Coalitional games

Based on Chapters 13 (The Core) and 15 (The Nash Solution) in Osborne and Rubinstein. *A Course in Game Theory*. The MIT Press, 1994.

In this talk:

- Introduction to coalitional games: basic definitions, and solution concepts (the core).
- Bargaining problems and the Nash solution.
- Applications of cryptography to game theory (finally!).

Introduction

Noncooperative games. (The games studied so far). Primitives:

- The players' set of possible actions.
- Their preferences over the possible outcomes.
- Each action is taken by a single player autonomously.

Coalitional games. Here, the players form groups or *coalitions*. Primitives:

- The set of joint actions that each coalition can take independently of the remaining players.
- An outcome of the game is a specification of the resulting coalition and the joint action it takes.
- The profile of the *individual* players' preferences over the outcomes.

Coalitional games

Definition 1. A *coalitional game* $\langle N, v\rangle$ *with transferable payoff* consists of

- A finite set N (the *players*).
- A function $v\colon 2^N \setminus \emptyset \to \mathbb{R}$ that assigns to every nonempty $S \subseteq N$ (a *coalition*) a real number $v(S)$ (the *worth* of S).

For convenience, we let $\mathcal{C} = 2^N \setminus \emptyset$ denote the set of possible coalitions.

Intuitively, $v(S)$ is the total payoff available to S. How this is divided between the members of S is not considered in this model.

Usually, the following restriction is made to the games under study.

Definition 2. A coalitional game $\langle N, v\rangle$ with transferable payoff is *cohesive* if

$$v(N) \geq \sum v(S_j)$$

for every partition $\{S_1, \dots, S_k\}$ of N.

Note that this is a special form of superadditivity, which require that $v(S \cup T) \geq v(S) + v(T)$ whenever $S \cap T = \emptyset$.

The core

Compare with the Nash equilibrium: an outcome is stable if no deviation is profitable (given the other players' actions).

For a coalitional game, the outcome is stable if no coalition can deviate and obtain a better outcome for all its members.

The basic solution concept for coalitional games is the *core*.

Definition 3. Let $\langle N, v\rangle$ be a coalitional game with transferable payoff. For any profile $(x_i)_{i\in N}, x_i \in \mathbb{R}$ and coalition S, define

$$x(S) = \sum_{i\in S} x_i.$$

The vector $(x_i)_{i\in S}$ is an S*-feasible payoff vector* if $x(S) = v(S)$. An N-feasible payoff vector is a *feasible payoff profile*.

The *core* of $\langle N, v\rangle$ is the set of feasible payoff profiles $(x_i)_{i\in N}$ for which there is no coalition S and S-feasible payoff vector $(y_i)_{i\in S}$ such that $y_i > x_i, \forall i \in S$.

Equivalently,

$$\mathrm{core}\langle N, v\rangle = \{(x_i)_{i\in N} \mid v(S) \leq x(S), \forall S\}.$$

Example 4. The cave people next door have forgotten to lock their doors. Since game theory seminar attenders are quite weak and cave technology is quite heavy, it takes two of us to carry one server. Since the cave people might come back, we cannot risk to enter the cave more than once. Thus, each pair of seminar attenders can steal one server.

This can be modeled as a coalitional game $\langle N, v\rangle$, where

$$v(S) = \begin{cases} |S|/2, & \text{if } |S| \text{ is even, and} \\ (|S|-1)/2, & \text{if } |S| \text{ is odd.} \end{cases}$$

Assume that $n = |N| \geq 3$.

Let $(x_1, \ldots, x_n) \in \mathrm{core}\langle N, v\rangle$. Since $v(\{i\}) = 0 \leq x(\{i\}) = x_i$, $x_i \geq 0$. Since $v(\{i,j\}) = 1 \leq x(\{i,j\}) = x_i + x_j$, at least one of $x_i, x_j \geq 1/2$. Since this holds for all pairs x_i, x_j, there can be at most one $x_i < 1/2$.

If n is even, $v(N) = n/2 = x(N) = \sum x_k$. Thus, $x_i + x_j = 1$, and all $x_i = 1/2$. Hence, $\mathrm{core}\langle N, v\rangle = \{(1/2, 1/2, \ldots, 1/2)\}$.

If n is odd, $v(N) = (n-1)/2 = x(N) = \sum x_k = (n-1)/2 + x_n$. Thus, $x_n = 0$, which implies that $x_i \geq 1$ for all $i \neq n$, a contradiction. Hence, $\mathrm{core}\langle N, v\rangle = \emptyset$.

Nonemptyness of the core

Definition 5. For each $S \in \mathcal{C}$, let $\chi^S \in \{0,1\}^{|N|}$ denote the characteristic vector $\chi_i^S = 1 \iff i \in S$. A collection $\{\lambda^S \in [0,1] \mid S \in \mathcal{C}\}$ is a *balanced collection of weights* if

$$\sum_{S \in \mathcal{C}} \lambda^S \chi^S = \chi^N.$$

A game $\langle N, v\rangle$ is *balanced* if

$$\sum_{S \in \mathcal{C}} \lambda^S v(S) \leq v(N)$$

for every balanced collection of weights.

Proposition 6. Let $\langle N, v\rangle$ be a coalitional game with transferable payoff. Then $\mathrm{core}\langle N, v\rangle \neq \emptyset$ if and only if the game is balanced.

Coalitional games without transferable payoff

A more general form of coalitional games is the following.

Definition 7. A *coalitional game* $\langle N, X, V, (\succsim_i)_{i \in N}\rangle$ (without transferable payoff) consists of

- A finite set N (the *players*).
- A set X (the *consequences*).
- A function $V\colon 2^S \setminus \emptyset \to 2^X$ that assigns to each nonempty $S \subseteq N$ (a *coalition*) a set of consequences $V(S) \subseteq X$.
- For each player $i \in N$ a preference relation $\succsim_i$ on X.

Definition 8. The *core* of a coalitional game $\langle N, X, V, (\succsim_i)_{i \in N}\rangle$ is the set of all $x \in V(N)$ for which there is no coalition S and $y \in V(S)$ such that $y \succ_i x$ for all $i \in S$.

Final remarks (on coalitional games)

Coalitional games are traditionally used to model certain types of *markets* and *exchange economies*.

There are other solution concepts for coalitional games and different ways to restrict the way that an objecting coalition can deviate.

A solution can be considered stabile if there is is no *valid objection* to the solution. Another stability condition is that every valid objection has a *valid counter objection*.

These concepts can be formalized in several ways and are discussed in Chapter 14 of Osborne and Rubinstein.

Bargaining problems

In many applications, two parties must agree on some common decision. Typically, the parties have different preferences, and must hence bargain to achieve a suitable compromise. This is formalized as follows.

Definition 9. A *bargaining problem* $\langle X, D, \succsim_1, \succsim_2 \rangle$ consists of

- A set X of the possible *consequences* the two parties can jointly achieve.
- An event $D \in X$ that occurs if the parties fail to agree.
- The two parties' preference relations $\succsim_1, \succsim_2$ on $\mathcal{L}(X)$ (the set of lotteries over X).

We refer to X as the set of possible *agreements*, and to D as the *disagreement* outcome.

Some additional restrictions are added to the definition to simplify the analysis, but we ignore them here.

Typically, $x \succsim_i D, \forall x \in X$. In this case, the utility functions representing $\succsim_i$ can usually be chosen so that $u_i(D) = 0$.

Definition 10. A *bargaining solution* is a function that assigns to every bargaining problem $\langle X, D, \succsim_1, \succsim_2 \rangle$ a unique member of X.

The Nash solution

Definition 11. The *Nash solution* is a bargaining solution that assigns to the bargaining problem $\langle X, D, \succsim_1, \succsim_2 \rangle$ an $x^* \in X$ such that

$$px \succ_i x^*, p \in [0, 1], x \in X \Longrightarrow px^* \succsim_j x, j \neq i.$$

Proposition 12. The agreement $x^* \in X$ is a Nash solution to $\langle X, D, \succsim_1, \succsim_2 \rangle$ if and only if

$$u_1(x^*)u_2(x^*) \geq u_1(x)u_2(x), \forall x \in X,$$

where u_i is the utility function that represents $\succsim_i$ and satisfies $u_i(D) = 0$. Furthermore, the Nash solution is well-defined.

(Here, the "additional restrictions" that we ignored are very relevant.)

Implementation

Unfortunately, we have not considered implementation theory in the seminar yet, but:

Let X and D be fixed. The following “protocol” implements the Nash solution (in the implementation theory sence) for all pairs $(\succsim_1, \succsim_2)$ for which $\langle X, D, \succsim_1, \succsim_2 \rangle$ is a bargaining problem.

1. Alice chooses $y \in X$.
2. Bob chooses $x \in X$ and $p \in [0, 1]$.
3. With probability $1 - p$, the game ends with outcome D. With probability p it continues.
4. Alice chooses either x or the lottery py. This choice is the outcome of the game.

Some cryptographic applications to game theory

What is really going on in the implementation of the bargaining problem? Why does it work, and in what sence?

Potential research problems:

- Design a generic and efficient cryptographic protocol that implements the Nash solution of bargaining problems.
- Implementation theory might provide several interesting interactions between cryptography and game theory. We should study at least the basic ideas behind implementation theory.

We will return to these issues during the next seminar