A Product Quasi-Interpolation Method for Weakly Singular Volterra Integral Equations

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Abstract. For a weakly singular Volterra integral equation, we propose a method of Nystrom type of accuracy $O(h^m)$ based on the smoothing change of variables and on the product quasi-interpolation by smooth splines of degree $m-1$ on the uniform grid.

Keywords: Volterra integral equation, weak singularities, spline quasi-interpolation, product integration, Nystrom type methods.

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1. INTRODUCTION

Different methods of Nystrom type for weakly singular Volterra and Fredholm integral equations have been constructed in [1], [2], [4]. In the present paper, we propose for a weakly singular Volterra integral equation a method of Nystrom type of accuracy $O(h^m)$ based on the smoothing change of variables and on the product quasi-interpolation by smooth splines of degree $m-1$ on the uniform grid. Similar method for weakly singular Fredholm equations has been developed in [5].

2. THE PROBLEM

Consider the weakly singular Volterra integral equation

$$u(x) = \int_0^x \left( a(x,y)(x-y)^{-\nu} + b(x,y) \right) u(y)dy + f(x), \quad 0 \leq x \leq 1,$$

where $0 < \nu < 1$, $a$ and $b$ are defined and $C^m$-smooth for $0 < x \leq 1$, $0 < y \leq x + \delta$, $\delta > 0$, $m \in \mathbb{N}$, and satisfy there for $k + l \leq m$ the inequalities

$$|\frac{\partial^k}{\partial y^k}a(x,y)| \leq c_{\nu} y^{-\nu - 1}, \quad |\frac{\partial^k}{\partial y^k}b(x,y)| \leq c_{\nu} y^{-\nu - l}, \quad \nu + l < 1, \quad \mu < 1.$$  

(2)

With the change of variables

$$x = r^\xi, \quad y = s^\xi, \quad 0 \leq t \leq 1, \quad 0 \leq s \leq t + \delta, \quad r \in \mathbb{N}, \quad (1 + \delta^\xi) = 1 + \delta,$$

(3)

equation (1) takes with respect to $v(t) = u(r^\xi)$ the form

$$v(t) = \int_0^t \left( \mathscr{A}(t,s)(t-s)^{-\nu} + \mathscr{B}(t,s) \right) v(s)ds + g(t), \quad 0 \leq t \leq 1,$$

(4)

which is similar to (1). Here

$$g(t) = f(t^\xi), \quad \mathscr{A}(t,s) = r^a(t^\xi,s^\xi) \Phi(t,s)^{-\nu}s^{\nu - 1}, \quad \mathscr{B}(t,s) = rb(t^\xi,s^\xi)s^{\nu - 1},$$

$$\Phi(t,s) = \left\{ \begin{array}{ll} t = s, & \text{if } s \neq s \in \mathbb{N}, \\ t \neq s \in \mathbb{N}, & \text{if } i = i \in \mathbb{N}, \end{array} \right. = \sum_{i=0}^{r-1} r^{r-1-k}s^k, \quad 0 \leq t \leq 1, \quad 0 < s \leq t + \delta.$$

We assume that the smoothing parameter $r \in \mathbb{N}$ satisfies the inequalities

$$r > (1 - \nu)/(1 - \nu - \lambda), \quad r > 1/(1 - \mu).$$  

(5)

Then $\mathscr{A}(t,s) \to 0$, $\mathscr{B}(t,s) \to 0$ as $s \to 0$, $0 \leq t \leq 1$. Extending $\mathscr{A}(t,s)$ and $\mathscr{B}(t,s)$ by the zero value for $s \leq 0$, the extended $\mathscr{A}(t,s)$ and $\mathscr{B}(t,s)$ are continuous for $0 \leq t \leq 1$, $-\infty < s \leq t + \delta$.  

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3. OPERATOR FORM OF THE METHOD

Let \( h = 1 / n, n \in \mathbb{N}, n \geq (m - 1) / \delta \). We call attention to a product quasi-interpolation method which we first present in the operator form:

\[
v_n(t) = \int_0^t (t - s)^{-1} Q_{h,m}(\gamma(t,s)v_n(s)) + Q_{h,m}(\gamma(t,s)v_n(s))|ds + g(t), \quad 0 < t < 1,
\]

\[
v_n(t) = (\Lambda_n v_n)(t) \quad \text{for } 1 \leq t \leq 1 + (m - 1)h.
\]

Here \( \Lambda_n v \) is the Lagrange interpolant of \( v \) by polynomials of degree \( m - 1 \) constructed in case of even \( m \), the knots \( l - jh, j = 0, \ldots, m-1 \), whereas \( Q_{h,m} w \) is the quasi-interpolant of \( w \) by polynomial splines of degree \( m - 1 \geq 2 \), defect 1, with spline knots \( jh, j \geq -m + 1 \) constructed in [3]. Namely, for a function \( w(s), s \in [- (m - 1)h, (n + 1)(m - 1))h \], depending on \( t, 0 < t \leq 1 \), as a parameter, the quasi-interpolant \( Q_{h,m} w \) is defined for \( s \in [0, t] \) by the formula:

\[
(Q_{h,m} w)(s) = \sum_{j=-m+1}^{[nt]-1} \left( \sum_{p=|m|}^{m-1} \alpha'_{p,m} w((j + p + m/2)h) \right) B_m(ns - f),
\]

where \([nt]\) is the smallest integer \( \geq nt \),

\[
m_1 = \begin{cases} \frac{m}{2} + 1, & m \text{ even } \\ \frac{m+1}{2}, & m \text{ odd } \end{cases}, \quad m_0 = \begin{cases} \frac{m}{2} - 1, & m \text{ even } \\ \frac{m-1}{2}, & m \text{ odd } \end{cases},
\]

\[
B_m(x) = \frac{1}{(m-1)!} \sum_{i=0}^{m} (-1)^i \binom{m}{i} (x - i)^{m-1}, \quad x \in \mathbb{R}, \text{ is the father B-spline,}
\]

\[
\alpha'_{p,m} = \sum_{q=|p|}^{m-1} (-1)^{k+q} \binom{2q}{k+q} \gamma_{q,m}, \quad |p| \leq m_1 - 1,
\]

\[
\gamma_{0,m} = 1, \quad \gamma_{q,m} = \sum_{l=1}^{m_0} \frac{(1 + z_{l,m})^{q+1}}{(1 - z_{l,m})^{2q+1} P_m(z_{l,m})}, \quad q \geq 1,
\]

\( z_{l,m} \in (-1,0), l = 1, \ldots, m_0, \) are roots of the characteristic polynomial \( P_m(z) = \sum_{k=0}^{m_0} B_m(k + m/2)z^k + m_0^m \) (they are simple; \( 1/z_{l,m} \in (-\infty, -1), l = 1, \ldots, m_0, \) are the other \( m_0 \) roots of \( P_m \in \mathcal{P}_{2m_0} \)).

4. MATRIX FORM OF THE METHOD

Note that \( v_n(0) = g(0) = f(0) \). The solution \( v_n \) of problem (6)-(7) is uniquely determined on \([0, 1]\) by the knot values \( v_n((i+j)h) \) for \( 0 < (i+j)h \leq 1 \). Collocating (6) at these points, the matrix form of method (6)-(7) follows. For even \( m \), we obtain with respect to \( v_{i,n} := v_n(ih), i = 1, \ldots, n + m, \) the system of linear equations

\[
v_{i,n} = \sum_{k=1}^{i+m-1} \tau_{i,k} v_{k,n} + g(\delta h), \quad i = 1, \ldots, n, \quad v_{i,n} = \sum_{j=0}^{m-1} \sigma_{i,j} v_{n-j,n}, \quad i = n+1, \ldots, n+m-1,
\]

where

\[
\sigma_{i,j} = \prod_{j' \neq j}^{m-1} \frac{f' + (i-n)}{j' - j}, \quad i = n+1, \ldots, n+m-1, \quad j = 0, \ldots, m-1,
\]

\[
\tau_{i,k} = a_{i,k} \sum_{j=k-m}^{\min{(i-k-1)}} \beta_{i,j} \alpha'_{j-k+m/2,m} + b_{i,k} \sum_{j=k-m}^{\min{(i-k-1)}} \beta_{i,j} \alpha'_{j-k+m/2,m}, \quad i = 1, \ldots, n, \quad k = 1, \ldots, n+m-1,
\]

\[
a_{i,k} = \gamma(\delta h, kh), \quad b_{i,k} = \gamma(\delta h, kh), \quad i = 1, \ldots, n, \quad k = 1, \ldots, n+m-1,
\]

\[
\beta_{i,j} = \int_0^\delta (\delta h-s)^{\gamma} B_m(ns - f) ds, \quad \beta_{i,j} = \int_0^\delta B_m(ns - f) ds, \quad i = 1, \ldots, n, \quad j = -m + 1, \ldots, i-1.
\]

The unknowns \( v_{i,n}, i = n+1, \ldots, n+m, \) can be eliminated from system (8).
5. FORMULAE FOR QUADRATURE COEFFICIENTS (9)

Again for even \( m \geq 3 \),

\[
\begin{align*}
\beta_{0,j}^0 &= \frac{h}{m!} \Delta^m \gamma^0_{0,j}, \\
\beta_{i,j}^0 &= h^{1-v} \Delta^m \gamma^0_{i,j}, \quad i = 1, \ldots, n, \quad j = -m + 1, \ldots, l - 1,
\end{align*}
\]

where \( \Delta^m \) is the forward difference of order \( m \), \( \Delta^m \gamma^0_{j,j+1} - \gamma^0_{j,j} \),

\[
\gamma^0_{0,j} = (j - l)^m - f^m, \quad \gamma^0_{j,j} = \sum_{k=0}^{m-1} \frac{(1)^{m-k} m-v-k}{k!(1-v) \ldots (m-k-v)} j^k \quad \text{for} \quad j = -m + 1, \ldots, -1,
\]

\[
\gamma^0_{j,j} = (j - l)^m, \quad \gamma^0_{j,j} = \frac{(1)^m}{(1-v) \ldots (m-v)} (i - f)^{m-v} \quad \text{for} \quad 0 \leq j \leq i - 1,
\]

\[
\gamma^0_{j,j} = \gamma^0_{i,j} = 0 \quad \text{for} \quad j \geq i.
\]

There are some symmetries for \( \beta_{i,j}^0 \) and \( \beta^0_{0,j} \); it holds \( \beta^0_{0,j} = h \) for \( 0 \leq j \leq i - m \).

6. CONVERGENCE AND ERROR ESTIMATES

Having solved system (8) we can use the Nyström extension to compute the solution \( v_n(t) \) of problem (6)-(7) for all \( t \in [0,1] \); a cheaper extension \( \bar{v}_n(t) \) can be constructed quasi-interpolating by splines of degree \( m - 1 \) the solution of system (8) completed by \( v_n = f(0) \) for \( i = -m + 1, \ldots, -1 \). Introduce the space

\[
C^m_w(0,1] = \{ f \in C[0,1] \cap C^m(0,1] : \| f^{(k)}(x) \| \leq c_j x^{-k}, \quad 0 < x < 1, \quad k = 0, \ldots, m \}.
\]

the smallest constant \( c_j \) defines the norm \( \| f \|_{C^m_w(0,1]} \).

**Theorem 1.**

(i) If \( f \in C[0,1] \), the functions \( a, b \) are continuous and satisfy (2) for \( k = l = 0 \), and \( r \in \mathbb{N} \) satisfies (5), then

\[
\max_{0 \leq t \leq 1} | v(t) - v_n(t) | \to 0 \quad \text{as} \quad n \to \infty \quad \text{where} \quad v \quad \text{and} \quad v_n \quad \text{are the solutions of (4) and (6)-(7), respectively.}
\]

(ii) If \( f \in C^m_w(0,1] \), the functions \( a, b \) are \( C^m \)-smooth for \( 0 \leq x \leq 1, 0 < y < x + \delta \) and satisfy (2) for \( k+l \leq m \), and \( r \in \mathbb{N} \) satisfies the inequalities \( r > (m + v)/(1 - \lambda), r > m/(1 - \mu) \), then

\[
\delta_{m,n,r} := \max_{0 \leq t \leq 1} f^{(r-1)v} | v(t) - v_n(t) | \leq c_{a,b,m,v,\lambda,\mu,r} h^m \| f \|_{C^m_w(0,1]}.
\]

(iii) Under the same conditions on \( f, a, b \) as in (ii) but \( r > m/(1 - v - \lambda), r > m/(1 - \mu) \), it holds

\[
\varepsilon_{m,n,r} := \max_{0 \leq t \leq 1} | v(t) - v_n(t) | \leq c_{a,b,m,v,\lambda,\mu,r} h^m \| f \|_{C^m_w(0,1]}.
\]

**Proof.** The proof is based on the compact convergence of operators and on the error estimates of quasi-interpolation established in [3].

**Remark 1.** Claim (i) is true also for \( \bar{v}_n \); error estimates like in (i) and (ii) hold for \( \bar{v}_n \) under a slightly strengthened condition on \( f \in C[0,1] \cap C^m(0,1] \).

**Remark 2.** If \( f(0) = 0 \), the first condition on \( r \) in (ii) and (iii) can be relaxed.

7. SOME EXTENSIONS OF THE CONSIDERATIONS

The results of Sections 2–6 have been extended in the following directions:

- in cases \( m = 1 \) and \( m = 2 \), the algorithms have a special treatment;
- in the case of odd \( m \geq 3 \), the algorithms are similar to those in Sections 4–5;
- equations with logarithmic diagonal singularity of the kernel are treated;
- the case of \( a \) and \( b \) in (1) given only for \( 0 \leq s \leq t \leq 1 \) is treated.
8. NUMERICAL TESTING

Method (6)-(7) and its modifications were tested numerically on the equation (1) with \( v = 1/2, \ a = 1, \ b = 0, \ f(x) = 1 - x^{1/2} - \frac{5}{2}x; \) the exact solution is then \( u(x) = 1 + x^{1/2}. \) About numerical results in the case of Fredholm equation, see [5].

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