Extensions of Lorentzian spacetime geometry

From Finsler to Cartan and vice versa

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This work:

MH,
“Extensions of Lorentzian spacetime geometry: from Finsler to Cartan and vice versa,”
arXiv:1304.5430 [gr-qc].

Cartan geometry of observer space:

S. Gielen and D. K. Wise,
“Lifting General Relativity to Observer Space,”
arXiv:1210.0019 [gr-qc].

Finsler spacetimes:

C. Pfeifer and M. N. R. Wohlfarth,
“Causal structure and electrodynamics on Finsler spacetimes,”

C. Pfeifer and M. N. R. Wohlfarth,
“Finsler geometric extension of Einstein gravity,”
Outline

1. Physical motivation
2. Cartan geometry on observer space
3. Finsler spacetimes
4. From Finsler geometry to Cartan geometry
5. From Cartan geometry to Finsler geometry
6. Closing the circle
7. Finsler-Cartan-Gravity
8. Conclusion
Outline

1 Physical motivation

2 Cartan geometry on observer space

3 Finsler spacetimes

4 From Finsler geometry to Cartan geometry

5 From Cartan geometry to Finsler geometry

6 Closing the circle

7 Finsler-Cartan-Gravity

8 Conclusion
Consider the following experiment:

- A supernova occurs in a far away galaxy.
- An astronomer points his telescope to the sky.
- He takes a picture of the supernova.
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How can we describe this experiment?
What does it tell us about “spacetime”?
The spacetime picture:

- Model spacetime as a Lorentzian manifold \((M, g)\).
- Supernova is a “beacon” at some event \(x_0 \in M\).
- Astronomer observes the light at another event \(x \in M\).
- Light follows a null geodesic \(\gamma\) from \(x_0\) to \(x\).
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Description of measured data:  
- Tangent vector to \(\gamma\) in \(x\) determines direction of light propagation.
- Distance determines apparent magnitude (observed brightness).
- Spacetime metric \(g\) determines redshift.
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How much imagination does this picture require?
Fix an observer frame:
- Location of the observer: spacetime event $x$.
- Four-velocity of the observer: future timelike unit tangent vector $f_0$.
- Coordinate axes of the observatory: spatial frame components $f_\alpha$. 

Perform the measurement:
- Direction of incoming light with respect to local frame.
- Brightness: photon rate using local clock.
- Redshift: frequency using local clock.

$\Rightarrow$ Variables split into two classes: $(x, f_\alpha) \in P$ describes the observer.
- Light direction, photon rate, frequency describe the observation.

$\Rightarrow$ "Beacon" event $x_0$ and geodesic $\gamma$ are part of the interpretation.
The actual measurement

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Spacetime vs. observer space

- The spacetime picture:
  - Spacetime geometry given by Lorentzian manifold \((M, g)\).
  - \(P\) is the space of orthonormal frames of \((M, g)\).
    \[ \Rightarrow \tilde{\pi} : P \rightarrow M \text{ is a principal } \text{SO}_0(3, 1)\text{-bundle.} \]
    \[ \Rightarrow \text{Observers } (x, f) \text{ and } (x, f') \text{ are related by Lorentz transform.} \]

- The observer space idea:
  - "Geometrodynamical" theories suggest split into space and time:
    - Loop quantum gravity
    - Spin foam models
    - Causal dynamical triangulations
  \[ \Rightarrow \text{Symmetry breaking } \text{SO}_0(3, 1) \rightarrow \text{SO}(3). \]
  \[ \Rightarrow \text{Possible dependence of physical quantities on } f_0. \]
  \[ \Rightarrow \text{Observers } (x, f_0, f_0) \text{ and } (x, f_0, f_0') \text{ are related by rotation.} \]
  \[ \Rightarrow \text{Consider a principal } \text{SO}(3)\text{-bundle } \pi : P \rightarrow O. \]
  \[ \Rightarrow \text{Describe experiments on } O \ni (x, f_0). \]
  \[ \Rightarrow \text{Advantage: No preferred observer. } O \text{ contains all observers.} \]
  \[ \Rightarrow \text{In general no (absolute) spacetime } M. \]
  \[ \Rightarrow \text{Geometry on observer space } O? \]
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  - In general no (absolute) spacetime \(M\).
  - Geometry on observer space \(O\)?
Cartan geometry

- Ingredients of a Cartan geometry:
  - A Lie group $G$ with a closed subgroup $H \subset G$.
  - A principal $H$-bundle $\pi : P \to M$.
  - A 1-form $A \in \Omega^1(P, g)$ on $P$ with values in $g$. 

Curvature of the Cartan connection:
- Curvature defined by $F = dA + \frac{1}{2} [A, A]$.
- Curvature measures deviation between $M$ and $G/H$. 

- Geometry of $M$:
  - Cartan connection describes geometry and parallel transport on $M$.
  - $M$ "locally looks like" homogeneous space $G/H$.
  - Tangent spaces $T_x M \cong z = g/h$.
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Conditions on the Cartan connection $A$:

- For each $p \in P$, $A_p : T_p P \to g$ is a linear isomorphism.
- $A$ is right-equivariant: $(R_h)^* A = \text{Ad}(h^{-1}) \circ A \quad \forall h \in H$.
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- $A$ has an “inverse” $\tilde{A} : g \to \Gamma(TP)$.
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  - A Lie group \( G \) with a closed subgroup \( H \subset G \).
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Choose Lie groups:

Let

\[ G = \begin{cases} 
  \text{SO}_0(4, 1) & \Lambda > 0 \\
  \text{ISO}_0(3, 1) & \Lambda = 0 \\
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\end{cases} \]

\[ H = \text{SO}_0(3, 1). \]

\[ \text{Coset spaces } G/H \text{ are the maximally symmetric spacetimes.} \]
Example: Cartan geometry of spacetime

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Let \( P \) be the oriented time-oriented orthonormal frames on \( M \).

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Choose Cartan connection:

\( g = \mathfrak{h} \oplus \mathfrak{z} \) splits into direct sum.

Let \( e \in \Omega^1(P, \mathfrak{z}) \) be the solder form of \( \tilde{\pi} : P \to M \).

Let \( \omega \in \Omega^1(P, \mathfrak{h}) \) be the Levi-Civita connection.

⇒ \( A = \omega + e \in \Omega^1(P, g) \) is a Cartan connection.
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  \[ \Rightarrow A = \omega + e \in \Omega^1(P, g) \] is a Cartan connection.

\[ \Rightarrow \] Spacetime \((M, g)\) can be reconstructed from Cartan geometry.
Example: Cartan geometry of observer space

- **Choose Lie groups:** [S. Gielen, D. Wise ’12]
  - Let
    \[
    G = \begin{cases} 
    \text{SO}_0(4, 1) & \Lambda > 0 \\
    \text{ISO}_0(3, 1) & \Lambda = 0, \quad K = \text{SO}(3) \\
    \text{SO}_0(3, 2) & \Lambda < 0
    \end{cases}
    \]
  - \(\Rightarrow\) Coset spaces \(G/K\) are the maximally symmetric observer spaces.

- **Choose principal \(K\)-bundle:**
  - Let \((M, g)\) be a Lorentzian manifold.
  - Let \(O\) be the future unit timelike vectors on \(M\).
  - Let \(P\) be the oriented time-oriented orthonormal frames on \(M\).
  - \(\Rightarrow\) \(\pi : P \rightarrow O\) is a principal \(K\)-bundle.

- **Choose Cartan connection:**
  - \(g = \mathfrak{h} \oplus \mathfrak{z}\) splits into direct sum.
  - Let \(e \in \Omega^1(P, \mathfrak{z})\) be the solder form of \(\tilde{\pi} : P \rightarrow M\).
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Proper time along a curve in Lorentzian spacetime:

$$\tau = \int_{t_1}^{t_2} \sqrt{-g_{ab}(x(t))\dot{x}^a(t)\dot{x}^b(t)} dt.$$
The clock postulate

- Proper time along a curve in Lorentzian spacetime:
  \[ \tau = \int_{t_1}^{t_2} \sqrt{-g_{ab}(x(t))\dot{x}^a(t)\dot{x}^b(t)} \, dt. \]

- Finsler geometry: use a more general length functional:
  \[ \tau = \int_{t_1}^{t_2} F(x(t), \dot{x}(t)) \, dt. \]

- Finsler function \( F : TM \to \mathbb{R}^+ \).

- Parametrization invariance requires homogeneity:
  \[ F(x, \lambda y) = \lambda F(x, y) \quad \forall \lambda > 0. \]
Definition of Finsler spacetimes

- Finsler geometries suitable for spacetimes exist. [C. Pfeifer, M. Wohlfarth '11]
  ⇒ Notion of timelike, lightlike, spacelike tangent vectors.
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Finsler metric with Lorentz signature:

$$g^F_{ab}(x, y) = \frac{1}{2} \frac{\partial}{\partial y^a} \frac{\partial}{\partial y^b} F^2(x, y).$$

Unit vectors $y \in T_x M$ defined by

$$F^2(x, y) = g^F_{ab}(x, y) y^a y^b = 1.$$
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⇒ Set \( \Omega_x \subset T_xM \) of unit timelike vectors at \( x \in M \).

\( \Omega_x \) contains a closed connected component \( S_x \subset \Omega_x \).

Causality: \( S_x \) corresponds to physical observers.
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Observer space

- Recall from the definition of Finsler spacetimes:
  - Set $\Omega_x \subset T_x M$ of unit timelike vectors at $x \in M$.
  - $\Omega_x$ contains a closed connected component $S_x \subseteq \Omega_x$. 

$\Rightarrow$ Tangent vectors $y \in S_x$ satisfy $g_{F}^{ab}(x,y)y^a y^b = 1$.

Complete $y = f_0$ to a frame $f^i$ with $g_{F}^{ab}(x,y)f^a_i f^b_j = -\eta^{ij}$.

Let $P$ be the space of all observer frames.

$\Rightarrow \pi: P \rightarrow O$ is a principal $SO(3)$-bundle.

In general no principal $SO_0(3,1)$-bundle $\tilde{\pi}: P \rightarrow M$. 

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$\Rightarrow$ $\pi : P \to O$ is a principal $\text{SO}(3)$-bundle.
- In general no principal $\text{SO}_0(3,1)$-bundle $\tilde{\pi} : P \to M$. 
Need to construct \( A \in \Omega^1(P, g) \).

Recall that

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\begin{align*}
g &= \mathfrak{h} \oplus \mathfrak{z} \\
A &= \omega + e
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\( \Rightarrow \) Need to construct \( \omega \in \Omega^1(P, \mathfrak{h}) \) and \( e \in \Omega^1(P, \mathfrak{z}) \).
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**Definition of $e$:** Use the *solder form*.

- Let $w \in T_{(x,f)}P$ be a tangent vector.
- Differential of the projection $\tilde{\pi} : P \to M$ yields $\tilde{\pi}^*(w) \in T_xM$.
- View frame $f$ as a linear isometry $f : \mathfrak{z} \to T_xM$.
- Solder form given by $e(w) = f^{-1}(\tilde{\pi}^*(w))$. 
Definition of $\omega$:

- Frames $(x, f)$ and $(x, f')$ related by generalized Lorentz transform.
  
  [C. Pfeifer, M. Wohlfarth '11]

- Relation between $f$ and $f'$ defined by parallel transport on $O$.

- Tangent vector $w \in T_{(x,f)}P$ “shifts” frame $f$ by small amount.

- Compare shifted frame with parallely transported frame.

- Measure the difference using the original frame:

  \[
  \Delta f^a_i = \epsilon f^a_j \omega^j_i(w).
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- Choose parallel transport on $O$ so that $g^F$ is covariantly constant.

- Connection on Finsler geometry: Cartan linear connection.
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Choose parallel transport on $O$ so that $g^F$ is covariantly constant.

Connection on Finsler geometry: Cartan linear connection.

$\Rightarrow$ Frames $f_i^a$ and $f_i^a + \Delta f_i^a$ are orthonormal wrt the same metric.

$\Rightarrow \omega(w) \in \mathfrak{h}$ is an infinitesimal Lorentz transform.
Complete Cartan connection

- Translational part $e \in \Omega^1(P, \mathfrak{h})$:
  \[
e^i = f^{-1}i_a dx^a.
  \]
Complete Cartan connection

- **Translational part** $e \in \Omega^1(P, \mathfrak{d})$:
  $$e^i = f^{-1} a dx^a.$$  

- **Boost / rotational part** $\omega \in \Omega^1(P, \mathfrak{h})$:
  $$\omega^i_{\ j} = f^{-1} a \left[ df^a + f^b_j \left( dx^c F^a_{\ bc} + (dx^d N^c_{\ d} + df^c_0) C^a_{\ bc} \right) \right].$$
Complete Cartan connection

- Translational part $e \in \Omega^1(P, \mathfrak{h})$:
  \[ e^i = f^{-1}_a dx^a. \]

- Boost / rotational part $\omega \in \Omega^1(P, \mathfrak{h})$:
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- Coefficients of Cartan linear connection:
  \[
  N^a_b = \frac{1}{4} \bar{\partial}_b \left[ g^F_{aq} \left( y^p \partial_p \bar{\partial}_q F^2 - \partial_q F^2 \right) \right], \\
  F^a_{bc} = \frac{1}{2} g^F_{ap} \left( \delta_b g^F_{pc} + \delta_c g^F_{bp} - \delta_p g^F_{bc} \right), \\
  C^a_{bc} = \frac{1}{2} g^F_{ap} \left( \bar{\partial}_b g^F_{pc} + \bar{\partial}_c g^F_{bp} - \bar{\partial}_p g^F_{bc} \right). 
  \]
Complete Cartan connection

- Translational part \( e \in \Omega^1(P, \mathfrak{h}) \):
  \[
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  \]
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  \]

\[\Rightarrow \ A = \omega + e \text{ is a Cartan connection on } \pi : P \rightarrow O.\]
Fundamental vector fields

- Let $a = z^i Z_i + \frac{1}{2} h^i_j \mathcal{H}_i^j \in \mathfrak{g}$.
- Define the vector field

$$A(a) = z^i f^a_i \left( \partial_a - f^b_j F_{ab}^c \bar{\partial}^j_c \right) + \left( h^i_j f^a_i - h^i_0 f^b_i f^c_j C_{abc}^a \right) \bar{\partial}^j_a.$$
Fundamental vector fields

- Let \( a = z^i Z_i + \frac{1}{2} h^i{}_j \mathcal{H}_{ij} \in \mathfrak{g} \).
- Define the vector field

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A(a) = z^i f^a_i \left( \partial_a - f^b_j F^c_{ab} \bar{\partial}^j_c \right) + \left( h^i{}_j f^a_i - h^i{}_0 f^b_i f^c_j C^a_{bc} \right) \bar{\partial}^j_a .
\]

\( \Rightarrow \) For all \( p \in P \) we find

\[
A(A(a)(p)) = a .
\]

\( \Rightarrow \) For all \( w \in T_p P \) we find

\[
A(A(w))(p) = w .
\]

\( \Rightarrow \) \( A_p : T_p P \to \mathfrak{g} \) and \( A_p : \mathfrak{g} \to T_p P \) complement each other.
Consider adjoint representation $\text{Ad} : K \subset G \to \text{Aut}(\mathfrak{g})$ of $K$ on $\mathfrak{g}$.

$\mathfrak{g}$ splits into irreducible subrepresentations of Ad.
Split of the tangent bundle $TP$

- Consider adjoint representation $\text{Ad} : K \subset G \rightarrow \text{Aut}(\mathfrak{g})$ of $K$ on $\mathfrak{g}$.
- $\mathfrak{g}$ splits into irreducible subrepresentations of $\text{Ad}$.
- Induced decompositions of $A$ and $TP$:

\[
\mathfrak{g} = \xi \oplus \eta \oplus \bar{\delta} \oplus \delta_0
\]

\[
A = \Omega + b + \bar{e} + e^0
\]

\[
TP_p = R_pP \oplus B_pP \oplus \bar{H}_pP \oplus H^0_pP
\]

- Subbundles of $TP$ spanned by fundamental vector fields $A$. 

- Rotations
- Boosts
- Spatial translations
- Temporal translations
Consider the fundamental vector field

\[ t = A(Z_0) = f_0^a \partial_a - f_j^a N^b_a \bar{\partial}_b \iff \omega^i_j(t) = 0, \quad e^i(t) = \delta^i_0. \]

Integral curve \( \Gamma : \mathbb{R} \to P, \lambda \mapsto (x(\lambda), f(\lambda)) \) of \( t \).
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From \( e^i(t) = \delta^i_0 \) follows:

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\( (x, f_0) \) is the canonical lift of a curve from \( M \) to \( O \).
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\Rightarrow \( (x, f_0) \) is a Finsler geodesic.

From \( \omega^\alpha_\beta(\mathbf{t}) = 0 \) follows:
\[ 0 = \dot{f}_\alpha^a + f_\alpha^b \left( \dot{x}^c F^a_{bc} + (\dot{x}^d N^c_d + \dot{f}_0^c) C^a_{bc} \right) = \nabla (\dot{x}, \dot{f}_0) f_\alpha^a. \]

\Rightarrow Frame \( f \) is parallely transported.
Curvature of the Cartan connection

- Curvature \( F \in \Omega^2(P, g) \) defined by

\[
F = dA + \frac{1}{2} [A, A].
\]
Curvature of the Cartan connection

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- Translational part $F_z \in \Omega^2(P, z)$ (“torsion”):

$$de^i + \omega^i{}_j \wedge e^j = - f^{-1}{}^i{}_a C^a{}_{bc} dx^b \wedge \delta f^c_0$$

with $\delta f^c_0 = N^c{}_d dx^d + df^c_0$. 

$R_d cab$, $P_d cab$, $S_d cab$: curvature of Cartan linear connection.
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  \]
  with $\delta f^c_0 = N^c_d dx^d + df^c_0$.

- Boost / rotational part $F_h \in \Omega^2(P, h)$:
  \[
  d\omega^i_j + \omega^i_k \wedge \omega^k_j = -\frac{1}{2}f^{-1}_{ia}f_j^c \left( R^d_{cab} dx^a \wedge dx^b 
  + 2P^d_{cab} dx^a \wedge \delta f^b_0 + S^d_{cab} \delta f^a_0 \wedge \delta f^b_0 \right).
  \]
Curvature of the Cartan connection

- Curvature $F \in \Omega^2(P, g)$ defined by
  \[ F = dA + \frac{1}{2} [A, A]. \]

- Translational part $F_\delta \in \Omega^2(P, \delta)$ ("torsion"):
  \[ de^i + \omega^i_j \wedge e^j = -f^{-1}A C^a_{bc} dx^b \wedge \delta f^c_0 \]
  with $\delta f^c_0 = N^c d dx^d + df^c_0$.

- Boost / rotational part $F_\mathfrak{h} \in \Omega^2(P, \mathfrak{h})$:
  \[ d\omega^i_j + \omega^i_k \wedge \omega^k_j = -\frac{1}{2} f^{-1}A f^c_j \left( R^d_{cab} dx^a \wedge dx^b \right. \]
  \[ + 2 P^d_{cab} dx^a \wedge \delta f^b_0 + S^d_{cab} \delta f^a_0 \wedge \delta f^b_0 \right). \]

- $R^d_{cab}, P^d_{cab}, S^d_{cab}$: curvature of Cartan linear connection.
1. Physical motivation
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Condition: boost distribution $VP = RP \oplus BP$ is integrable.

$\Rightarrow VP$ can be integrated to a foliation $\mathcal{F}$ with leaf space $M$. 
Condition: boost distribution $VP = RP \oplus BP$ is integrable.
⇒ $VP$ can be integrated to a foliation $\mathcal{F}$ with leaf space $M$.
Condition: foliation $\mathcal{F}$ is strictly simple.
⇒ Leaf space $M$ is a smooth manifold.
⇒ Canonical projection $\tilde{\pi}: P \rightarrow M$ is a submersion.
Spacetime

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- Condition: foliation $\mathcal{F}$ is strictly simple.
  $\Rightarrow$ Leaf space $M$ is a smooth manifold.
  $\Rightarrow$ Canonical projection $\tilde{\pi} : P \to M$ is a submersion.
- Canonical projections $\tilde{\pi} = \pi' \circ \pi$:

\[
P \xrightarrow{\pi} O \xrightarrow{\pi'} M
\]
Condition: boost distribution \( VP = RP \oplus BP \) is integrable.

\[ \Rightarrow \quad VP \text{ can be integrated to a foliation } \mathcal{F} \text{ with leaf space } M. \]

Condition: foliation \( \mathcal{F} \) is strictly simple.

\[ \Rightarrow \quad \text{Leaf space } M \text{ is a smooth manifold.} \]

\[ \Rightarrow \quad \text{Canonical projection } \tilde{\pi} : P \rightarrow M \text{ is a submersion.} \]

Canonical projections \( \tilde{\pi} = \pi' \circ \pi \):

\[ P \xrightarrow{\pi} O \xrightarrow{\pi'} M \]

Tangent spaces (with \( o = \pi(p) \) and \( x = \pi'(o) = \tilde{\pi}(p) \)):

\[ R_pP \oplus B_pP \oplus H_pP = T_pP \]

\[ 0 \xrightarrow{\pi_*} B_oO \oplus H_oO = T_oO \]

\[ 0 \xrightarrow{\pi_*} T_xM = T_xM \]
Observer trajectories

- Embedding of observer space $O$ into $TM$?
- Four-velocity of an observer?

Fundamental vector field $t = A(Z^0) \in \Gamma(TP)$ of time translation.

$ \Rightarrow $ Vector field $r \in \Gamma(TO)$ independent of $p \in \pi^{-1}(o)$:

$$ r(o) = \pi^* (t(p)) $$

Relation of $t$ and $r$:

$$ P_{\pi} \rightarrow t \downarrow \downarrow O \rightarrow r \downarrow \downarrow TP_{\pi} \rightarrow \rightarrow TO $$

Define the map $\sigma = \pi' \ast \circ r$.

$\sigma$ is in general not an embedding.

Impose this as another condition.
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$\pi$}

\[ \begin{array}{ccc}
P & \xrightarrow{\pi} & O \\
\downarrow & & \downarrow \\
TP & \xrightarrow{\pi_*} & TO \\
\end{array} \]
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\downarrow \mathbf{t} \quad \downarrow \mathbf{r} \\
TP \xrightarrow{\pi_*} TO \xrightarrow{\pi'_*} TM \\
\end{array} \]

- Define the map $\sigma = \pi'_* \circ \mathbf{r}$.
- $\sigma$ is in general not an embedding.
- Impose this as another condition.
Finsler geometry

- Finsler function must be positively homogeneous of degree one:
  \[ F(x, \lambda y) = |\lambda|F(x, y) \]

- Unit timelike condition: \( F(\sigma(o)) = 1 \) for all observers \( o \in O \).
Finsler geometry

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  \[ \Rightarrow \text{Define } F(\lambda \sigma(o)) = |\lambda| \text{ on double cone } \mathbb{R}\sigma(O). \]
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- Condition: \( \sigma(O) \) must intersect each line \((x, \mathbb{R}y)\) at most once.

- Condition: Finsler metric \( g^F_{ab} \) must have Lorentz signature:
  \[ g^F_{ab} = \frac{1}{2} \bar{\partial}_a \bar{\partial}_b F^2 \]
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- Condition: Finsler metric \( g^F_{ab} \) must have Lorentz signature:
  \[ g^F_{ab} = \frac{1}{2} \partial_a \partial_b F^2 \]

- No Finsler geometry on \( TM \setminus \mathbb{R}\sigma(O) \).
- Cartan geometry describes only geometry visible to observers.
Outline

1 Physical motivation
2 Cartan geometry on observer space
3 Finsler spacetimes
4 From Finsler geometry to Cartan geometry
5 From Cartan geometry to Finsler geometry
6 Closing the circle
7 Finsler-Cartan-Gravity
8 Conclusion
Reconstruction of a given Finsler spacetime

- Idea:
  - Start from a Finsler spacetime \((M, F)\).
  - Construct a Cartan observer space \((\pi : P \rightarrow O, A)\).
  - Construct a new Finsler spacetime \((\hat{M}, \hat{F})\).
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- **Idea:**
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- **Equivalence of Finsler spacetimes** $(M, F)$ and $(\hat{M}, \hat{F})$?
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- Equivalence of Finsler spacetimes \((M, F)\) and \((\hat{M}, \hat{F})\)?
- There exists a diffeomorphism \(\mu\):

![Diagram showing the relationships between the spacetimes and observer spaces.]
Reconstruction of a given Finsler spacetime

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  - Start from a Finsler spacetime \((M, F)\).
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- Equivalence of Finsler spacetimes \((M, F)\) and \((\hat{M}, \hat{F})\)?
- There exists a diffeomorphism \(\mu\):

\[
\begin{array}{cccc}
TM & \rightarrow & M \\
\pi_* & \downarrow & \downarrow & \pi' \\
\hat{\pi}_* & \downarrow & \downarrow & \hat{\pi}' \\
\mu_* & \downarrow & \downarrow & \mu \\
T\hat{M} & \rightarrow & \hat{M}
\end{array}
\]

- \(\mu\) preserves the Finsler function on timelike vectors.

\(\Rightarrow\) Reconstruction of the original Finsler geometry.
Reconstruction of a given Cartan observer space

Idea:
- Start from a Cartan observer space \((\pi : P \to O, A)\).
- Construct a Finsler spacetime \((M, F)\).
- Construct a new Cartan observer space \((\hat{\pi} : \hat{P} \to \hat{O}, \hat{A})\).
Reconstruction of a given Cartan observer space

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  - Construct a Finsler spacetime \((M, F)\).
  - Construct a new Cartan observer space \((\hat{\pi} : \hat{P} \rightarrow \hat{O}, \hat{A})\).
- Equivalence of \((\pi : P \rightarrow O, A)\) and \((\hat{\pi} : \hat{P} \rightarrow \hat{O}, \hat{A})\)?
- Only if a “Cartan morphism” \(\chi\) exists:

\[
\begin{array}{cccccc}
M & \xleftarrow{\pi'} & O & \xleftarrow{\pi} & P & \xrightarrow{A(a)} & TP \\
\downarrow{\hat{\pi}'} & \downarrow{\sigma} & \downarrow{\chi} & \downarrow{\chi^*} & \downarrow & \\
\hat{O} & \xleftarrow{\hat{\pi}} & \hat{P} & \xrightarrow{\hat{A}(a)} & T\hat{P} \\
\end{array}
\]
Reconstruction of a given Cartan observer space

- **Idea:**
  - Start from a Cartan observer space \((\pi : P \to O, A)\).
  - Construct a Finsler spacetime \((M, F)\).
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- **Equivalence of** \((\pi : P \to O, A)\) **and** \((\hat{\pi} : \hat{P} \to \hat{O}, \hat{A})\) **?**
  - Only if a “Cartan morphism” \(\chi\) exists:

\[
\begin{align*}
M & \xleftarrow{\pi'} O \xleftarrow{\pi} P \xrightarrow{A(a)} TP \xrightarrow{\pi_*} TO \xrightarrow{\pi'_*} TM \\
\hat{O} & \xleftarrow{\hat{\pi}} \hat{P} \xrightarrow{A(a)} T\hat{P} \xrightarrow{\hat{\pi}_*} T\hat{O}
\end{align*}
\]

- **Every Cartan morphism** \(\chi = (x, f)\) **takes the form**

\[
x(p) = \pi'(\pi(p)) , \quad f_i(p) = \pi'_*(\pi_*(A(Z_i))(p))
\]

\(\Rightarrow\) **Simple test for equivalence of** \((\pi : P \to O, A)\) **and** \((\hat{\pi} : \hat{P} \to \hat{O}, \hat{A})\).
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MacDowell-Mansouri gravity on observer space: [S. Gielen, D. Wise ’12]

\[ S_G = \int_{\mathcal{O}} \epsilon_{\alpha\beta\gamma} \text{tr}_{\mathfrak{h}}(F_{\mathfrak{h}} \wedge \star F_{\mathfrak{h}}) \wedge b^\alpha \wedge b^\beta \wedge b^\gamma \]

- Hodge operator \(*\) on \(\mathfrak{h}\).
- Non-degenerate \(H\)-invariant inner product \(\text{tr}_{\mathfrak{h}}\) on \(\mathfrak{h}\).
MacDowell-Mansouri gravity on observer space: [S. Gielen, D. Wise ’12]

\[ S_G = \int \epsilon_{\alpha\beta\gamma} \text{tr}_{\mathfrak{h}}(F_{\mathfrak{h}} \wedge \star F_{\mathfrak{h}}) \wedge b^\alpha \wedge b^\beta \wedge b^\gamma \]

- Hodge operator \( \star \) on \( \mathfrak{h} \).
- Non-degenerate \( H \)-invariant inner product \( \text{tr}_{\mathfrak{h}} \) on \( \mathfrak{h} \).

Translate terms into Finsler language (with \( R = d\omega + \frac{1}{2}[\omega, \omega] \)):

- Curvature scalar:
  \[ [e, e] \wedge \star R \rightsquigarrow g^{F \, ab} R^c_{\, abc} \, dV \, . \]

- Cosmological constant:
  \[ [e, e] \wedge \star [e, e] \rightsquigarrow dV \, . \]

- Gauss-Bonnet term:
  \[ R \wedge \star R \rightsquigarrow \epsilon^{abcd} \epsilon^{efgh} R_{abef} R_{cdgh} \, dV \, . \]
MacDowell-Mansouri gravity on observer space: [S. Gielen, D. Wise ’12]

\[ S_G = \int_O \epsilon_{\alpha\beta\gamma} \text{tr}_h(F_h \wedge \star F_h) \wedge b^\alpha \wedge b^\beta \wedge b^\gamma \]

- Hodge operator \( \star \) on \( h \).
- Non-degenerate \( H \)-invariant inner product \( \text{tr}_h \) on \( h \).

Translate terms into Finsler language (with \( R = d\omega + \frac{1}{2} [\omega, \omega] \)):

- Curvature scalar:
  \[ [e, e] \wedge \star R \leadsto g^{F \, ab} R^{c \, acb} dV . \]

- Cosmological constant:
  \[ [e, e] \wedge \star [e, e] \leadsto dV . \]

- Gauss-Bonnet term:
  \[ R \wedge \star R \leadsto \epsilon^{abcd} \epsilon^{efgh} R_{abef} R_{cdgh} dV . \]

⇒ Gravity theory on Finsler spacetime.
Finsler gravity action: [C. Pfeifer, M. Wohlfarth '11]

\[ S_G = \int_O d^4 x \, d^3 y \sqrt{-\tilde{G} R^a_{\,ab} y^b}. \]

- Sasaki metric \( \tilde{G} \) on \( O \).
- Non-linear curvature \( R^a_{\,ab} \).
Gravity from Finsler to Cartan

- Finsler gravity action: [C. Pfeifer, M. Wohlfarth '11]
  \[ S_G = \int_O d^4x \, d^3y \sqrt{-\tilde{G} R^a_{\, \, ab} y^b}. \]

  - Sasaki metric \( \tilde{G} \) on \( O \).
  - Non-linear curvature \( R^a_{\, \, ab} \).

- Translate terms into Cartan language:
  \[ d^4x \, d^3y \sqrt{-\tilde{G}} = \epsilon_{ijkl} \epsilon_{\alpha\beta\gamma} \, e^i \wedge e^j \wedge e^k \wedge e^l \wedge b^\alpha \wedge b^\beta \wedge b^\gamma, \]
  \[ R^a_{\, \, ab} y^b = b^\alpha [A(Z_\alpha), A(Z_0)]. \]
Finsler gravity action: [C. Pfeifer, M. Wohlfarth '11]

\[ S_G = \int_O d^4x \, d^3y \sqrt{-\tilde{G} R^a_{\, \, \, ab} y^b}. \]

- Sasaki metric $\tilde{G}$ on $O$.
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$\Rightarrow$ Gravity theory on observer space.
Summary

Observer space:
- Lift physics from spacetime to the space of observers.
- Describe observer space geometry using Cartan geometry.
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From Finsler to Cartan:
- Cartan geometry on observer space derived from Finsler geometry.
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From Cartan to Finsler:
- Spacetime can (sometimes) be constructed from Cartan geometry.
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- Observers have timelike four-velocities in $\mathcal{T}M$.

Both constructions complement each other.

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Outlook

- Current projects:
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  - Translate more terms between both languages.

- Future projects:
  - Consistent matter coupling.
  - Study of exact solutions.
  - Effects of deviations from metric geometry?
  - Geometrodynamics of Finsler spacetimes.
  - …