Extensions of Lorentzian spacetime geometry

From Finsler to Cartan and vice versa

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References

- This work:
  - MH,
    "Extensions of Lorentzian spacetime geometry:
    from Finsler to Cartan and vice versa,"
    arXiv:1304.5430 [gr-qc].
- Cartan geometry of observer space:
  - S. Gielen and D. K. Wise,
    "Lifting General Relativity to Observer Space,"
    arXiv:1210.0019 [gr-qc].
- Finsler spacetimes:
  - C. Pfeifer and M. N. R. Wohlfarth,
    "Causal structure and electrodynamics on Finsler spacetimes,"
    Phys. Rev. D **84** (2011) 044039 [arXiv:1104.1079 [gr-qc]].
  - C. Pfeifer and M. N. R. Wohlfarth,
    "Finsler geometric extension of Einstein gravity,"
    Phys. Rev. D **85** (2012) 064009 [arXiv:1112.5641 [gr-qc]].

Outline

1. Physical motivation
2. Cartan geometry on observer space
3. Finsler spacetimes
4. From Finsler geometry to Cartan geometry
5. From Cartan geometry to Finsler geometry
6. Closing the circle
7. Finsler-Cartan-Gravity
8. Conclusion

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A simple experiment

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  - A supernova occurs in a far away galaxy.
  - An astronomer points his telescope to the sky.
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  - An astronomer points his telescope to the sky.
  - He takes a picture of the supernova.
- How can we describe this experiment?
- What does it tell us about "spacetime"?

The spacetime picture

- The spacetime picture:
  - Model spacetime as a Lorentzian manifold (M, g).
  - Supernova is a “beacon” at some event $x_0 \in M$.
  - Astronomer observes the light at another event $x \in M$.
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  - Tangent vector to γ in x determines direction of light propagation.
  - Distance determines apparent magnitude (observed brightness).
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- How much imagination does this picture require?

The actual measurement

- Fix an observer frame:
  - Location of the observer: spacetime event x.
  - Four-velocity of the observer: future timelike unit tangent vector f_0.
  - Coordinate axes of the observatory: spatial frame components f_α.

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- $(x, f) \in P$ describes the *observer*.
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$\Rightarrow$ "Beacon" event x_0 and geodesic γ are part of the interpretation.

Spacetime vs. observer space

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  - P is the space of orthonormal frames of (M, g).
  - $\Rightarrow$ $\tilde{\pi} : P \to M$ is a principal $SO_0(3, 1)$-bundle.
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  - $\Rightarrow$ Consider a principal $SO(3)$-bundle $\pi : P \to O$.
  - Describe experiments on *observer space* $O \ni (x, f_0)$.
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  - In general no (absolute) spacetime M.
  - Geometry on observer space O?

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Cartan geometry

- Ingredients of a Cartan geometry:
  - A Lie group G with a closed subgroup $H \subset G$.
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  - M "locally looks like" homogeneous space G/H.
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- Curvature of the Cartan connection:
  - Curvature defined by $F = dA + \frac{1}{2}[A, A]$.
  - Curvature measures deviation between M and G/H.

Example: Cartan geometry of spacetime

- Choose Lie groups:
  - Let

$$G = \begin{cases} \mathrm{SO}_0(4,1) & \Lambda > 0 \\ \mathrm{ISO}_0(3,1) & \Lambda = 0 \\ \mathrm{SO}_0(3,2) & \Lambda < 0 \end{cases}, \quad H = \mathrm{SO}_0(3,1)\,.$$

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- Choose Cartan connection:
  - $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ splits into direct sum.
  - Let $e \in \Omega^1(P, \mathfrak{z})$ be the solder form of $\tilde{\pi} : P \to M$.
  - Let $\omega \in \Omega^1(P, \mathfrak{h})$ be the Levi-Civita connection.

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$\Rightarrow$ Spacetime (M, g) can be reconstructed from Cartan geometry.

Example: Cartan geometry of observer space

- Choose Lie groups: [S. Gielen, D. Wise '12]
  - Let
$$G = \begin{cases} SO_0(4,1) & \Lambda > 0 \\ ISO_0(3,1) & \Lambda = 0 \\ SO_0(3,2) & \Lambda < 0 \end{cases}, \quad K = SO(3)\,.$$
  - $\Rightarrow$ Coset spaces G/K are the maximally symmetric observer spaces.
- Choose principal K-bundle:
  - Let (M, g) be a Lorentzian manifold.
  - Let O be the future unit timelike vectors on M.
  - Let P be the oriented time-oriented orthonormal frames on M.
  - $\Rightarrow$ $\pi : P \to O$ is a principal K-bundle.
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The clock postulate

- Proper time along a curve in Lorentzian spacetime:

$$\tau = \int_{t_1}^{t_2} \sqrt{-g_{ab}(x(t))\dot{x}^a(t)\dot{x}^b(t)}dt\,.$$

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- Finsler geometry: use a more general length functional:

$$\tau = \int_{t_1}^{t_2} F(x(t),\dot{x}(t))dt\,.$$

- Finsler function $F : TM \to \mathbb{R}^+$.
- Parametrization invariance requires homogeneity:

$$F(x,\lambda y) = \lambda F(x,y) \quad \forall \lambda > 0\,.$$

Definition of Finsler spacetimes

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$$g^F_{ab}(x,y) = \frac{1}{2}\frac{\partial}{\partial y^a}\frac{\partial}{\partial y^b}F^2(x,y)\,.$$

- Unit vectors $y \in T_xM$ defined by

$$F^2(x,y) = g^F_{ab}(x,y)y^a y^b = 1\,.$$

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$\Rightarrow$ Set $\Omega_x \subset T_xM$ of unit timelike vectors at $x \in M$.

- Ω_x contains a closed connected component $S_x \subseteq \Omega_x$.
- Causality: S_x corresponds to physical observers.

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- Complete $y = f_0$ to a frame f_i with $g^F_{ab}(x, y)f_i^a f_j^b = -\eta_{ij}$.
- Let P be the space of all observer frames.

$\Rightarrow$ $\pi : P \to O$ is a principal SO(3)-bundle.

- In general no principal $\mathrm{SO}_0(3, 1)$-bundle $\tilde{\pi} : P \to M$.

- Need to construct $A \in \Omega^1(P, \mathfrak{g})$.
- Recall that

$$\begin{array}{ccccc} \mathfrak{g} & = & \mathfrak{h} & \oplus & \mathfrak{z} \\ A & = & \omega & + & e \end{array}$$

$\Rightarrow$ Need to construct $\omega \in \Omega^1(P, \mathfrak{h})$ and $e \in \Omega^1(P, \mathfrak{z})$.

Cartan connection - translational part

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- Definition of e: Use the *solder form*.
  - Let $w \in T_{(x,f)}P$ be a tangent vector.
  - Differential of the projection $\tilde{\pi} : P \to M$ yields $\tilde{\pi}_*(w) \in T_xM$.
  - View frame f as a linear isometry $f : \mathfrak{z} \to T_xM$.
  - Solder form given by $e(w) = f^{-1}(\tilde{\pi}_*(w))$.

Cartan connection - boost / rotational part

- Definition of ω:
  - Frames (x, f) and (x, f') related by generalized Lorentz transform.
    [C. Pfeifer, M. Wohlfarth '11]
  - Relation between f and f' defined by parallel transport on O.
  - Tangent vector $w \in T_{(x,f)}P$ "shifts" frame f by small amount.
  - Compare shifted frame with parallely transported frame.
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- Choose parallel transport on O so that g^F is covariantly constant.
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$\Rightarrow$ Frames f_i^a and $f_i^a + \Delta f_i^a$ are orthonormal wrt the same metric.

$\Rightarrow$ $\omega(w) \in \mathfrak{h}$ is an infinitesimal Lorentz transform.

Complete Cartan connection

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- Coefficients of Cartan linear connection:

$$N^a{}_b = \frac{1}{4}\bar{\partial}_b \left[g^{F\,aq}\left(y^p \partial_p \bar{\partial}_q F^2 - \partial_q F^2\right)\right] \,,$$

$$F^a{}_{bc} = \frac{1}{2} g^{F\,ap} \left(\delta_b g^F_{pc} + \delta_c g^F_{bp} - \delta_p g^F_{bc}\right) \,,$$

$$C^a{}_{bc} = \frac{1}{2} g^{F\,ap} \left(\bar{\partial}_b g^F_{pc} + \bar{\partial}_c g^F_{bp} - \bar{\partial}_p g^F_{bc}\right) \,.$$

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- Coefficients of Cartan linear connection:

$$N^a{}_b = \frac{1}{4}\bar{\partial}_b\left[g^{F\,aq}\left(y^p\partial_p\bar{\partial}_q F^2 - \partial_q F^2\right)\right]\,,$$
$$F^a{}_{bc} = \frac{1}{2}g^{F\,ap}\left(\delta_b g^F_{pc} + \delta_c g^F_{bp} - \delta_p g^F_{bc}\right)\,,$$
$$C^a{}_{bc} = \frac{1}{2}g^{F\,ap}\left(\bar{\partial}_b g^F_{pc} + \bar{\partial}_c g^F_{bp} - \bar{\partial}_p g^F_{bc}\right)\,.$$

$\Rightarrow$ $A = \omega + e$ is a Cartan connection on $\pi : P \to O$.

Fundamental vector fields

- Let $a = z^i \mathcal{Z}_i + \frac{1}{2} h^i{}_j \mathcal{H}_i{}^j \in \mathfrak{g}$.
- Define the vector field

$$\underline{A}(a) = z^i f_i^a \left(\partial_a - f_j^b F^c{}_{ab} \bar{\partial}_c^j \right) + \left(h^i{}_j f_i^a - h^i{}_0 f_i^b f_j^c C^a{}_{bc} \right) \bar{\partial}_a^j \,.$$

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⇒ For all $p \in P$ we find

$$A(\underline{A}(a)(p)) = a\,.$$

⇒ For all $w \in T_pP$ we find

$$\underline{A}(A(w))(p) = w\,.$$

⇒ $A_p : T_pP \to \mathfrak{g}$ and $\underline{A}_p : \mathfrak{g} \to T_pP$ complement each other.

Split of the tangent bundle TP

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- $\mathfrak{g}$ splits into irreducible subrepresentations of Ad.
- Induced decompositions of A and TP:

$$
\begin{array}{ccccccccc}
\mathfrak{g} & = & \mathfrak{k} & \oplus & \mathfrak{y} & \oplus & \vec{\mathfrak{z}} & \oplus & \mathfrak{z}0 \\
\uparrow A & = & \uparrow \Omega & + & \uparrow b & + & \uparrow \vec{e} & + & \uparrow e^0 \\
T_pP & = & R_pP & \oplus & B_pP & \oplus & \vec{H}_pP & \oplus & H^0_pP \\
 & & \Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow \\
 & & \text{rotations} & & \text{boosts} & & \text{spatial translations} & & \text{temporal translations}
\end{array}
$$

- Subbundles of TP spanned by fundamental vector fields $\underline{A}$.

Time translation

- Consider the fundamental vector field

$$\mathbf{t} = \underline{A}(\mathcal{Z}_0) = f_0^a \partial_a - f_j^a N^b{}_a \bar{\partial}_b^j \qquad \Leftrightarrow \qquad \omega^i{}_j(\mathbf{t}) = 0\,, \quad e^i(\mathbf{t}) = \delta_0^i\,.$$

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- From $\omega^\alpha{}_\beta(\mathbf{t}) = 0$ follows:

$$0 = \dot{f}_\alpha^a + f_\alpha^b \left(\dot{x}^c F^a{}_{bc} + (\dot{x}^d N^c{}_d + \dot{f}_0^c) C^a{}_{bc}\right) = \nabla_{(\dot{x}, \dot{f}_0)} f_\alpha^a\,.$$

⇒ Frame f is parallely transported.

Curvature of the Cartan connection

- Curvature $F \in \Omega^2(P, \mathfrak{g})$ defined by

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- Boost / rotational part $F_{\mathfrak{h}} \in \Omega^2(P, \mathfrak{h})$:

$$d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j = -\frac{1}{2} f^{-1\,i}{}_d f_j^c \Big(R^d{}_{cab} dx^a \wedge dx^b + 2P^d{}_{cab} dx^a \wedge \delta f_0^b + S^d{}_{cab} \delta f_0^a \wedge \delta f_0^b \Big)\,.$$

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- $R^d{}_{cab}$, $P^d{}_{cab}$, $S^d{}_{cab}$: curvature of Cartan linear connection.

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- Tangent spaces (with $o = \pi(p)$ and $x = \pi'(o) = \tilde{\pi}(p)$):

$$\begin{array}{ccccccc}
R_pP & \oplus & B_pP & \oplus & H_pP & = & T_pP \\
\downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \\
0 & & B_oO & \oplus & H_oO & = & T_oO \\
 & & \downarrow{\scriptstyle \pi'_*} & & \downarrow{\scriptstyle \pi'_*} & & \\
 & & 0 & & T_xM & = & T_xM
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Observer trajectories

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$$\mathbf{r}(o) = \pi_*(\mathbf{t}(p)) \, .$$

- Relation of $\mathbf{t}$ and $\mathbf{r}$:

$$\begin{array}{ccc} P & \xrightarrow{\pi} & O \\ \mathbf{t} \downarrow & & \downarrow \mathbf{r} \\ TP & \xrightarrow[\pi_*]{} & TO \end{array}$$

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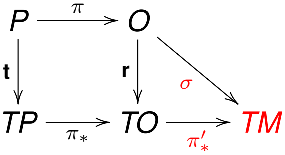

- Define the map $\sigma = \pi'_* \circ \mathbf{r}$.

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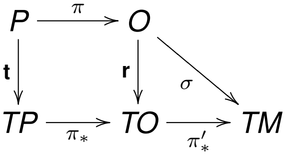

- Define the map $\sigma = \pi'_* \circ \mathbf{r}$.
- σ is in general not an embedding.
- Impose this as another condition.

Finsler geometry

- Finsler function must be positively homogeneous of degree one:

$$F(x, \lambda y) = |\lambda| F(x, y)$$

- Unit timelike condition: $F(\sigma(o)) = 1$ for all observers $o \in O$.

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- Condition: $\sigma(O)$ must intersect each line $(x, \mathbb{R}y)$ at most once.
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- No Finsler geometry on $TM \setminus \mathbb{R}\sigma(O)$.
- Cartan geometry describes only geometry visible to observers.

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Reconstruction of a given Finsler spacetime

- Idea:
  - Start from a Finsler spacetime (M, F).
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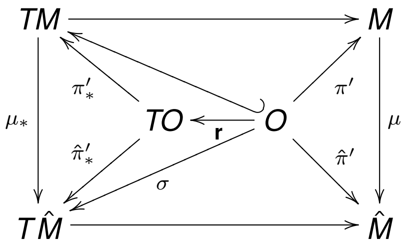

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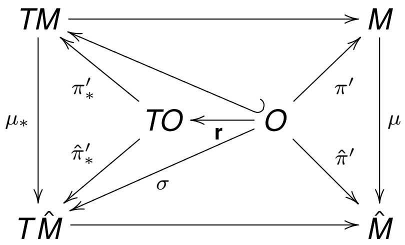

- μ preserves the Finsler function on timelike vectors.

$\Rightarrow$ Reconstruction of the original Finsler geometry.

Reconstruction of a given Cartan observer space

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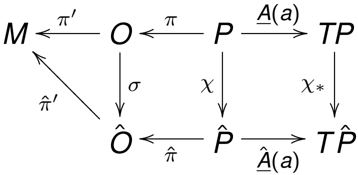

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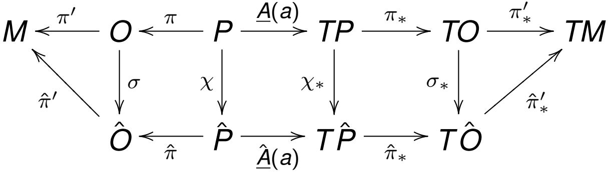

- Every Cartan morphism $\chi = (x, f)$ takes the form

$$x(p) = \pi'(\pi(p)), \quad f_i(p) = \pi'_*(\pi_*(\underline{A}(\mathcal{Z}_i)(p)))$$

$\Rightarrow$ Simple test for equivalence of $(\pi : P \to O, A)$ and $(\hat{\pi} : \hat{P} \to \hat{O}, \hat{A})$.

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Gravity from Cartan to Finsler

- MacDowell-Mansouri gravity on observer space: [S. Gielen, D. Wise '12]

$$S_G = \int_O \epsilon_{\alpha\beta\gamma}\, \mathrm{tr}_{\mathfrak{h}}(F_{\mathfrak{h}} \wedge \star F_{\mathfrak{h}}) \wedge b^\alpha \wedge b^\beta \wedge b^\gamma$$

  - Hodge operator $\star$ on $\mathfrak{h}$.
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- Translate terms into Finsler language (with $R = d\omega + \frac{1}{2}[\omega, \omega]$):
  - Curvature scalar:

$$[e, e] \wedge \star R \rightsquigarrow g^{F\,ab} R^c{}_{acb}\, dV\,.$$

  - Cosmological constant:

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$\Rightarrow$ Gravity theory on Finsler spacetime.

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  - …