Matemaatiline analüüs I
20. praktikum

Olgu (a_n) arvjada.

Kirjutist $\sum\limits_{n=1}^{\infty} a_n$ ehk $\sum\limits_n a_n$ ehk $\sum a_n$ nimetatakse *arvreaks* ehk *reaks.*

Arve $S_n = \sum\limits_{k=1}^{n} a_k$ nimetatakse rea $\sum\limits_n a_n$ *osasummadeks.*

Rea $\sum\limits_n a_n$ *summaks* nimetatakse piirväärtust $\sum\limits_n a_n := \lim\limits_n S_n = \lim\limits_n \sum\limits_{k=1}^{n} a_k$.

Rida $\sum\limits_n a_n$ on *koonduv* $\overset{\text{def}}{\iff}$ jada (S_n) on koonduv.

Kui rida ei ole koonduv, siis nimetatakse teda *hajuvaks.*

Rida $\sum\limits_n a_n$ on positiivne $\overset{\text{def}}{\iff}$ $\forall n \in \mathbb{N}\ \ a_n \geqslant 0$.

Lause 9.9. Positiivne rida $\sum\limits_n a_n$ koondub parajasti siis, kui tema osasummade jada (S_n) on (ülalt) tõkestatud.

Lause 9.3. Rida $\sum\limits_n a_n$ on koonduv $\implies$ $\lim\limits_n a_n = 0$.

Lause 9.2. Olgu $m \in \mathbb{N}$ ja olgu antud rida $\sum\limits_{n=1}^{\infty} a_n$. Tähistame $b_n = a_{n+m}$ iga $n \in \mathbb{N}$ korral.

Rida $\sum\limits_{n=1}^{\infty} a_n$ on koonduv $\iff$ rida $\sum\limits_{n=1}^{\infty} b_n$ on koonduv.

Lause 9.4. *Geomeetriline rida* $\sum\limits_{k=0}^{\infty} q^k = 1 + q + q^2 + \ldots$ koondub parajasti siis, kui $|q| < 1$.

Lause 9.7. *Harmooniline rida* $\sum\limits_{k=1}^{\infty} \frac{1}{k^\alpha} = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \ldots$ koondub parajasti siis, kui $\alpha > 1$.

Lause 9.8. (Ridade aritmeetika) Kui $\sum_{n=1}^{\infty} a_n = A$ ja $\sum_{n=1}^{\infty} b_n = B$, siis

(a) $\sum_{n=1}^{\infty} (a_n \pm b_n) = A \pm B$,

(b) $\forall \lambda \in \mathbb{R} \left(\sum_{n=1}^{\infty} (\lambda \cdot a_n) = \lambda \cdot A \right)$.

Lause 9.10. (I võrdluslause)**.**

$$\left.\begin{array}{r} \forall n \in \mathbb{N}\ \ b_n \geqslant a_n \geqslant 0, \\ \text{rida } \sum_n b_n \text{ on koonduv} \end{array}\right\} \implies \text{rida } \sum_n a_n \text{ on koonduv,}$$

$$\left.\begin{array}{r} \forall n \in \mathbb{N}\ \ b_n \geqslant a_n \geqslant 0, \\ \text{rida } \sum_n a_n \text{ on hajuv} \end{array}\right\} \implies \text{rida } \sum_n b_n \text{ on hajuv.}$$

Lause 9.12. (II võrdluslause)**.**

$$\left.\begin{array}{r} \forall n \in \mathbb{N}\ \ a_n, b_n \geqslant 0, \\ \lim\limits_n \dfrac{a_n}{b_n} \in (0, \infty) \end{array}\right\} \implies \left(\text{rida } \sum_n a_n \text{ on koonduv} \Leftrightarrow \text{rida } \sum_n b_n \text{ on koonduv} \right).$$

Rida $\sum_n a_n$ on *absoluutselt koonduv* $\overset{\text{def}}{\iff}$ rida $\sum_n |a_n|$ on koonduv.

Lause 9.13. Rida $\sum_n a_n$ on absoluutselt koonduv $\implies$ rida $\sum_n a_n$ on koonduv.

Teoreem 10.1. (Cauchy tunnus)**.** Olgu $\sum_n a_n$ arvrida ning $D = \lim\limits_n \sqrt[n]{|a_n|}$.

$$\begin{array}{rcl} 0 \leqslant D < 1 & \implies & \text{rida } \sum_n a_n \text{ on absoluutselt koonduv,} \\ D > 1 & \implies & \text{rida } \sum_n a_n \text{ on hajuv.} \end{array}$$

Teoreem 10.2. (D'Alembert'i tunnus)**.** Olgu $\sum_n a_n$ arvrida ning $D = \lim\limits_n \left| \dfrac{a_{n+1}}{a_n} \right|$.

$$\begin{array}{rcl} 0 \leqslant D < 1 & \implies & \text{rida } \sum_n a_n \text{ on absoluutselt koonduv,} \\ D > 1 & \implies & \text{rida } \sum_n a_n \text{ on hajuv.} \end{array}$$

Teoreem 10.3. (Leibnizi tunnus)**.**

$$\left.\begin{array}{r} \forall n \in \mathbb{N}\ \ a_n \geqslant a_{n+1} \geqslant 0, \\ \lim\limits_n a_n = 0 \end{array}\right\} \implies \text{rida } \sum_n (-1)^n a_n \text{ on koonduv.}$$