Data dissemination problem in wireless networks

Ivo Kubjas, Vitaly Skachek

University of Tartu
Tartu, Estonia

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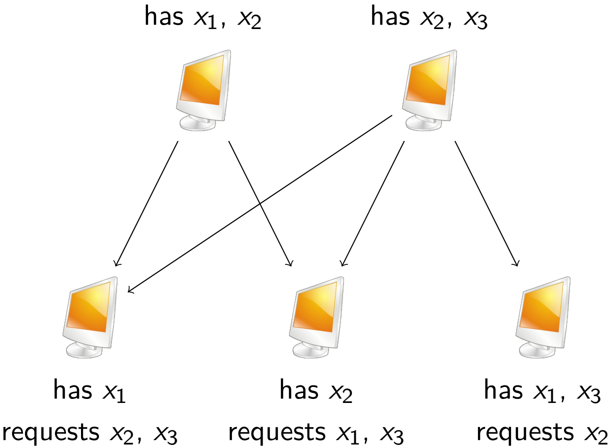
Motivation



has x_1, x_2
has x_2, x_3
has x_1
requests x_2, x_3
has x_2
requests x_1, x_3
has x_1, x_3
requests x_2

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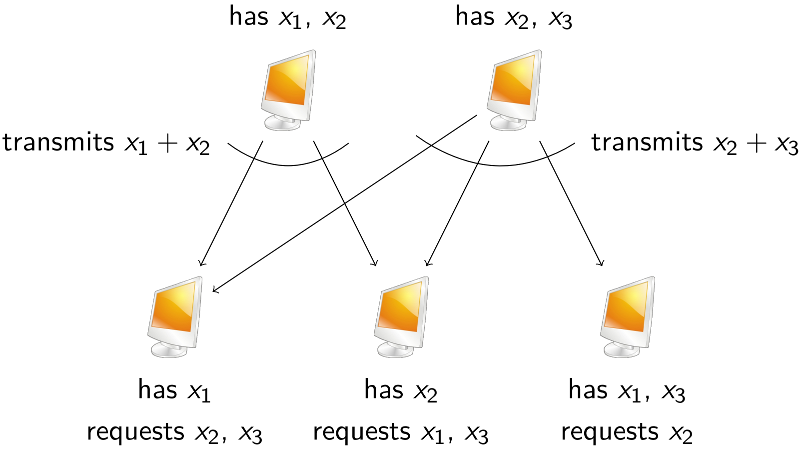
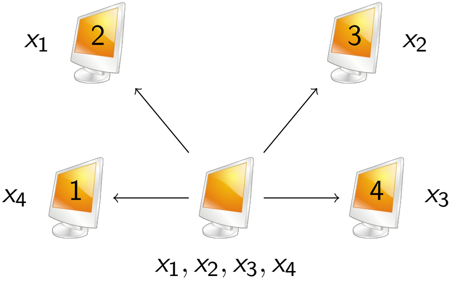


has x_1, x_2
has x_2, x_3
transmits $x_1 + x_2$
transmits $x_2 + x_3$
has x_1
requests x_2, x_3
has x_2
requests x_1, x_3
has x_1, x_3
requests x_2

Index coding

Let $\mathcal{P}_\ell$ be the set of indices of possessed bits by node ℓ. Transmitter wants to transmit linear combination of bits $x_1, \ldots, x_n$ such that each node ℓ recovers bit x_ℓ.



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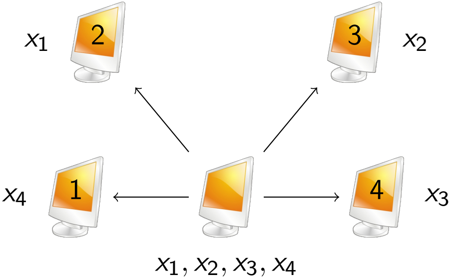

Solution (Bar-Yossef, Birk, Jayram, Kol '06)

Side information graph $\mathcal{H}$ is a graph with an edge from ℓ to j iff $j \in \mathcal{P}_\ell$. Then the optimum number of transmitted bits is $\text{minrank}_2(\mathcal{H})$.

minrank_2

Definition

Let $\mathcal{H}$ be a directed graph of n vertices without self-loops. We say that a $0-1$ matrix $\boldsymbol{A} = (a_{ij})$ fits $\mathcal{H}$ if for all i and j:

- $a_{ii} = 1$,
- $a_{ij} = 0$ whenever (i, j) is not an edge of $\mathcal{H}$.

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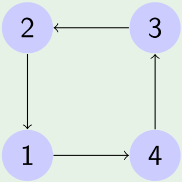
Definition

$\text{minrank}_2(\mathcal{H}) := \min\{\text{rank}_2(\boldsymbol{A}) : \boldsymbol{A} \text{ fits } \mathcal{H}\}.$

Side information graph

Example

The side information graph for graph $\mathcal{H}$ from the previous example is



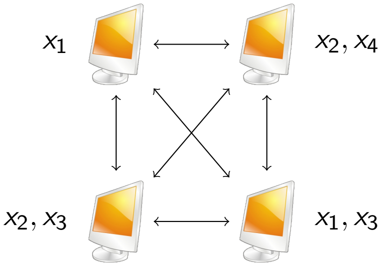
Matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

fits $\mathcal{H}$ and $\operatorname{rank}_2(\boldsymbol{A}) = 3$. Thus, the transmitter needs at least 3 transmissions. The transmissions are $x_1 + x_2$, $x_2 + x_3$ and $x_3 + x_4$.

Data Exchange Protocol

The indices of the bits possessed by the node ℓ are given by $\mathcal{P}_\ell$. Every node wishes to recover all other bits.



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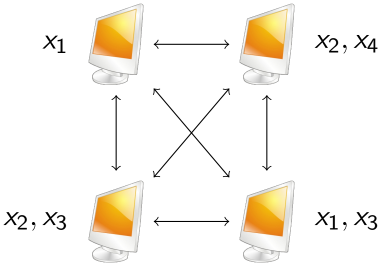

Solution (El Rouayheb, Sprintson, Sadeghi '10)

Let $\mathbb{A}$ be a family of matrices corresponding to the bits the nodes have and $\boldsymbol{P}_\ell$ be a matrix denoting the bits the node ℓ has. Then the number of transmissions is $\tau = \min_{\boldsymbol{A} \in \mathbb{A}} rank(\boldsymbol{A})$ *such that* $\operatorname{rank}\left(\begin{bmatrix} \boldsymbol{A} \\ \hline \boldsymbol{P}_\ell \end{bmatrix}\right) = n, \forall \ell \in \mathcal{V}$.

Definition

For each node ℓ, the possession matrix $\mathbb{A}_\ell$ is a $n \times n$ matrix over $\mathbb{F} \cup \{'\star'\}$, where $'\star'$ is a symbol which can take any value in $\mathbb{F}$. It is defined as

$$(\mathbb{A}_\ell)_{i,j} = \begin{cases} '\star' & \text{if } j \in \mathcal{P}_\ell \\ 0 & \text{otherwise} \end{cases} .$$

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The possession matrix of the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $k = |\mathcal{V}|$, is the $(kn \times n)$-dimensional matrix

$$\mathbb{A} = \begin{bmatrix} \mathbb{A}_1 \\ \hline \mathbb{A}_2 \\ \hline \vdots \\ \hline \mathbb{A}_k \end{bmatrix},$$

where $\mathbb{A}_\ell$ is the possession matrix family corresponding to the node ℓ, $\ell \in \mathcal{V}$.

Given $\boldsymbol{A} \in \mathbb{A}$, the j-th $n \times n$ sub-matrix of $\boldsymbol{A}$ will be denoted as $\boldsymbol{A}_j$.

- For each $\ell \in \mathcal{V}$, the $n \times n$ *information matrix* $\boldsymbol{P}_\ell = (\boldsymbol{P}_\ell)_{i \in [n], j \in [n]}$ is

$$(\boldsymbol{P}_\ell)_{i,j} = \begin{cases} 1 & \text{if } i = j \text{ and } i \in \mathcal{P}_\ell \\ 0 & \text{otherwise} \end{cases} .$$

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- Let $\mathcal{T}_\ell$ be the set of indices of information bits requested by node ℓ. For each $\ell \in \mathcal{V}$, the $n \times n$ *query matrix* $\boldsymbol{T}_\ell = (\boldsymbol{T}_\ell)_{i \in [n], j \in [n]}$ is

$$(\boldsymbol{T}_\ell)_{i,j} = \begin{cases} 1 & \text{if } i = j \text{ and } i \in \mathcal{T}_\ell \\ 0 & \text{otherwise} \end{cases} .$$

- The set of in-neighbours for node ℓ is $\mathcal{N}_{in}(\ell)$ and

$$\boldsymbol{A}_{\mathcal{N}_{in}(\ell)} = \left[\begin{array}{c} \boldsymbol{A}_{i_1} \\ \hline \boldsymbol{A}_{i_2} \\ \hline \vdots \\ \hline \boldsymbol{A}_{i_d} \end{array} \right] ,$$

where $\mathcal{N}_{in}(\ell) = \{i_1, i_2, \cdots, i_d\}$, and d is an in-degree of ℓ in $\mathcal{G}$.

Data dissemination problem

Protocol

for every round $i = 1$ to r **do**
  for every node $\ell \in \mathcal{V}$ **do**
    for $j = 1$ to $\tau_{i,\ell}$ **do** ▷ transmitting phase
      broadcast $z_{i,\ell,j} = \sum_{x \in \mathcal{Q}_\ell^{(i-1)}} \mu_{x,i,\ell,j} \cdot x$
    set $\mathcal{Q}_\ell^{(i)} = \mathcal{Q}_\ell^{(i-1)} \cup \{z_{i,v,j}\}_{v \in \mathcal{N}_{in}(\ell),\, j=1,2,\cdots,\tau_{i,v}}$ ▷ receiving phase
for each node $\ell \in \mathcal{V}$ **do** ▷ recovery phase
  compute $x_j = \sum_{x \in \mathcal{Q}_\ell^{(r)}} \mu_{x,\ell,j} \cdot x$ for all $j \in \mathcal{T}_\ell$

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Definition

The network based on the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be *r-solvable*, $r \in \mathbb{N}$, if it is strongly connected and for any feasible assignment of the sets $\mathcal{P}_\ell$ and $\mathcal{T}_\ell$, $\ell \in \mathcal{V}$, r communications rounds are sufficient for the protocol to satisfy all the node requests, but $r - 1$ rounds are not sufficient. If the network is not r-solvable for any $r \in \mathbb{N}$, then we say that it is *not solvable*.

Theorem

Consider a wireless network defined by the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The minimal number of transmissions needed to satisfy the demands of all nodes ***in one round*** *of communications is*

$$\tau = \min_{\boldsymbol{A} \in \mathbb{A}} \left\{ \sum_{\ell \in \mathcal{V}} \operatorname{rank}\left(\boldsymbol{A}_\ell\right) \right\} ,$$

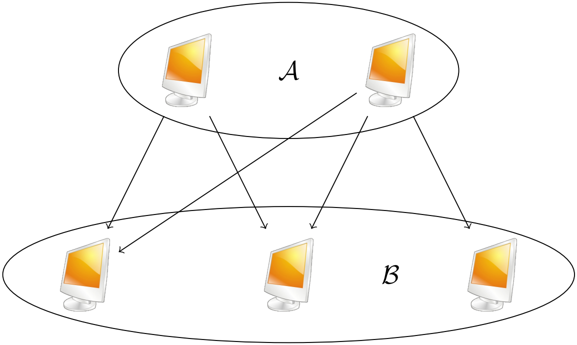
where for all $\ell \in \mathcal{V}$

$$\operatorname{rowspace}\left(\left[\frac{\boldsymbol{A}_{\mathcal{N}_{in}(\ell)}}{\boldsymbol{P}_\ell} \right] \right) \supseteq \operatorname{rowspace}(\boldsymbol{T}_\ell) .$$

If the above matrix $\boldsymbol{A} \in \mathbb{A}$ as above does not exist then there is no algorithm that satisfies all requests in one round.

Graph-theoretic bounds for bipartite networks

- The network is defined by the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \mathcal{A} \cup \mathcal{B}$, $\mathcal{A} \cap \mathcal{B} = \varnothing$ and $\mathcal{E} \subseteq \mathcal{A} \times \mathcal{B}$.
- Nodes in $\mathcal{A}$ are transmitters, i.e. $\mathcal{P}_\ell = [n]$ and $\mathcal{T}_\ell = \varnothing$ for all $\ell \in \mathcal{A}$.
- Nodes in $\mathcal{B}$ are receivers. Assume w.l.o.g. that $\mathcal{B} = [n]$ and $\mathcal{T}_\ell = \{\ell\}$ for all $\ell \in \mathcal{B}$.
- Side information graph $\mathcal{H}$ is a graph with a vertex set $\mathcal{B}$. There is an edge from ℓ to j iff $j \in \mathcal{P}_\ell$.



Graph-theoretic lower bound

Proposition

The optimal number of transmissions for a bipartite data dissemination problem is at least $\mathrm{minrank}_2(\mathcal{H})$, *which is in turn bounded from below by* $\Theta(\mathcal{H})$ *and by* $\alpha(\mathcal{H})$, *where* $\Theta(\mathcal{H})$ *is the Shannon capacity and* $\alpha(\mathcal{H})$ *is the independence number of the graph.*

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Proof idea

- *Merge transmitters into a single transmitter.*
- *Add missing edges from the transmitter to every receiver.*
- *Solve the corresponding index coding problem.*
- *The solution to the initial data dissemination problem is the solution to the new index coding problem.*

Graph-theoretic upper bound

Bipartite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ can be partitioned into disjoint subgraphs $\mathfrak{P} = \{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_t\}$, $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$, where $|\mathcal{A}| = t$ and $|\mathcal{A} \cap \mathcal{V}_i| = 1$ for every $i \in [t]$.

Lemma

For all $\mathcal{G}_i$ as above, $i \in [t]$, consider an induced instance of index coding problem with a transmitter in $\mathcal{A} \cap \mathcal{V}_i$ and the set of the receivers $\mathcal{B} \cap \mathcal{V}_i$. For each $\ell \in \mathcal{V}_i$, the sets $\mathcal{P}_\ell$ and $\mathcal{T}_\ell$ are defined exactly as above. Denote by $\mathcal{H}_i$, $i \in [t]$, the corresponding side information graph. Then, the optimum number of transmissions for the given bipartite data dissemination problem is less or equal to $\sum_{i=1}^{t} \mathrm{minrank}_2(\mathcal{H}_i)$.

Graph-theoretic upper bound

Proof idea

- *Every subgraph $\mathcal{G}_i$ induces an index coding instance.*
- *Solve the corresponding index coding problem for every subgraph $\mathcal{G}_i$.*
- *The solutions to each sub-problem together define a solution to the initial problem.*

Notation

- The matrix $\boldsymbol{I}$ is a $n \times n$ unity matrix.
- The matrix $\boldsymbol{E}$ is a $n \times n$ ones matrix.
- The vector $\boldsymbol{e}_j$ is canonical vector of length n.
- The operator diag is the diagonalization operator.
- The vector $\boldsymbol{D}^{[j]}$ is the j-th row vector of $\boldsymbol{D}$.
- The operator $\Gamma(\cdot)$ replaces the symbols '$\star$' in the maximal number of the first rows with linearly independent canonical vectors, and replaces the symbols '$\star$' in the remaining rows with zeros.
- The operator $\Gamma_\ell(\cdot)$ is defined as $\Gamma_\ell(\mathbb{A}) = \Gamma(\mathbb{A}_\ell)$.
- The operator max-rank$(\cdot)$ returns the rank of the matrix with the maximal rank in the matrix family.

General case

Theorem

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $|\mathcal{V}| = k$, be the underlying directed graph of a r_0-solvable network defined by the adjacency matrix $\boldsymbol{D}^T$. Then there exists an iterated data exchange protocol with r rounds, for any $r \geq r_0$, and τ transmissions, where

$$\tau = \sum_{i=1}^{r} \left(\min_{\boldsymbol{A}^{(i)} \in (\boldsymbol{D}^{i-1} \otimes \boldsymbol{E}) \cdot \mathbb{A}} \left\{ \sum_{\ell \in \mathcal{V}} \operatorname{rank}\left(\boldsymbol{A}_{\ell}^{(i)}\right) \right\} \right)$$

for matrices $\boldsymbol{A}^{(i)}$ which are subject to

$$\forall \ell \in \mathcal{V} : \operatorname{rank}\left(\left[\frac{\left(\operatorname{diag}\left(\boldsymbol{D}^{[\ell]}\right) \otimes \boldsymbol{I}\right) \cdot \boldsymbol{A}^{(i)}}{\Gamma_{\ell}((\boldsymbol{D}^{i-1} \otimes \boldsymbol{E}) \cdot \mathbb{A})} \right] \right) = \text{max-rank}\left((\operatorname{diag}(\boldsymbol{e}_{\ell}) \otimes \boldsymbol{I}) \cdot \left(\boldsymbol{D}^{i} \otimes \boldsymbol{E}\right) \cdot \mathbb{A} \right).$$

Thanks!

Questions?

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