$\mathbb{Z}_2 \times \mathbb{Z}_3 = \mathbb{Z}_6$

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The cyclic group $\mathbb{Z}_6$ is the direct product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$. How are their representations related?

Representations of $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are related to representations of $\mathbb{Z}_6$ by

$$\begin{aligned}(0,0) &\to 0,\\ (1,1) &\to 1,\\ (0,2) &\to 2,\\ (1,0) &\to 3,\\ (0,1) &\to 4,\\ (1,2) &\to 5,\end{aligned} \tag{1}$$

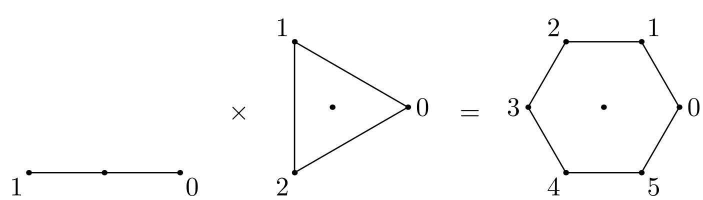
or in general

$$(x,y) \to (ax+by) \mod 6, \tag{2}$$

where $a = 3$ and $b = 4$. As explained in [1], the coefficients a and b arise from noting that if $(1,0) \to a$ and $(0,1) \to b$, then $(x,y) \to (ax+by) \mod 6$. Thus a and b are found from the equations

$$a = 1 \mod 2, \quad a = 0 \mod 3, \quad b = 0 \mod 2, \quad b = 1 \mod 3. \tag{3}$$

We can visualise the relation $\mathbb{Z}_2 \times \mathbb{Z}_3 = \mathbb{Z}_6$ as



(4)

Note that $4 = -2 \mod 6$ and $5 = -1 \mod 6$ is consistent with the representation 4 of $\mathbb{Z}_6$ being the complex conjugate of 2 and 5 being the conjugate of 1.

References

[1] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*. Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA, 2nd ed., 1994.