In this note, we will model the partially polarizing beam splitter (PPBS) (with the surrounding half-wave plates) in the qutrit SIC storage loop experiment [1] as a binary weak measurement with operators \( \{ M_0, M_1 \} \).

From [2], we know that we can define a weak measurement operator

\[
M_0 = \alpha (I + \epsilon \Pi), \quad |\phi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},
\]

where \( 0 \leq \alpha \leq 1 \), \( \Pi \) is the projection onto the SIC fiducial state \( |\phi\rangle \), and \( \epsilon \ll 1 \) implies that the measurement is weak. \( M_0 \) would be the operator for the outcome when the detector clicks.

Since \( M_0^\dagger = M_0 \), the corresponding POVM operators are given by

\[
E_0 = M_0^2 = \alpha^2 \left( I + (2\epsilon + \epsilon^2)\Pi \right),
E_1 = I - E_0 = (1 - \alpha^2)I - \alpha^2 (2\epsilon + \epsilon^2)\Pi.
\]

Since \( E_0 \) and \( E_1 \) have the same form, we can try to find \( M_1 \) with

\[
M_1 = \beta (I + \gamma \Pi),
\]

where \( 0 \leq \beta \leq 1 \). For simplicity, we choose \( \gamma \in \mathbb{R} \) but in general this can be complex-valued.

This leads to

\[
E_1 = M_1^2 = \beta^2 \left( I + (2\gamma + \gamma^2)\Pi \right)
\]

and comparing to the expression above we get the conditions

\[
\beta^2 = 1 - \alpha^2, \quad 2\gamma + \gamma^2 = \frac{\alpha^2}{\alpha^2 - 1} (2\epsilon + \epsilon^2).
\]

Therefore, if we define \( \alpha = \cos \theta, \beta = \sin \theta \) for \( \theta \in \left( 0, \frac{\pi}{2} \right) \), we get

\[
M_0 = \cos \theta (I + \epsilon \Pi),
M_1 = \sin \theta (I + \gamma \Pi), \quad \gamma = -1 + \frac{1}{\sin \theta} \sqrt{1 - \cos^2 \theta (1 + \epsilon)^2},
\]

which is valid as long as \( \cos \theta \leq \frac{1}{1 + \epsilon} \). For example, if \( \theta = \frac{\pi}{4} \), this gives

\[
M_0 = \frac{1}{\sqrt{2}} (I + \epsilon \Pi),
M_1 = \frac{1}{\sqrt{2}} \left[ I + \left( \sqrt{1 - 2\epsilon - \epsilon^2} - 1 \right) \Pi \right] \approx \frac{1}{\sqrt{2}} (I - \epsilon \Pi),
\]

where the approximation holds when \( \epsilon \) is small enough. In general, for small \( \epsilon \) we would get

\[
\gamma \approx -1 + \frac{1}{\sin \theta} \sqrt{1 - \cos^2 \theta (1 + 2\epsilon)} = -1 + \frac{1}{\sin \theta} \sqrt{\sin^2 \theta - 2\epsilon \cos^2 \theta} = -1 + \sqrt{1 - 2\epsilon \cot^2 \theta} \approx -1 + (1 - \epsilon \cot^2 \theta)
\]

Thus, for small enough \( \epsilon \), if the pre-measurement state is \( |\psi\rangle \) we know that the post-measurement state will be roughly proportional to

\[
|\psi\rangle + O(\epsilon)(\Pi|\psi\rangle).
\]
In the experiment, the probability of a detector click is proportional to the reflectivity of the PPBS. If we suppose that $\epsilon = \cos^2 \theta$, which effectively equates the detector-click probability to the reflectivity, then the measurement operators in the experiment would be

\[
M_0 = \sqrt{\epsilon} (I + \epsilon \Pi),
\]

\[
M_1 \approx \sqrt{1 - \epsilon} \left[ I - \left( \frac{\epsilon^2}{1 - \epsilon} \right) \Pi \right].
\] (10)

This shows that when the reflectivity $\epsilon$ is low, then the amount of disturbance added to the pre-measurement state is $O(\epsilon^2)$, so it does not change too much.

References