Combinatorics.

Problem set 1: Counting I.

1. (MN 3.1.6). Show that a natural number $n \geq 1$ has an odd number of divisors (including 1 and itself) if and only if $\sqrt{n}$ is an integer.

2. (MN 3.2.3) Let p be a permutation, and let p^k be the k-fold composition of p. By the *order* of the permutation p we mean the smallest natural number $k \geq 1$ such that $p^k = id$, where id denotes the identity permutation (mapping each element onto itself).

(a) Determine the order of the permutation (2 3 1 5 4 7 8 9 6).

(b) Show that each permutation p of a finite set has a well-defined finite order, and show how to compute the order using the lengths of the cycles of p.

3. (MN 3.2.7)

(a) Find out what is the largest power of 10 dividing the number 70! (i.e. the number of trailing zeros in the decimal notation for 70!).

(b) Find a general formula for the highest power k such that $n!$ is divisible by p^k, where p is a given prime number.

4. (MN 3.3.4) For natural numbers $m \leq n$ calculate (i.e. express by a simple formula not containing a sum) $\sum_{k=m}^{n} \binom{k}{m} \cdot \binom{n}{k}$.

5. (MN 3.3.7) How many functions $f : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ are there that are monotone; that is, for $i < j$ we have $f(i) \leq f(j)$?

6. (MN 3.4.2) Find positive and nondecreasing functions $f(n)$, $g(n)$ defined for all natural numbers such that neither $f(n) = O(g(n))$ nor $g(n) = O(f(n))$ holds.

7. (MN 3.5.5) Decide which of the following statements are true:
(a) $n! \sim ((n+1)/2)^n$, (b) $n! \sim ne(n/e)^n$, (c) $n! = O((n/e)^n)$,
(d) $\ln(n!) = \Omega(n \ln n)$, (e) $\ln(n!) \sim n \ln n$.

8. (MN 3.5.9) Prove the lower bound $n! \geq e(\frac{n}{e})^n$ in Thm. 3.5.5 by induction.

9*. (MN 3.3.6) Prove that $\sum_{k=0}^{m} \binom{m}{k} \cdot \binom{n+k}{m} = \sum_{k=0}^{m} \binom{n}{k} \cdot \binom{m}{k} \cdot 2^k$.