m+1-\nu)/(1-\mu_1) \end{aligned}$$

the convergence order (4.4) of method (4.2) can be improved (cf. [13]):

$$\| v - v_n \|_\infty \le ch^{m+1-\nu} \| f \|_{C^{m+1}_\star(0,1)} . \tag{4.15}$$

The outlines of the proof are as follows. Due to (2.12), $\alpha^{(p)}_{k,m} = \alpha^{(p)}_{-k,m}$. Together with the symmetry property (2.2) of B_m this implies that for a function e which if odd w.r.t. a point $(j+\frac{1}{2})h$, also $Q'_{h,m}e$ has the same parity property; a consequence is that $\int_{jh}^{(j+1)h}(e_m - Q'_{h,m}e_m)ds = 0$ for odd m and polynomials e_m of degree $\le m$ (note that, due to (2.9) and (2.14), $e_{m-1} - Q'_{h,m}e_{m-1} = 0$ for polynomials of degree $\le m-1$). Hence for odd m,

$$\left| \int_{jh}^{(j+1)h} (w - Q'_{h,m}w)ds \right| \le ch^{m+2} \max_{(j-m+\frac{1}{2})h \le s \le (j+m-\frac{1}{2})h} | w^{(m+1)}(s) | . \tag{4.16}$$

For $w(s) = \mathcal{A}(t,s)v(s)$, $0 < s < 1$, $0 \le t \le 1$, the conditions on r_0, r_1 yield

$$| w^{(m)}(s) | \le c \parallel v \parallel_{C^m_\star(0,1)}, \ \ | w^{(m+1)}(s) | \le c[s^{-1} + (1-s)^{-1}] \parallel v \parallel_{C^{m+1}_\star(0,1)},$$

and with the help of (2.16) and (4.16) we obtain

$$\varepsilon_j := \max_{jh \le s \le (j+1)h} | w(s) - (Q'_{h,m}w)(s) | \le ch^m \parallel v \parallel_{C^m_\star(0,1)}, \ \ 0 \le j \le n-1,$$

$$\varepsilon'_j := \left| \int_{jh}^{(j+1)h} (w - Q'_{h,m}w)ds \right| \le \ ch^{m+2}[(jh)^{-1} + ((n-j)h)^{-1}] \parallel v \parallel_{C^{m+1}_\star(0,1)},$$

$$j = m+1, ..., n-m-2.$$

With $i \in \mathbb{Z}$, $0 \le i \le n-1$, such that $| t - (i + \frac{1}{2})h | \le \frac{h}{2}$, we get

$$\left| \int_0^1 | t-s |^{-\nu} [w(s) - (Q'_{h,m}w)(s)]ds \right| \le \left(\sum_{j=0}^{m} + \sum_{j=n-m-1}^{n-1} \right) \varepsilon_j \int_{jh}^{(j+1)h} | t-s |^{-\nu} ds$$

$$+ \sum_{j=m+1}^{n-m-2} \left\{ \varepsilon_j \int_{jh}^{(j+1)h} \left| | t-s |^{-\nu} - | (i + \frac{1}{2} - j)h |^{-\nu} \right| ds + \varepsilon'_j \ | (i + \frac{1}{2} - j)h |^{-\nu} \right\}$$

$$\le ch^{m+1-\nu} \parallel v \parallel_{C^{m+1}_\star(0,1)}, \ 0 \le t \le 1.$$

For $w(s) = \mathcal{B}(t,s)v(s)$ we get in a straightforward manner, like in the proof of Theorem 4.1, that $| \int_0^1 (w - Q'_{h,m}w)ds | \le ch^{m+1-\nu} \parallel v \parallel_{C^{m+1}_\star(0,1)}$, and now (4.15) follows.

For even m, an inequality like (4.16) is violated, and estimate (4.4) for the method (4.2) cannot be improved; for $m = 4$, a numerical confirmation can be seen from Table 6.1, column $r = 9$. There are possibilities to modify the basic method (4.2) so that (4.15) will hold for even m; we do not go into the details here.

4.2. Matrix form of the method. With the help of (4.1) we rewrite equation (4.2) in the form

$$(4.17)\ v_n(t) = \sum_{j=-m+1}^{n-1} \ \sum_{p \in \mathbb{Z}:\, |p| < m_1,\, 0 < j-p+\frac{m}{2} < n} \alpha'_{p,m} \Big[\mathcal{A}(t, (j-p+\frac{m}{2})h)\beta_j(t)$$

$$+ \mathcal{B}(t, (j-p+\frac{m}{2})h)\beta^0_j \Big] \, v_n((j-p+\frac{m}{2})h) + g(t), \ \ 0 \le t \le 1,$$

where

$$\beta_j(t) = \int_0^1 |t-s|^{-\nu} B_m(ns-j)ds, \ \ \beta^0_j = \int_0^1 B_m(ns-j)ds;$$

we took into account that $\mathcal{A}(t,s) = 0$, $\mathcal{B}(t,s) = 0$ for $s \le 0$ and for $s \ge 1$. The solution v_n of (4.17) is determined by its values at the points $(k + \frac{m}{2})h$ for k satisfying $0 < (k + \frac{m}{2})h < 1$, that is, for $k = -m_0, ..., n - m_1$ with m_0 and m_1 defined in (2.8), (2.15). Collocating (4.17) at $(i + \frac{m}{2})h$ and denoting

$$v_{i,n} = v_n((i+\frac{m}{2})h), \;\; g_i = g((i+\frac{m}{2})h), \;\; i = -m_0, ..., n-m_1,$$

$$a_{i,k} = \mathcal{A}((i+\frac{m}{2})h, (k+\frac{m}{2})h), \;\; b_{i,k} = \mathcal{B}((i+\frac{m}{2})h, (k+\frac{m}{2})h), \;\; i,k = -m_0, ..., n-m_1,$$

we arrive at the system of linear algebraic equations

$$(4.18)\; v_{i,n} = \sum_{j=-m+1}^{n-1} \sum_{p\in\mathbb{Z}:\, |p|<m_1,\, 0<j-p+\frac{m}{2}<n} \alpha'_{p,m}(\beta_{i,j}a_{i,j-p} + \beta_j^0 b_{i,j-p})v_{j-p,n}$$

$$+g_i, \quad i = -m_0, ..., n-m_1,$$

where for $i = -m_0, ..., n-m_1$, $j = -m+1, ..., n-1$,

$$(4.19) \quad \beta_{i,j} = \int_0^1 |(i+\frac{m}{2})h - s|^{-\nu} B_m(ns-j)ds, \;\; \beta_j^0 = \int_0^1 B_m(ns-j)ds.$$

To present (4.18) in a standard form, we change the summation ordering:

$$\sum_{j=-m+1}^{n-1} \sum_{p\in\mathbb{Z}:\, |p|<m_1,\, 0<j-p+\frac{m}{2}<n} \alpha'_{p,m}(\beta_{i,j}a_{i,j-p} + \beta_j^0 b_{i,j-p})v_{j-p,n}$$

$$= \sum_{j=-m+1}^{n-1} \sum_{k=\max\{-m_0,\, j-m_1+1\}}^{\min\{n-m_1,\, j+m_1-1\}} \alpha'_{j-k,m}(\beta_{i,j}a_{i,k} + \beta_j^0 b_{i,k})v_{k,n}$$

$$= \sum_{k=-m_0}^{n-m_1} \sum_{j=k-m_1+1}^{k+m_1-1} \alpha'_{j-k,m}(\beta_{i,j}a_{i,k} + \beta_j^0 b_{i,k})v_{k,n}$$

(the summation goes over $(j,k) \in D\cap\mathbb{Z}^2$ where $D \subset \mathbb{R}^2$ is the parallelogram restricted by the lines $k = j-m_1+1$, $k = j+m_1-1$, $k = -m_0$, $k = n-m_1$; note that $m_0 + m_1 = m$). We arrive at the following form of system (4.18):

$$(4.20) \qquad v_{i,n} = \sum_{k=-m_0}^{n-m_1} \tau_{i,k} v_{k,n} + g_i, \quad i = -m_0, ..., n-m_1,$$

where

$$(4.21) \quad \tau_{i,k} = \sum_{j=k-m_1+1}^{k+m_1-1} \alpha'_{j-k,m}(\beta_{i,j}a_{i,k} + \beta_j^0 b_{i,k}), \quad i,k = -m_0, ..., n-m_1.$$

The dimension of system (4.20) is, respectively, $n-1$ and n for even and odd m. The computation of matrix elements (4.21) from $\alpha'_{j,m}$, $\beta_{i,j}$, β_j^0, $a_{i,k}$, $b_{i,k}$ costs approximately $3mn^2$ flops.

4.3. Quasi-interpolation extension of the discrete solution. Having found the solution $v_{i,n}$, $i = -m_0, ..., n - m_1$, of system (4.20) we can use (4.17) to compute $v_n(t)$ for any $t \in [0, 1]$. A cheaper way is to construct on every $[ih, (i+1)h]$, $i = 0, ..., n-1$, the quasi-interpolant $\widetilde{v}_n(t)$ of degree $m-1$ using the solution of (4.20) completed by values $v_{i,n} = v_{-m_0,n}$ for $i < -m_0$ and $v_{i,n} = v_{n-m_1,n}$ for $i > n - m_1$; a slightly more accurate way is to compute $v_n(0)$ and $v_n(1)$ from (4.17) and set $v_{i,n} = v_n(0)$ for $i < -m_0$ and $v_{i,n} = v_n(1)$ for $i > n - m_1$. To maintain the convergence orders (4.4) and (4.5) for $\widetilde{v}_n$, we have to strengthen conditions on f and parameters r_0 and r_1.

THEOREM 4.2. *(i) Under assumptions of Theorem 4.1(i),* $\| v - \widetilde{v}_n \|_\infty \to 0$ *as* $n \to \infty$.

(ii) Under assumptions of Theorem 1.3, $f \in C^{m,\theta_0,\theta_1}(0,1)$ *with* $\theta_0 = \max\{\lambda_0 + \nu, \mu_0\}$, $\theta_1 = \max\{\lambda_1 + \nu, \mu_1\}$, (3.17), (3.20) *and* $r_0 > m$, $r_1 > m$, *it holds*

$$\| v - \widetilde{v}_n \|_\infty \leq ch^m \| f \|_{C^{m,\theta_0,\theta_1}(0,1)} . \tag{4.22}$$

(iii) Under assumptions of Theorem 1.3, $f \in C^{m,\theta_0,\theta_1}(0,1)$ *as in (ii),* (3.20), (3.23) *and* $r_0 > (m+\nu)/(1+\nu)$, $r_1 > (m+\nu)/(1+\nu)$, *it holds*

$$\max_{0 \leq t \leq 1} \varphi'(t)^\nu |v(t) - \widetilde{v}_n(t)| \leq ch^m \| f \|_{C^{m,\theta_0,\theta_1}(0,1)} . \tag{4.23}$$

Proof. Accept the assumptions of (ii). Due to (3.9) and conditions on r_0, r_1, it holds $v^{(j)}(0) = v^{(j)}(1) = 0$, $j = 1, ..., m$. Extending $v(t) = v(0)$ for $t < 0$, $v(t) = v(1)$ for $t > 1$, we have $v \in C^m(\mathbb{R})$. The equality $v - \widetilde{v}_n = v - Q'_{h,m} v + Q'_{h,m}(v - v_n)$ together with (2.16), (3.9) and (4.4) implies

$$\| v - \widetilde{v}_n \|_\infty \leq \| v - Q'_{h,m} v \|_\infty + q'_m \| v - v_n \|_\infty$$

$$\leq ch^m \| v^{(m)} \|_\infty + ch^m \| f \|_{C^m_\star(0,1)} \leq ch^m \| u \|_{C^{m,\theta_0,\theta_1}(0,1)} + ch^m \| f \|_{C^m_\star(0,1)}$$

where u is the solution of (1.1), $\| u \|_{C^{m,\theta_0,\theta_1}(0,1)} \leq c \| f \|_{C^{m,\theta_0,\theta_1}(0,1)}$ by Theorem 1.3. This completes the proof of (4.22). The proof of (4.23) is similar. □

4.4. Modification: the product interpolation method. The considerations of Sections 4.1–4.3 can be easily modified for the case of the product interpolation method defined by

$$v_n(t) = \int_0^1 [|t-s|^{-\nu} Q_{h,m}(\mathcal{A}(t,s)v_n(s)) + Q_{h,m}(\mathcal{B}(t,s)v_n(s))]ds + g(t) \tag{4.24}$$

(the spline interpolation projector $Q_{h,m}$ is used instead of the quasi-interpolation operator $Q'_{h,m}$, cf. (4.2); $\mathcal{A}(t,\cdot)v$ and $\mathcal{B}(t,\cdot)v$ are still extended from $[0,1]$ to $\mathbb{R}$ with the zero value). For $0 \leq s \leq 1$ and a function w defined on $\mathbb{R}$ we have (see (2.5), (2.7))

$$(Q_{h,m}w)(s) = \sum_{j=-m+1}^{n-1} \sum_{k \in \mathbb{Z}} \alpha_{j-k,m} w((k + \frac{m}{2})h) B_m(ns - j).$$

Since in the functions $w(s) = \mathcal{A}(t,s)v(s)$ and $w(s) = \mathcal{B}(t,s)v(s)$ we have $w(s) = 0$ for $s \leq 0$ and for $s \geq 1$, the infinite sum over k reduces to the sum over $k =$

$-m_0, ..., n-1$, and the matrix form of method (4.24) takes the form (4.20) with (cf. (4.21))

$$\tau_{i,k} = \sum_{j=-m+1}^{n-1} \alpha_{j-k,m}(\beta_{i,j} a_{i,k} + \beta_j^0 b_{i,k})$$
$$= a_{i,k} \sum_{j=-m+1}^{n-1} \alpha_{j-k,m}\beta_{i,j} + b_{i,k} \sum_{j=-m+1}^{n-1} \alpha_{j-k,m}\beta_j^0, \quad i,k = -m_0, ..., n-m_1.$$

Now the computation of the elements of the matrix costs $O(n^3)$ flops, but using the convolution structure of the sums in the last representation form of $\tau_{i,k}$, the cost can be reduced to $O(n^2 \log n)$ flops with the help of FFT. Nevertheless, this is more complicated and slightly more expensive than in the case of the product quasi-interpolation method.

Theorem 4.1 remains to be valid also for method (4.24), in the proof we simply exploit Lemma 2.1 instead of Lemma 2.3.

5. Some computational details.

5.1. A numerically stable evaluation of $\Phi(t,s)$. The computation of the divided difference $\Phi(t,s) = (\varphi(t) - \varphi(s))/(t-s)$ (see (3.5) and (3.2)) for small $|t-s| \neq 0$ may cause a loss of accuracy. For $r_0, r_1 \in \mathbb{N}$, the division by $t-s$ can be performed analytically. For $0 \le s < t \le 1$ we have

$$\Phi(t,s) = \frac{1}{t-s}\int_s^t \varphi'(\sigma)d\sigma = \frac{1}{c_\star(t-s)}\int_s^t \sigma^{r_0-1}(1-\sigma)^{r_1-1}d\sigma$$

$$= \frac{1}{c_\star}\int_0^1 [(t-s)\xi + s]^{r_0-1}[(1-t) + (t-s)(1-\xi)]^{r_1-1}d\xi \tag{5.1}$$

where we undertook the change of variables $\sigma = (t-s)\xi + s$. The last integral can be computed in a fast and stable way by an exact Gauss rule since the integrand is a polynomial of degree r_0+r_1-2 in ξ. Alternatively, applying the Newton binomial formula we obtain the expansion

$$\Phi(t,s) = (r_0+r_1-1)!\sum_{p=0}^{r_0-1}\sum_{q=0}^{r_1-1} \frac{s^{r_0-1-p}(1-t)^{r_1-1-q}(t-s)^{p+q}}{(r_0-1-p)!(r_1-1-q)!(p+q+1)!}$$

with non-negative terms for $0 \le s \le t \le 1$. Nevertheless, the use of this stable formula is more expensive than the use of an exact Gauss rule in (5.1). For $0 \le t < s \le 1$, the symmetry $\Phi(t,s) = \Phi(s,t)$ can be used.

5.2. Computation of the quadrature coefficients. The integrals $\beta_j(t)$ in (4.17), in particular the quadrature coefficients $\beta_{i,j}$ and β_j^0 defined in (4.19) can be evaluated on the basis of the following two lemmas.

LEMMA 5.1. *For a locally integrable function $w \in L^1_{\text{loc}}(\mathbb{R})$, it holds*

$$\int_{\mathbb{R}} w(s)B_m(ns-j)ds = h(D_h^m w^{(-m)})(jh), \quad h = 1/n, \quad j \in \mathbb{Z}, \tag{5.2}$$

where $w^{(-m)}$ is an integral function of w of order m, i.e., in the sense of distributions, $(d/ds)^m w^{(-m)}(s) = w(s)$, and $D_h^m = (D_h)^m$ is defined by

$$(D_h u)(x) = h^{-1}(u(x+h) - u(x)), \quad (D_h^m u)(x) = h^{-m}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}u(x+kh).$$

Proof. Integrating $m-1$ times by parts and using (2.2)–(2.4) we find that

$$\int_{\mathbb{R}} w(s)B_m(ns-j)ds = (-1)^{m-1}n^{m-1}\int_{\mathbb{R}} w^{-(m-1)}(s)B_m^{(m-1)}(ns-j)ds$$

$$= (-1)^{m-1}n^{m-1}\sum_{l=0}^{m-1}(-1)^l \binom{m-1}{l} \int_{(j+l)h}^{(j+l+1)h} w^{-(m-1)}(s)ds$$

$$= (-1)^{m-1}h^{-m+1}\sum_{l=0}^{m-1}(-1)^l \binom{m-1}{l} h(D_h w^{(-m)})((j+l)h).$$

We get (5.2) since $(-1)^{m-1}h^{-m+1}\sum_{l=0}^{m-1}(-1)^l \binom{m-1}{l} u(x+lh) = (D_h^{m-1}u)(x)$. ▯

Of course, the r.h.s. of (5.2) is independent of the choice of a particular integral function $w^{(-m)}$ since D_h^m nullifies the difference of any two integral functions – those differ by a polynomial of degree $\le m-1$.

LEMMA 5.2. *For $w \in L^1(0,1)$, let $w^{(-m)}(s)$ be its integral function of order m on $[0,1]$; put $w^{(k-m)}(s) = (d/ds)^k w^{(-m)}(s)$, $0 \le s \le 1$, $k = 0, ..., m-1$. Then*

$$\int_0^1 w(s)B_m(ns-j)ds = h(D_h^m \overline{w}^{(-m)})(jh), \quad h = 1/n, \quad j \in \mathbb{Z}, \tag{5.3}$$

where

$$\overline{w}^{(-m)}(s) = \begin{cases} \sum_{k=0}^{m-1} \frac{w^{(k-m)}(0)}{k!} s^k & \text{for } s < 0, \\ w^{(-m)}(s) & \text{for } 0 \le s \le 1, \\ \sum_{k=0}^{m-1} \frac{w^{(k-m)}(1)}{k!}(s-1)^k & \text{for } s > 1. \end{cases}$$

Proof. This immediately follows from (5.2) representing

$$\int_0^1 w(s)B_m(ns-j)ds = \int_{\mathbb{R}} \overline{w}(s)B_m(ns-j)ds$$

where $\overline{w}$ is the extension of w by the zero value outside $(0,1)$. The integral functions of $\overline{w}$ and w can be taken equal on $[0,1]$; for $s<0$ the integral function of $\overline{w}$ must be a polynomial of degree $m-1$, hence the Taylor polynomial of $w^{(-m)}$ with the expansion centre 0; similarly for $s>1$. ▯

To compute $\beta_j^0 = \int_0^1 B_m(ns-j)ds$, Lemma 5.2 can be applied with $\overline{w}^{(-m)}(s) = \frac{1}{m!}s_+^m$ for $-\infty < s \le 1$. According to (5.3),

$$\beta_j^0 = h\frac{1}{m!}\triangle^m\gamma_j^0, \ j = -m+1, ..., n-m+1, \ \gamma_j^0 = \begin{cases} 0, & j<0, \\ j^m, & 0 \le j \le n, \end{cases} \tag{5.4}$$

where $\triangle^m$ is the forward difference operator of order m defined for $(\gamma_j)_{j\in\mathbb{Z}}$ by

$$\Delta\gamma_j = \gamma_{j+1} - \gamma_j, \ \ \triangle^m\gamma_j = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\gamma_{j+k}, \ \ j \in \mathbb{Z}.$$

Actually it is sufficient to compute $\beta_j^0 = h\frac{1}{m!}\triangle^m\gamma_j^0$ by (5.4) only for $j = -m+1, ..., -1$, since $\beta_j^0 = h$ for $0 \le j \le n-m$ and by a symmetry argument $\beta_{n-m+k}^0 = \beta_{-k}^0$, $k = 1, 2, ...$, see (2.2).

For $w_t(s) = |t-s|^{-\nu}$, $t \in (0,1)$, an integral function $w_t^{(-m)}(s)$ on $[0,1]$ is given by

$$\overline{w}^{(-m)}(s) = w_t^{(-m)}(s) = \frac{1}{(1-\nu)...(m-\nu)} \begin{cases} (-1)^m (t-s)^{m-\nu}, & 0 \le s \le t, \\ (s-t)^{m-\nu}, & t \le s \le 1. \end{cases}$$

Respectively,

$$\text{for } s \le 0, \ \ \overline{w}^{(-m)}(s) = \sum_{k=0}^{m-1} \frac{1}{k!} \frac{(-1)^{m-k} t^{m-k-\nu}}{(1-\nu)...(m-k-\nu)} s^k,$$

$$\text{for } s \ge 1, \ \ \overline{w}^{(-m)}(s) = \sum_{k=0}^{m-1} \frac{1}{k!} \frac{(1-t)^{m-k-\nu}}{(1-\nu)...(m-k-\nu)} (s-1)^k.$$

For $\beta_{i,j} = \int_0^1 |(i+\frac{m}{2})h - s|^{-\nu} B_m(ns-j)ds$, Lemma 5.2 yields the formula

$$\beta_{i,j} = h^{1-\nu} \triangle_j^m \gamma_{i,j}, \ \ i = -m_0, ..., n-m_1, \ \ j = -m+1, ..., n-1, \tag{5.5}$$

where $\triangle^m$ is still applied to $(\gamma_{i,j})$ with respect to j and

$$\gamma_{i,j} = \begin{cases} \sum_{k=0}^{m-1} \frac{(-1)^{m-k}}{k!(1-\nu)...(m-k-\nu)} (i+\frac{m}{2})^{m-k-\nu} j^k, \ \ j < 0, \\ \frac{(-1)^m}{(1-\nu)...(m-\nu)} (i-j+\frac{m}{2})^{m-\nu}, \ \ 0 \le j < i + \frac{m}{2}, \\ \frac{1}{(1-\nu)...(m-\nu)} (j-i-\frac{m}{2})^{m-\nu}, \ \ i + \frac{m}{2} \le j \le n, \\ \sum_{k=0}^{m-1} \frac{1}{k!(1-\nu)...(m-k-\nu)} (n-i-\frac{m}{2})^{m-k-\nu} (j-n)^k, \ \ j > n. \end{cases} \tag{5.6}$$

There are some symmetries which enable to reduce the computation cost of (5.5)–(5.6): due to (2.2), for $\widetilde{\beta}_{i,j} := \int_{\mathbb{R}} |(i+\frac{m}{2})h-s|^{-\nu} B_m(ns-j)ds$ we have $\widetilde{\beta}_{i,j} = \widetilde{\beta}_{0,|i-j|}$, $i,j \in \mathbb{Z}$, and $\beta_{i,j} = \widetilde{\beta}_{i,j} = \widetilde{\beta}_{0,|i-j|}$ for $i = -m_0, ..., n-m_1$, $j = 0, ..., n-m$.

One must be careful using exact formulae like (5.5)–(5.6): the differences $\triangle^m \gamma_{i,j}$ may cause a loss of accuracy in a standard floating point arithmetic. For instance, for $j - i \ge m$ we have

$$\triangle^m (j-i-\frac{m}{2})^{m-\nu} = (\frac{d}{dz})^m z^{m-\nu}|_{z=\xi} = (1-\nu)...(m-\nu)\xi^{-\nu}$$

where ξ is a point from the interval $(j-i-\frac{m}{2}, j-i+\frac{m}{2})$, thus $\triangle^m(j-i-\frac{m}{2})^{m-\nu}$ is approximately $(j-i)^m/[(1-\nu)...(m-\nu)]$ times smaller than $(j-i-\frac{m}{2})^{m-\nu}$ itself. To avoid the loss of accuracy, $\gamma_{i,j}$ and their differences should be presented with sufficiently long mantissas. A simple rule is as follows: in case of single precision computations, the 7 decimal digits of $\triangle^m \gamma_{i,j}$ are correct if $\gamma_{i,j}$ and their differences are computed with the double precision and $n^m \le 10^7 \cdot [(1-\nu)...(m-\nu)]$; in case of double precision computations, the 15 decimal digits of $\triangle^m \gamma_{i,j}$ are correct if $\gamma_{i,j}$ and their differences are computed with the quadruple precision and

$$n^m \le 10^{15} \cdot [(1-\nu)...(m-\nu)]. \tag{5.7}$$

Actually, condition (5.7) can be somewhat relaxed, since usually we do not need maximally high precision of $\beta_{i,j}$.

Forgetting exact formulae, $\beta_{i,j}$ can be computed in the framework of a standard arithmetic approximately using suitable quadratures, with a given accuracy; this is more laborious than the use of (5.5)–(5.6). Exact formulae for the quadrature coefficient seem to be a delicate problem also in other methods of Nyström type for weakly singular integral equations. For instance, in [9] an unstable recurrence formula is proposed and successfully used for the computations of the quadrature coefficients in a Gauss type quadrature.

5.3. Solving the system. Due to (4.8) and (4.13), the condition number of the matrix $(\delta_{i,k}-\tau_{i,k})_{i,k=-m_0}^{n-m_1}$ of system (4.20) is bounded uniformly in n, therefore standard methods are expected to work well when solving (4.20). The Gauss elimination with its $O(n^3)$ operations can be recommended for moderate n. For large n, iteration methods with stopping when the discrepancy achieves a value δn^{-r}, $\delta>0$, $r\geq m$, are preferable. The two grid iteration method with a fixed coarse level [2] is of complexity $O(n^2\log n)$ flops; with suitable enlarging strategy for the dimension of the coarse level, the complexity can be reduced to $O(n^2)$ flops, cf. [20]; here n^2 is the cost of one matrix to vector multiplication. The complexity of GMRES is $o(n^2\log n)$ flops due to (4.6), (4.7), see [19]. In our numerical computations treated in Section 6, rarely more than 20 GMRES iterations were needed to achieve the discrepancy level 10^{-14}–10^{-16}.

5.4. Approximate solution of equation (1.1). The solutions of equations (1.1) and (3.4) are in the relation $v(t)=u(\varphi(t))$, or $u(x)=v(\varphi^{-1}(x))$, where $\varphi^{-1}:[0,1]\to[0,1]$ is the inverse function of φ defined in (3.2)/(3.3). Having solved the system (4.20), $v_{i,n}=v_n((i+\frac{m}{2})h)$ are approximations to $u(x_{i,n})$ at $x_{i,n}=\varphi((i+\frac{m}{2})h)$, $0<(i+\frac{m}{2})h<1$, and a local polynomial interpolation of degree $m-1$ can be used to obtain an approximation for $u(x)$ between the knots. A more precise way is to determine $t=\varphi^{-1}(x)$ and put $\tilde u_n(x)=\widetilde v_n(\varphi^{-1}(x))$, see Section 4.3. Under conditions of Theorem 4.2(ii), we have the uniform error bound

$$\max_{0\leq x\leq 1}|u(x)-\tilde u_n(x)|=\max_{0\leq t\leq 1}|v(t)-\widetilde v_n(t)|\leq ch^m \parallel f\parallel_{C^{m,\theta_0,\theta_1}(0,1)};$$

under conditions of Theorem 4.2(iii) we have the weighted error bound

$$\max_{0\leq x\leq 1}x^{\nu(r_0-1)/r_0}(1-x)^{\nu(r_1-1)/r_1}|u(x)-\tilde u_n(x)|$$

$$\leq c'\max_{0\leq t\leq 1}\varphi'(t)^{\nu}|v(t)-\widetilde v_n(t)|\leq ch^m\parallel f\parallel_{C^{m,\theta_0,\theta_1}(0,1)}.$$

In general φ is too complicated to present a closed formula for $t=\varphi^{-1}(x)$. For given $x\in(0,1)$, $t=\varphi^{-1}(x)$ can be approximated by the Newton method

$$t_k=t_{k-1}-(\varphi(t_{k-1})-x)/\varphi'(t_{k-1}),\ \ k=1,2,...;$$

note that according to (3.2) $\varphi'(t)=(1/c_\star)t^{r_0-1}(1-t)^{r_1-1}$. For any $x\in(0,1)$, starting from the initial guess $t_0=(r_0-1)/(r_0+r_1-2)$ which is the only flex point of φ for $r_0,r_1>1$, the iterations converge monotonically to $t=\varphi^{-1}(x)$. For x close to 0 or 1, the initial guess $t_0=(c_\star r_0x)^{1/r_0}$, respectively, $t_0=1-(c_\star r_1(1-x))^{1/r_1}$ may be preferable.

6. Numerical example. For the testing of the algorithms we took a simple equation (1.1) with $\nu=1/2$:

$$u(x)=\int_0^1|x-y|^{-1/2}u(y)dy+f(x),\ \ 0\leq x\leq 1.$$

We put $u(x)=1+x^{1/2}+(1-x)^{1/2}$ to be the exact solution; it corresponds to $f(x)=1-\frac{\pi}{2}-2x^{1/2}-2(1-x)^{1/2}-x\log(1+(1-x)^{1/2})-(1-x)\log(1+x^{1/2})+\frac{1}{2}x\log x+\frac{1}{2}(1-x)\log(1-x)$. We composed system (4.20), (4.21) for $m=3,4,...,10$, $n=2^k$ with $k=4,5,...,12$, and the smoothing parameters $r_0=r_1=:r\in\mathbb{N}$ with $2\leq r\leq 2m+1$; we solved the system by GMRES. The double precision was used everywhere except in the computation of $\beta_{i,j}$ by formulae (5.5)–(5.6) where the quadruple precision was involved (see Section 5.2). The errors

$$\varepsilon_{m,n,r}:=\max_{-m_0\leq i\leq n-m_1}|v((i+\frac{m}{2})h)-v_{i,n}|,$$

$$\bar{\varepsilon}_{m,n,r} := \max_{-m_0 \le i \le n-m_1} [\varphi'((i+\frac{m}{2})h)]^{1/2} |v((i+\frac{m}{2})h) - v_{i,n}|$$

are presented in Tables 6.1 and 6.2 for $m = 4$, different n and r. The error $\varepsilon_{m,n,r}$ for $m = 6$, different r is presented graphically on Figure 6.1 as a function of $\log_2 n$. On Figure 6.2, the error $\varepsilon_{m,n,2m}$ is presented graphically for different m. The numerical results are in a good accordance with Theorem 4.1 which in the present case ($\nu = 1/2$, $\lambda_0 = \lambda_1 = 0$, $b(x,y) \equiv 0$) claims that (i) $\varepsilon_{m,n,r} \to 0$ as $n \to \infty$ for $r > 1$; (ii) $\varepsilon_{m,n,r} = O(n^{-m})$ for $r > 2m$; (iii) $\bar{\varepsilon}_{m,n,r} = O(n^{-m})$ for $r > m + \frac{1}{2}$. We can see from Tables 6.1 and 6.2 that in the reality, the best results for $m = 4$ in the uniform norm are obtained for $r = 2m = 8$, whereas in the weighted norm, the best results are obtained for $r = m + 1 = 5$; for the uniform norm, the empirical convergence order is $\varepsilon_{4,n,r} = O(n^{-4})$ for $r \ge 8$ and lower for $r \le 7$; for the weighted norm, the empirical convergence order is $\bar{\varepsilon}_{4,n,r} = O(n^{-4})$ for $r \ge 5$ and lower for $m \le 4$; on the other hand, the logarithmic factor of estimate (4.14) for $r = 2m = 8$ cannot be observed in the present example.

For $r > 2m$ recommended by Theorem 4.1(ii), and also for $r = 2m$, the computations occurred to be rather stable in a wide scale of n, see Figure 6.2. A side remark is that the numerical results presented on Figure 6.2 encourage to use high order splines in computations: in the present example, already beginning from $n = 32$, the use of tenth order splines gave more precise numerical results than the use of splines of lower order.

$n \setminus r$	2	4	6	7	8	9
16	9.145e-02	1.036e-02	4.228e-03	5.434e-03	8.920e-03	1.322e-02
32	3.269e-02	2.318e-03	2.994e-04	2.786e-04	3.003e-04	4.377e-04
64	1.335e-02	5.648e-04	3.382e-05	1.843e-05	1.466e-05	1.930e-05
128	5.950e-03	1.407e-04	4.005e-06	1.353e-06	8.458e-07	1.113e-06
256	2.794e-03	3.521e-05	4.842e-07	1.085e-07	5.206e-08	6.994e-08
512	1.350e-03	8.812e-06	5.936e-08	8.876e-09	3.321e-09	4.460e-09
1024	6.632e-04	2.204e-06	7.339e-09	7.372e-10	2.114e-10	2.839e-10
2048	3.285e-04	5.513e-07	9.121e-10	6.218e-11	1.341e-11	1.809e-11
4096	1.634e-04	1.379e-07	1.137e-10	5.709e-12	8.606e-13	1.163e-12

Table 6.1: Uniform errors $\varepsilon_{4,n,r}$

$n \setminus r$	2	3	4	5	6	7
16	5.422e-02	9.333e-03	2.967e-03	1.093e-03	3.582e-03	8.021e-03
32	1.393e-02	1.591e-03	1.872e-04	1.868e-05	1.356e-04	3.039e-04
64	4.055e-03	2.203e-04	1.004e-05	1.746e-06	8.275e-06	1.540e-05
128	1.283e-03	2.865e-05	5.313e-07	1.788e-07	5.478e-07	9.365e-07
256	4.268e-04	3.640e-06	2.924e-08	1.437e-08	3.566e-08	5.952e-08
512	1.460e-04	4.581e-07	1.684e-09	1.019e-09	2.287e-09	3.799e-09
1024	5.074e-05	5.744e-08	1.004e-10	6.794e-11	1.454e-10	2.414e-10
2048	1.777e-05	7.196e-09	6.101e-12	4.372e-12	9.160e-12	1.519e-11
4096	6.254e-06	9.164e-10	4.716e-13	3.281e-13	6.533e-13	9.770e-13

Table 6.2: Weighted errors $\bar{\varepsilon}_{4,n,r}$

For $m = 4$ and fixed $r \ge 2$, we can see from Table 6.1 a monotone decrease of errors $\varepsilon_{4,n,r}$ in n. For large m but r significantly smaller than $2m$, the error $\varepsilon_{m,n,r}$ turns to grow beginning from some (relatively large) $n = n_{m,r}$, see Figure 6.1 ($m = 6$). We are in difficulties trying to explain this kind of instability for small r ($r = 2$ on Figure 6.1 for $m = 6$). Condition (5.7) guaranteeing a safe use

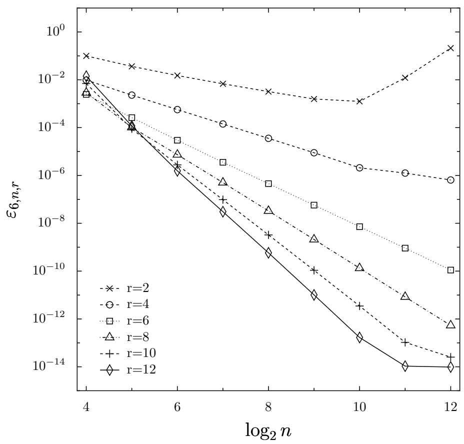

Figure 6.1: Errors $\varepsilon_{6,n,r}$

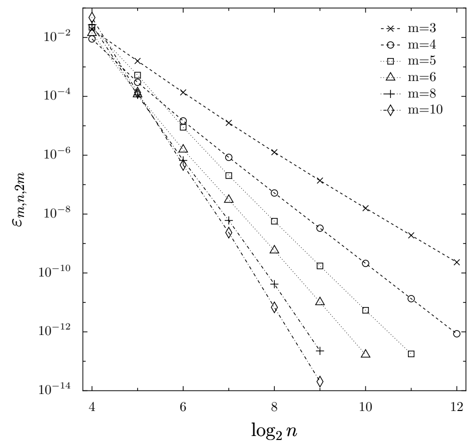

Figure 6.2: Errors $\varepsilon_{m,n,2m}$

of the quadruple precision in formulae (5.5)–(5.6) is violated for $m = 6$, $n \geq 739$. Nevertheless, when the preciseness of the computations in (5.5)–(5.6) was increased, no essential change in the behaviour of $\varepsilon_{6,n,r}$ has been observed compared with Figure 6.1. Hence the instability for large m, n, small r cannot be explained by the violation of (5.7) only.

All programs for given computations were written using the SciPy package (Scientific Tools for Python, `http://www.scipy.org`). For carrying out higher precision floating point arithmetics, a Python interface to a Fortran95 subroutine was used, which uses the `quadruple precision` datatype for internal calculations. As an alternative to Fortran95, we used also Python Decimal datatype for higher precision calculations. With the Decimal type there is a performance penalty compared to Frotran95 calculations. But the flexibility of specifying an arbitrary length of the mantissa in floating point calculations proved to be very helpful in finding out the sensitivity of our algorithms to precision changes.

The source code and numerical results in more detail of provided tests can be found at the address `http://www.ut.ee/~eero/WSIE/Fredholm/` .

Further prospects. We solved the equation $\lambda u(x) = \int_0^1 \mid x - y \mid^{-\nu} u(y)du + f(x)$ numerically also for some other values of $\nu \in (0,1)$. For small and moderate ν, the method behaves as expected by Theorem 4.1. For ν close to 1, the uniform accuracy becomes lower due to great values of the smoothing parameter $r \sim m/(1-\nu)$, see Theorem 4.1(ii).

Returning to general problem (1.1), it is possible to modify our approach towards smaller values of the smoothing parameters r_0 and r_1. In the arguments of Section 4 we exploited the fact that the smoothing change of variables suppresses the coefficient function by $\mid t - s \mid^{-\nu}$ but since $v(0) \neq 0$, $v(1) \neq 0$ in general, we were not able to use the fact that also the derivatives of the solution v of (3.4) are suppressed if $f \in C^{m,\theta_0,\theta_1}(0,1)$ with $\theta_0 < 1$, $\theta_1 < 1$. The situation changes if $u(0) = u(1) = 0$ for the solution of (1.1), then also $v(0) = v(1) = 0$. The idea of the modification is to rewrite equation (1.1) w.r.t. unknowns $u(0)$, $u(1)$ and $U(x) = u(x) - u(0) - (u(1) - u(0))\psi(x)$ where ψ is a function of type (3.2), achieving $V^{(j)}(0) = V^{(j)}(1) = 0$, $j = 0, ..., m$, for $V(t) = U(\varphi(t)$. In the algorithm and its justification, some new problems arrive. We cannot go into details here because of the length of the paper but we will return to the question elsewhere.

Acknowledgement. The authors are grateful to the referees for the useful discussion and valuable comments that enabled to improve some details of the paper.

REFERENCES

[1] P. M. Anselone, *Collectively Compact Operator Approximation Theory,* Prentice-Hall, New Jersey, 1971.

[2] K. E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind,* Cambridge University Press, Cambridge, 1997.

[3] C. de Boor, *A practical Guide to Splines*, Springer, New York, 2001.

[4] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Equations*, Cambridge University Press, 2004.

[5] W. Hackbusch, *Integral Equations,* Birkhäuser, Basel, 1995.

[6] N. P. Korneychuk, *Splines in the Approximation Theory*, Nauka, Moscow, 1984 (in Russian).

[7] R. Kress, *Linear Integral Equations*, Springer, Berlin, 1989.

[8] E. Leetma, G. Vainikko, Quasi-interpolation by splines on the uniform knot sets, *Math. Mod. Anal.* 12 (2007), 107–120.

[9] G. Monegato, L. Scuderi, High order methods for weakly singular integral equations with nonsmooth input functions. *Math. Comp.* 67 (1998), 1493–1515.

[10] A. Pedas, G. Vainikko, Smoothing transformation and piecewise polynomial collocation for weakly singular integral equations, *Computing* 73 (2004), 271–293.

[11] A. Pedas, G. Vainikko, Integral equations with diagonal and boundary singularities of the kernel, *ZAA* 25 (2006), 487–516.

[12] A.Pedas, G. Vainikko, On the regularity of solutions to integral equations with nonsmooth kernels (submitted).

[13] C. Schneider, Product integration for weakly singular integral equations, *Math. Comp.* 36 (1981), 207–213.

[14] I. J. Schoenberg, *Cardinal Interpolation*, SIAM, Philadelphia, 1973.

[15] L. L. Schumaker, *Spline Functions*: *Basic Theory*, Krieger Publ., Malabar, Florida, 1993.

[16] S. B. Stechkin, Yu. N. Subbotin, *Splines in Numerical Mathematics*, Nauka, Moscow, 1976 (in Russian).

[17] E. Vainikko, G. Vainikko, A product quasi-interpolation method for weakly singular Volterra integral equations, *AIP Conference Proceedings* 936 (2007), Numerical Analysis and Applied Mathematics, ed T. E. Simos, 570–573.

[18] G. Vainikko, *Funktionalanalysis der Diskretisierungsmethoden*, Teubner, Leipzig, 1976.

[19] G. Vainikko, GMRES and discrete approximation of operators, *Proc. Estonian Acad Sci. Phys. Math.* 53 (2004), 124–131.

[20] G. Vainikko, Fast solvers of integral equations of the second kind: quadrature methods, *J. Integr. Eq. Appl.* 17 (2005), 91–120.

[21] G. Vainikko, On the best approximation of function classes from values on a uniform grid in the real line, *WSEAS Transactions on Math.* 4/6 (2006), 523–528.

[22] G. Vainikko, Error estimates for the cardinal spline interpolation, *ZAA* (to appear).