Extensions of Lorentzian spacetime geometry
From Finsler to Cartan and vice versa

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Outline

1. Introduction
2. Cartan geometry on observer space
3. Finsler spacetimes
4. From Finsler geometry to Cartan geometry
5. From Cartan geometry to Finsler geometry
6. Closing the circle
7. Finsler-Cartan-Gravity
8. Conclusion
This work:


Cartan geometry of observer space:


Finsler spacetimes:


Physical motivation

- A simple experiment: light propagation in spacetime $M$.
  - A supernova occurs at some “beacon” event $x_0 \in M$.
  - Light from the supernova follows a null geodesic $\gamma$ in $M$.
  - An astronomer observes the light at another event $x \in M$. 

The measured data:
- Light intensity: photon rate measured with local clock.
- Spectrum: photon frequency measured with local clock.
- Location of the source: direction of incoming light in local frame.

Mathematical description of the measured data:
- General covariance: Physical quantities are tensors.
  - Tensor components are measured with respect to local frame.
  - No measurement without a frame.
  - ⇒ Consider observer frames as more fundamental than spacetime.
  - Geometric theory based on this assumption?
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Geometric theory based on this assumption?
Quantum gravity may suggest breaking of general covariance:
- Loop quantum gravity
- Spin foam models
- Causal dynamical triangulations
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Possible implications: Existence of...
- ...preferred observers / timelike vector fields.
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Problems:
- Breaking of Copernican principle.
- No observation of (strongly) broken symmetry.
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- Problems:
  - Breaking of Copernican principle.
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- Solution:
  - Consider space $O$ of all allowed observers.
  - Describe experiments on observer space instead of spacetime.
    $\Rightarrow$ Observer dependence of physical quantities follows naturally.
    $\Rightarrow$ No preferred observers.
  - Geometry of observer space modeled by Cartan geometry.
Why Finsler geometry of spacetimes?

- Finsler geometry of space widely used in physics:
  - Approaches to quantum gravity
  - Electrodynamics in anisotropic media
  - Modeling of astronomical data

Finsler geometry generalizes Riemannian geometry:
- Clock postulate: proper time equals arc length along trajectories.
- Geometry described by Finsler metric.
- Well-defined notions of connections, curvature, parallel transport.

Finsler spacetimes are suitables backgrounds for:
- Gravity
- Electrodynamics
- Other matter field theories

Possible explanations of yet unexplained phenomena:
- Fly-by anomaly
- Galaxy rotation curves
- Accelerating expansion of the universe

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Ingredients of a Cartan geometry:
- A Lie group $G$ with a closed subgroup $H \subset G$.
- A principal $H$-bundle $\pi : P \rightarrow M$.
- A 1-form $A \in \Omega^1(P, g)$ on $P$ with values in $g$. 

Curvature of the Cartan connection:
The curvature is defined by $F = dA + \frac{1}{2}[A, A]$.

Curvature measures deviation between $M$ and $G/H$.
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Conditions on the Cartan connection $A$:
- For each $p \in P$, $A_p : T_p P \to g$ is a linear isomorphism.
- $A$ is right-equivariant: $(R_h)^* A = \text{Ad}(h^{-1}) \circ A \quad \forall h \in H$.
- $A$ restricts to the Maurer-Cartan form of $H$ on $\ker \pi^*$.
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- Fundamental vector fields:
  - $A$ has an “inverse” $\overline{A} : g \rightarrow \Gamma(TP)$.
  - Vector fields $\overline{A}(a)$ for $a \in g$ are nowhere vanishing.

Geometry of $M$:
- Cartan connection describes geometry and parallel transport on $M$.
- $M$ “locally looks like” homogeneous space $G/H$.
- Tangent spaces $T_xM \sim = z = g/h$.
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Example: Cartan geometry of spacetime

Choose Lie groups:
- Let

\[
G = \begin{cases} 
\text{SO}_0(4, 1) & \Lambda > 0 \\
\text{ISO}_0(3, 1) & \Lambda = 0 \\
\text{SO}_0(3, 2) & \Lambda < 0 
\end{cases}
\]

\[ H = \text{SO}_0(3, 1). \]

\[ \Rightarrow \text{Coset spaces } G/H \text{ are the maximally symmetric spacetimes.} \]
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⇒ Coset spaces \( G/H \) are the maximally symmetric spacetimes.

Choose principal \( H \)-bundle:

Let \( (M, g) \) be a Lorentzian manifold.

Let \( P \) be the oriented time-oriented orthonormal frames on \( M \).

⇒ \( \tilde{\pi} : P \rightarrow M \) is a principal \( H \)-bundle.
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Choose Cartan connection:

\[ g = \mathfrak{h} \oplus \mathfrak{z} \text{ splits into direct sum.} \]
Let \(e \in \Omega^1(P, \mathfrak{z})\) be the solder form of \(\tilde{\pi} : P \to M\).
Let \(\omega \in \Omega^1(P, \mathfrak{h})\) be the Levi-Civita connection.

\[ A = \omega + e \in \Omega^1(P, g) \text{ is a Cartan connection.} \]
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⇒ Spacetime \((M, g)\) can be reconstructed from Cartan geometry.
Choose Lie groups:  [S. Gielen, D. Wise ’12]

\[ G = \begin{cases} 
    \text{SO}_0(4, 1) & \Lambda > 0 \\
    \text{ISO}_0(3, 1) & \Lambda = 0 , \quad K = \text{SO}(3) . \\
    \text{SO}_0(3, 2) & \Lambda < 0 
\end{cases} \]

\( \Rightarrow \) Coset spaces \( G/K \) are the maximally symmetric observer spaces.

Choose principal \( K \)-bundle:

\( \Rightarrow \) Let \( (M, g) \) be a Lorentzian manifold. 
\( \Rightarrow \) Let \( O \) be the future unit timelike vectors on \( M \). 
\( \Rightarrow \) Let \( P \) be the oriented time-oriented orthonormal frames on \( M \). 
\( \Rightarrow \pi : P \to O \) is a principal \( K \)-bundle.

Choose Cartan connection:

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Proper time along a curve in Lorentzian spacetime:

\[
\tau = \int_{t_1}^{t_2} \sqrt{-g_{ab}(x(t))\dot{x}^a(t)\dot{x}^b(t)} dt.
\]
The clock postulate

- Proper time along a curve in Lorentzian spacetime:
\[ \tau = \int_{t_1}^{t_2} \sqrt{-g_{ab}(x(t)) \dot{x}^a(t) \dot{x}^b(t)} \, dt . \]

- Finsler geometry: use a more general length functional:
\[ \tau = \int_{t_1}^{t_2} F(x(t), \dot{x}(t)) \, dt . \]

- Finsler function \( F : TM \to \mathbb{R}^+ \).

- Parametrization invariance requires homogeneity:
\[ F(x, \lambda y) = \lambda F(x, y) \quad \forall \lambda > 0 . \]
Finsler geometries suitable for spacetimes exist. [C. Pfeifer, M. Wohlfarth ’11]

→ Notion of timelike, lightlike, spacelike tangent vectors.
Definition of Finsler spacetimes

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⇒ Notion of timelike, lightlike, spacelike tangent vectors.

- Finsler metric with Lorentz signature:
  \[
g_{ab}^F(x, y) = \frac{1}{2} \frac{\partial}{\partial y^a} \frac{\partial}{\partial y^b} F^2(x, y).
\]

- Unit vectors \( y \in T_x M \) defined by
  \[
  F^2(x, y) = g_{ab}^F(x, y)y^a y^b = 1.
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\[ \implies \] Set \( \Omega_x \subset T_x M \) of unit timelike vectors at \( x \in M \).

- \( \Omega_x \) contains a closed connected component \( S_x \subseteq \Omega_x \).

- Causality: \( S_x \) corresponds to physical observers.
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Observer space

- Recall from the definition of Finsler spacetimes:
  - Set $\Omega_x \subset T_x M$ of unit timelike vectors at $x \in M$.
  - $\Omega_x$ contains a closed connected component $S_x \subseteq \Omega_x$. 

$O = \bigcup_{x \in M} S_x$.

$\Rightarrow$ Tangent vectors $y \in S_x$ satisfy $g_{F^{ab}}(x, y) y^a y^b = 1$.

Complete $y = f_0$ to a frame $f_i$ with $g_{F^{ab}}(x, y) f^a_i f^b_j = -\eta_{ij}$.

Let $P$ be the space of all observer frames.

$\Rightarrow \pi: P \to O$ is a principal $SO(3)$-bundle.

In general no principal $SO(0(3,1))$-bundle $\tilde{\pi}: P \to M$. 

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  - In general no principal $\text{SO}_0(3, 1)$-bundle $\tilde{\pi} : P \to M$. 
Need to construct \( A \in \Omega^1(P, g) \).

Recall that

\[
\begin{align*}
g & = h \oplus \mathcal{Z} \\
A & = \omega + e
\end{align*}
\]

\[\Rightarrow\] Need to construct \( \omega \in \Omega^1(P, h) \) and \( e \in \Omega^1(P, \mathcal{Z}) \).
Need to construct $A \in \Omega^1(P, g)$.

Recall that

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g = h \oplus z
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A = \omega + e
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\[\Rightarrow\]

Need to construct $\omega \in \Omega^1(P, h)$ and $e \in \Omega^1(P, z)$.

Definition of $e$: Use the solder form.

- Let $w \in T_{(x,f)}P$ be a tangent vector.
- Differential of the projection $\tilde{\pi} : P \rightarrow M$ yields $\tilde{\pi}^* (w) \in T_x M$.
- View frame $f$ as a linear isometry $f : z \rightarrow T_x M$.
- Solder form given by $e(w) = f^{-1}(\tilde{\pi}_* (w))$. 
Definition of $\omega$:

- Frames $(x, f)$ and $(x, f')$ related by generalized Lorentz transform. 
  [C. Pfeifer, M. Wohlfarth '11]
- Relation between $f$ and $f'$ defined by parallel transport on $O$.
- Tangent vector $w \in T_{(x,f)}P$ "shifts" frame $f$ by small amount.
- Compare shifted frame with parallely transported frame.
- Measure the difference using the original frame:

$$\Delta f_i^a = \epsilon f_i^a \omega^i_j(w).$$
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Choose parallel transport on $O$ so that $g^F$ is covariantly constant.

Connection on Finsler geometry: Cartan linear connection.
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Connection on Finsler geometry: Cartan linear connection.

⇒ Frames $f^a_i$ and $f^a_i + \Delta f^a_i$ are orthonormal wrt the same metric.

⇒ $\omega(w) \in \mathfrak{h}$ is an infinitesimal Lorentz transform.
Complete Cartan connection

- Translational part $e \in \Omega^1(P, \delta)$:
  
  $$e^i = f^{-1}_{i}a dx^a.$$
Complete Cartan connection

- Translational part $e \in \Omega^1(P, \mathfrak{e})$:
  \[ e^i = f^{-1}i_a dx^a. \]

- Boost / rotational part $\omega \in \Omega^1(P, \mathfrak{h})$:
  \[ \omega^i_j = f^{-1}i_a \left[ df^a_j + f^b_j \left( dx^c F^a_{bc} + (dx^d N^c_d + df^c_0) C^a_{bc} \right) \right]. \]
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- Translational part \( e \in \Omega^1(P, \mathfrak{h}) \):
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- Coefficients of Cartan linear connection:
  \[
  \begin{align*}
    N^a_b &= \frac{1}{4} \bar{\partial}_b \left[ g^F_{aq} \left( y^p \partial_p \bar{\partial}_q F^2 - \partial_q F^2 \right) \right], \\
    F^a_{bc} &= \frac{1}{2} g^F_{ap} \left( \delta_b g^F_{pc} + \delta_c g^F_{bp} - \delta_p g^F_{bc} \right), \\
    C^a_{bc} &= \frac{1}{2} g^F_{ap} \left( \bar{\partial}_b g^F_{pc} + \bar{\partial}_c g^F_{bp} - \bar{\partial}_p g^F_{bc} \right).
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  \[
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$\Rightarrow A = \omega + e$ is a Cartan connection on $\pi : P \rightarrow O$. 
Fundamental vector fields

- Let \( a = z^i Z_i + \frac{1}{2} h^i_j \mathcal{H}_j \in \mathfrak{g} \).
- Define the vector field
  \[
  A(a) = z^i f^a_i \left( \partial_a - f^b_j F^c_{ab} \bar{\partial}^j_c \right) + \left( h^i_j f^a_i - h^i_0 f^b_j f^c_j C^a_{bc} \right) \bar{\partial}^j_a.
  \]
Let \( a = z^i Z_i + \frac{1}{2} h^i_j \mathcal{H}_i \overline{j} \in \mathfrak{g} \).

Define the vector field

\[
A(a) = z^i f_i^a \left( \partial_a - f_j^b F^c_{ab} \overline{j}^i \right) + \left( h^i_j f_i^a - h^i_0 f_i^b f^c_j C^a_{bc} \right) \overline{j}^a.
\]

\[\Rightarrow\] For all \( p \in P \) we find

\[A(A(a)(p)) = a.\]

\[\Rightarrow\] For all \( w \in T_p P \) we find

\[A(A(w))(p) = w.\]

\[\Rightarrow\] \( A_p : T_p P \to g \) and \( A_p : g \to T_p P \) complement each other.
Consider adjoint representation $\text{Ad} : K \subset G \rightarrow \text{Aut}(\mathfrak{g})$ of $K$ on $\mathfrak{g}$.

$\mathfrak{g}$ splits into irreducible subrepresentations of $\text{Ad}$.
Split of the tangent bundle $TP$

- Consider adjoint representation $\text{Ad} : K \subset G \rightarrow \text{Aut}(\mathfrak{g})$ of $K$ on $\mathfrak{g}$.
- $\mathfrak{g}$ splits into irreducible subrepresentations of $\text{Ad}$.
- Induced decompositions of $A$ and $TP$:

$$
\mathfrak{g} = \xi \oplus \eta \oplus \vec{\delta} \oplus \delta_0 \\
A = \Omega \oplus b \oplus \vec{e} \oplus e^0 \\
TP = R_pP \oplus B_pP \oplus H_pP \oplus H^0_pP
$$

- Subbundles of $TP$ spanned by fundamental vector fields $A$. 

rotations boosts spatial translations temporal translations
Consider the fundamental vector field
\[ t = A(\mathcal{Z}_0) = f_0^a \partial_a - f_j^a N_a^b \bar{j}_b \quad \Leftrightarrow \quad \omega^i_j(t) = 0, \quad e^i(t) = \delta^i_0. \]

Integral curve \( \Gamma : \mathbb{R} \to P, \lambda \mapsto (x(\lambda), f(\lambda)) \) of \( t \).
Consider the fundamental vector field
\[ t = A(Z_0) = f_0^a \partial_a - f_j^a N^b_{a\bar{b}} \bar{\partial}_b \iff \omega^i_j(t) = 0, \quad e^i(t) = \delta^i_0. \]

Integral curve \( \Gamma : \mathbb{R} \rightarrow P, \lambda \mapsto (x(\lambda), f(\lambda)) \) of \( t \).

From \( e^i(t) = \delta^i_0 \) follows:
\[ \dot{x}^a = f_0^a. \]

\( \Rightarrow (x, f_0) \) is the canonical lift of a curve from \( M \) to \( O \).
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From \( \omega^i_0(t) = 0 \) follows:
\[ 0 = \dot{f}_0^a + N^a_b \dot{x}^b = \ddot{x}^a + N^a_b \dot{x}^b. \]

\( (x, f_0) \) is a Finsler geodesic.
Consider the fundamental vector field 
\[ t = A(Z_0) = f_0^a \partial_a - f_j^a N_b^a \tilde{\partial}_b \quad \iff \quad \omega^i_j(t) = 0, \quad e^i(t) = \delta_0^i. \]

Integral curve \( \Gamma : \mathbb{R} \to P, \lambda \mapsto (x(\lambda), f(\lambda)) \) of \( t \).

From \( e^i(t) = \delta_0^i \) follows:
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\( (x, f_0) \) is a Finsler geodesic.

From \( \omega^\alpha_\beta(t) = 0 \) follows:
\[ 0 = \dot{f}_\alpha^a + f_\alpha^b \left( \dot{x}^c F^a_{bc} + (\dot{x}^d N^c_d + \dot{f}_0^c) C^a_{bc} \right) = \nabla_{(\dot{x}, \dot{f}_0)} f_\alpha^a. \]

\( \Rightarrow \) Frame \( f \) is parallely transported.
Curvature of the Cartan connection

- Curvature \( F \in \Omega^2(P, g) \) defined by

\[
F = dA + \frac{1}{2}[A, A].
\]
Curvature of the Cartan connection

- Curvature $F \in \Omega^2(P, \mathfrak{g})$ defined by

$$F = dA + \frac{1}{2}[A, A].$$

- Translational part $F_\xi \in \Omega^2(P, \xi)$ (“torsion”):

$$de^i + \omega^i_{\;j} \wedge e^j = -f^{-1i}_a C^{a}_{bc} dx^b \wedge \delta f^c_0$$

with $\delta f^c_0 = N^c_d dx^d + df^c_0$. 

$R^d_{cab}$, $P^d_{cab}$, $S^d_{cab}$: curvature of Cartan linear connection.
Curvature of the Cartan connection

- Curvature $F \in \Omega^2(P, g)$ defined by
  \[
  F = dA + \frac{1}{2}[A, A].
  \]

- Translational part $F_\delta \in \Omega^2(P, \delta)$ ("torsion"): \[
  de^i + \omega^i{}_j \wedge e^j = -f^{-1i}{}_a C^a{}_{bc} dx^b \wedge \delta f^c_0
  \]
  with $\delta f^c_0 = N^c{}_d dx^d + df^c_0$.

- Boost / rotational part $F_h \in \Omega^2(P, h)$:

  \[
  d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j = -\frac{1}{2}f^{-1i}{}_d f^c_0 \left( R^d{}_{cab} dx^a \wedge dx^b 
  + 2P^d{}_{cab} dx^a \wedge \delta f^b_0 + S^d{}_{cab} \delta f^a_0 \wedge \delta f^b_0 \right).
  \]
Curvature of the Cartan connection

- Curvature $F \in \Omega^2(P, g)$ defined by
  \[ F = dA + \frac{1}{2}[A, A]. \]

- Translational part $F_z \in \Omega^2(P, z)$ ("torsion"):  
  \[ de^i + \omega^i_j \wedge e^j = - f^{-1} i_a C_{abc} dx^b \wedge \delta f^c_0 \]
  with $\delta f^c_0 = N^c d dx^d + df^c_0$.

- Boost / rotational part $F_h \in \Omega^2(P, h)$:  
  \[ d\omega^i j + \omega^i_k \wedge \omega^k j = - \frac{1}{2} f^{-1} i_d f^c j \left( R^d_{cab} dx^a \wedge dx^b \right. \]
  \[ \left. + 2 P^d_{cab} dx^a \wedge \delta f^b_0 + S^d_{cab} \delta f^a_0 \wedge \delta f^b_0 \right). \]

- $R^d_{cab}, P^d_{cab}, S^d_{cab}$: curvature of Cartan linear connection.
Condition: boost distribution $VP = RP \oplus BP$ is integrable.
$\Rightarrow$ $VP$ can be integrated to a foliation $\mathcal{F}$ with leaf space $M$. 
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Condition: foliation $\mathcal{F}$ is strictly simple.

$\Rightarrow$ Leaf space $M$ is a smooth manifold.

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Canonical projections $\tilde{\pi} = \pi' \circ \pi$:

$$
\begin{array}{ccc}
P & \xrightarrow{\pi} & O & \xrightarrow{\pi'} & M \\
\text{\rotatebox[origin=c]{90}{$\tilde{\pi}$}} & & \text{\rotatebox[origin=c]{90}{$\tilde{\pi}$}} & & \\
\end{array}
$$
Condition: boost distribution $VP = RP \oplus BP$ is integrable.

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Canonical projections $\tilde{\pi} = \pi' \circ \pi$:

$$
\begin{array}{c}
P \xrightarrow[\tilde{\pi}]{\pi} O \xrightarrow[\pi']{\pi'} M \\
\end{array}
$$

Tangent spaces (with $o = \pi(p)$ and $x = \pi'(o) = \tilde{\pi}(p)$):

$$
\begin{align*}
R_pP & \oplus B_pP \oplus H_pP = T_pP \\
0 & \oplus B_oO \oplus H_oO = T_oO \\
0 & = T_xM
\end{align*}
$$
Observer trajectories

- Embedding of observer space $O$ into $TM$?
- Four-velocity of an observer?

Fundamental vector field $t = A(Z_0) \in \Gamma(TP)$ of time translation.

$\Rightarrow$ Vector field $r \in \Gamma(TO)$ independent of $p \in \pi^{-1}(o)$:

$r(o) = \pi^*(t(p))$.

Relation of $t$ and $r$:

$\Pi \pi^{-1} \rightarrow t \downarrow \downarrow O \rightarrow r \downarrow \downarrow TP \pi^* \rightarrow TO$

Define the map $\sigma = \pi' \ast \circ r$.

$\sigma$ is in general not an embedding.

Impose this as another condition.
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- Relation of $t$ and $r$:

\[
\begin{array}{c}
\text{P} \xrightarrow{\pi} \text{O} \\
\downarrow t \quad \downarrow r \\
\text{TP} \xrightarrow{\pi_*} \text{TO}
\end{array}
\]

Define the map $\sigma = \pi' \circ r$. $\sigma$ is in general not an embedding.
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$$\mathbf{r}(o) = \pi_*(\mathbf{t}(p)).$$

- Relation of $\mathbf{t}$ and $\mathbf{r}$:

\[ P \overset{\pi}{\longrightarrow} O \]
\[ TP \overset{\pi_*}{\longrightarrow} TO \]
\[ TO \overset{\pi'_*}{\longrightarrow} TM \]

- Define the map $\sigma = \pi'_* \circ \mathbf{r}$. 
Observer trajectories

- Embedding of observer space $O$ into $TM$?
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\[ P \xrightarrow{\pi} O \]
\[ TP \xrightarrow{\pi_*} TO \xrightarrow{\pi'_*} TM \]

- Define the map $\sigma = \pi'_* \circ r$.
- $\sigma$ is in general not an embedding.
- Impose this as another condition.
Finsler geometry

- Finsler function must be positively homogeneous of degree one:
  \[ F(x, \lambda y) = |\lambda|F(x, y) \]
- Unit timelike condition: \( F(\sigma(o)) = 1 \) for all observers \( o \in O \).
Finsler function must be positively homogeneous of degree one:

\[ F(x, \lambda y) = |\lambda| F(x, y) \]

Unit timelike condition: \( F(\sigma(o)) = 1 \) for all observers \( o \in O \).

\[ \Rightarrow \text{ Define } F(\lambda \sigma(o)) = |\lambda| \text{ on double cone } \mathbb{R}\sigma(O). \]
Finsler geometry

- Finsler function must be positively homogeneous of degree one:
  \[ F(x, \lambda y) = |\lambda|F(x, y) \]

- Unit timelike condition: \( F(\sigma(o)) = 1 \) for all observers \( o \in O \).
  \[ \Rightarrow \] Define \( F(\lambda \sigma(o)) = |\lambda| \) on double cone \( \mathbb{R}\sigma(O) \).

- Condition: \( \sigma(O) \) must intersect each line \((x, \mathbb{R}y)\) at most once.

- Condition: Finsler metric \( g_{ab}^F \) must have Lorentz signature:
  \[ g_{ab}^F = \frac{1}{2} \tilde{\partial}_a \tilde{\partial}_b F^2 \]
Finsler geometry

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- Condition: Finsler metric \( g^F_{ab} \) must have Lorentz signature:
  \[ g^F_{ab} = \frac{1}{2} \partial_a \partial_b F^2 \]

- No Finsler geometry on \( TM \setminus \mathbb{R}\sigma(O) \).

- Cartan geometry describes only geometry visible to observers.
1. Introduction
2. Cartan geometry on observer space
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5. From Cartan geometry to Finsler geometry
6. Closing the circle
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8. Conclusion
Reconstruction of a given Finsler spacetime

- Idea:
  - Start from a Finsler spacetime \((M, F)\).
  - Construct a Cartan observer space \((\pi : P \rightarrow O, A)\).
  - Construct a new Finsler spacetime \((\hat{M}, \hat{F})\).
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- **Equivalence of Finsler spacetimes** \((M, F)\) and \((\hat{M}, \hat{F})\)?
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- Equivalence of Finsler spacetimes \((M, F)\) and \((\hat{M}, \hat{F})\)?
- There exists a diffeomorphism \(\mu\):

\[
\begin{array}{c}
\text{T}M & \xrightarrow{\mu^*} & M \\
\downarrow & & \downarrow \\
T\hat{M} & \xrightarrow{\hat{\pi}^*} & \hat{M}
\end{array}
\]

\[
\begin{array}{c}
\pi^* & \xleftarrow{\pi'} & \pi^* \\
\downarrow & & \downarrow \\
\sigma & \xleftarrow{r} & \hat{\pi}^*
\end{array}
\]
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- Equivalence of Finsler spacetimes \((M, F)\) and \((\hat{M}, \hat{F})\)?
- There exists a diffeomorphism \(\mu:\)

\[
\begin{array}{ccc}
TM & \xrightarrow{\mu} & M \\
\downarrow \pi^* & \downarrow \pi' & \downarrow \pi \\
\hat{T}\hat{M} & \xleftarrow{\hat{\pi}^*} & \hat{M}
\end{array}
\]

- \(\mu\) preserves the Finsler function on timelike vectors.

\(\Rightarrow\) Reconstruction of the original Finsler geometry.
Reconstruction of a given Cartan observer space

Idea:
- Start from a Cartan observer space $\left( \pi : P \rightarrow O, A \right)$.
- Construct a Finsler spacetime $\left( M, F \right)$.
- Construct a new Cartan observer space $\left( \hat{\pi} : \hat{P} \rightarrow \hat{O}, \hat{A} \right)$. 

Equivalence of $\left( \pi : P \rightarrow O, A \right)$ and $\left( \hat{\pi} : \hat{P} \rightarrow \hat{O}, \hat{A} \right)$?

Only if a "Cartan morphism" $\chi$ exists:

\[\begin{array}{ccc}
M & \overset{\sigma}{\leftarrow} & O \\
\downarrow & & \downarrow \\
\pi & \leftarrow & \pi' \\
\downarrow & & \downarrow \\
\sigma & \leftarrow & \chi \\
\downarrow & & \downarrow \\
A(p) & \rightarrow & \chi_A(p) \\
\downarrow & & \downarrow \\
\hat{P} & \leftarrow & \hat{O} \\
\downarrow & & \downarrow \\
\hat{\pi} & \leftarrow & \hat{\pi}' \\
\downarrow & & \downarrow \\
\hat{A}(p) & \rightarrow & \hat{A}(p) \\
\end{array}\]

Every Cartan morphism $\chi = (\chi_x, \chi_A)$ takes the form $\chi_x(p) = \pi'(\pi(p))$, $\chi_A(p) = \pi'\circ \pi_A(Z_i(p))$. 

Simple test for equivalence of $\left( \pi : P \rightarrow O, A \right)$ and $\left( \hat{\pi} : \hat{P} \rightarrow \hat{O}, \hat{A} \right)$. 

Reconstruction of a given Cartan observer space

- **Idea:**
  - Start from a Cartan observer space \((\pi: P \rightarrow O, A)\).
  - Construct a Finsler spacetime \((M, F)\).
  - Construct a new Cartan observer space \((\hat{\pi}: \hat{P} \rightarrow \hat{O}, \hat{A})\).
  - **Equivalence of** \((\pi: P \rightarrow O, A)\) **and** \((\hat{\pi}: \hat{P} \rightarrow \hat{O}, \hat{A})\) **?**
Reconstruction of a given Cartan observer space

Idea:
- Start from a Cartan observer space \((\pi : P \rightarrow O, A)\).
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Equivalence of \((\pi : P \rightarrow O, A)\) and \((\hat{\pi} : \hat{P} \rightarrow \hat{O}, \hat{A})\)?

Only if a “Cartan morphism” \(\chi\) exists:

\[
\begin{array}{c}
M & \xleftarrow{\pi'} & O & \xleftarrow{\pi} & P & \xrightarrow{A(a)} & TP \\
\hat{O} & \xleftarrow{\hat{\pi}} & \hat{P} & \xrightarrow{\hat{A}(a)} & T \hat{P}
\end{array}
\]
Reconstruction of a given Cartan observer space

- **Idea:**
  - Start from a Cartan observer space \((\pi : P \rightarrow O, A)\).
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- **Only if a “Cartan morphism” \(\chi\) exists:**

\[
\begin{array}{ccccccccc}
M & \xleftarrow{\pi'} & O & \xleftarrow{\pi} & P & \xrightarrow{A(a)} & TP & \xrightarrow{\pi_*} & TO & \xrightarrow{\pi'_*} & TM \\
\downarrow{\hat{\pi}'} & & \downarrow{\sigma} & & \downarrow{\chi} & & \downarrow{\chi_*} & & \downarrow{\sigma_*} & & \downarrow{\hat{\pi}'_*} \\
\hat{O} & \xleftarrow{\hat{\pi}} & \hat{P} & \xrightarrow{\hat{A}(a)} & T\hat{P} & \xrightarrow{\hat{\pi}_*} & T\hat{O} & & & & \\
\end{array}
\]

- **Every Cartan morphism** \(\chi = (x, f)\) **takes the form**

\[
x(p) = \pi'(\pi(p)), \quad f_i(p) = \pi'_*(\pi_*(A(Z_i)(p)))
\]

\Rightarrow \text{ Simple test for equivalence of } (\pi : P \rightarrow O, A) \text{ and } (\hat{\pi} : \hat{P} \rightarrow \hat{O}, \hat{A}).
MacDowell-Mansouri gravity on observer space: [S. Gielen, D. Wise ’12]

\[ S_G = \int_O \epsilon_{\alpha\beta\gamma} \text{tr}_\mathcal{H}(F_\mathcal{H} \wedge \star F_\mathcal{H}) \wedge b^\alpha \wedge b^\beta \wedge b^\gamma \]

- Hodge operator \( \star \) on \( \mathcal{H} \).
- Non-degenerate \( H \)-invariant inner product \( \text{tr}_\mathcal{H} \) on \( \mathcal{H} \).
MacDowell-Mansouri gravity on observer space: \cite{GielenWise2012}

\[
S_G = \int_O \epsilon_{\alpha\beta\gamma} \text{tr}_h(F_h \wedge \star F_h) \wedge b^\alpha \wedge b^\beta \wedge b^\gamma
\]

- Hodge operator $\star$ on $h$.
- Non-degenerate $H$-invariant inner product $\text{tr}_h$ on $h$.
- Translate terms into Finsler language (with $R = d\omega + \frac{1}{2}[\omega, \omega]$):
  - Curvature scalar:
    \[
    [e, e] \wedge \star R \leadsto g^{F \, ab} R^c_{\, acb} \, dV.
    \]
  - Cosmological constant:
    \[
    [e, e] \wedge \star [e, e] \leadsto dV.
    \]
  - Gauss-Bonnet term:
    \[
    R \wedge \star R \leadsto \epsilon^{abcd} \epsilon^{efgh} R_{abef} R_{cdgh} \, dV.
    \]
Gravity from Cartan to Finsler

- MacDowell-Mansouri gravity on observer space: [S. Gielen, D. Wise ’12]

\[ S_G = \int O \epsilon_{\alpha\beta\gamma} \text{tr}_\mathfrak{h}(F_\mathfrak{h} \wedge \star F_\mathfrak{h}) \wedge b^\alpha \wedge b^\beta \wedge b^\gamma \]

- Hodge operator \( \star \) on \( \mathfrak{h} \).
- Non-degenerate \( H \)-invariant inner product \( \text{tr}_\mathfrak{h} \) on \( \mathfrak{h} \).

- Translate terms into Finsler language (with \( R = d\omega + \frac{1}{2}[[\omega, \omega]] \)):
  - Curvature scalar:
    \[ [e, e] \wedge \star R \leadsto g^{F ab} R^c_{acb} dV . \]
  - Cosmological constant:
    \[ [e, e] \wedge \star [e, e] \leadsto dV . \]
  - Gauss-Bonnet term:
    \[ R \wedge \star R \leadsto \epsilon^{abcd} \epsilon^{efgh} R_{abef} R_{cdgh} dV . \]

\( \Rightarrow \) Gravity theory on Finsler spacetime.
Finsler gravity action: [C. Pfeifer, M. Wohlfarth '11]

\[ S_G = \int_O d^4x \, d^3y \, \sqrt{-\tilde{G} R^a_{ab} y^b}. \]

- Sasaki metric \( \tilde{G} \) on \( O \).
- Non-linear curvature \( R^a_{ab} \).
Finsler gravity action: [C. Pfeifer, M. Wohlfarth '11]

\[ S_G = \int_O d^4x \: d^3y \: \sqrt{-\tilde{G}} R^a_{\;ab} y^b. \]

- Sasaki metric \( \tilde{G} \) on \( O \).
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Translate terms into Cartan language:

\[ d^4x \: d^3y \: \sqrt{-\tilde{G}} = \epsilon_{ijkl} \epsilon_{\alpha\beta\gamma} e^i \wedge e^j \wedge e^k \wedge e^l \wedge b^{\alpha} \wedge b^{\beta} \wedge b^{\gamma}, \]

\[ R^a_{\;ab} y^b = b^{\alpha} [A(Z_\alpha), A(Z_0)]. \]
Finsler gravity action: [C. Pfeifer, M. Wohlfarth '11]

\[ S_G = \int_O d^4 x \, d^3 y \sqrt{-\tilde{G}} R^a_{\ ab} y^b. \]

- Sasaki metric $\tilde{G}$ on $O$.
- Non-linear curvature $R^a_{\ ab}$.

Translate terms into Cartan language:

\[
d^4 x \, d^3 y \sqrt{-\tilde{G}} = \epsilon_{ijkl} \epsilon_{\alpha\beta\gamma} e^i \wedge e^j \wedge e^k \wedge e^l \wedge b^\alpha \wedge b^\beta \wedge b^\gamma,\]

\[
R^a_{\ ab} y^b = b^\alpha [A(\mathcal{Z}_\alpha), A(\mathcal{Z}_0)].
\]

⇒ Gravity theory on observer space.
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Summary

- Observer space:
  - Lift physics from spacetime to the space of observers.
  - Describe observer space geometry using Cartan geometry.
Summary

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  - Lift physics from spacetime to the space of observers.
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  - Based on generalized length functional.
  - Finsler metric is observer dependent.
Summary

- **Observer space:**
  - Lift physics from spacetime to the space of observers.
  - Describe observer space geometry using Cartan geometry.

- **Finsler spacetime:**
  - Based on generalized length functional.
  - Finsler metric is observer dependent.

- **From Finsler to Cartan:**
  - Cartan geometry on observer space derived from Finsler geometry.
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  - Parallely transported observer frames given by the “flow of time”.

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  - Observers have timelike four-velocities in $TM$.

- **Gravity:**
  - MacDowell-Mansouri gravity from Cartan to Finsler.
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Both constructions complement each other.
Summary

Observer space:
- Lift physics from spacetime to the space of observers.
- Describe observer space geometry using Cartan geometry.

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Outlook

Current projects:
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- Translate more terms between both languages.

Future projects:
- Consistent matter coupling.
- Study of exact solutions.
- Effects of deviations from metric geometry?
- Geometrodynamics of Finsler spacetimes.
- …