# Extensions of Lorentzian spacetime geometry

## From Finsler to Cartan and vice versa

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13. November 2013

# Outline


1. Introduction
2. Cartan geometry on observer space
3. Finsler spacetimes
4. From Finsler geometry to Cartan geometry
5. From Cartan geometry to Finsler geometry
6. Closing the circle
7. Finsler-Cartan-Gravity
8. Conclusion

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# References


- This work:
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- Cartan geometry of observer space:
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- Finsler spacetimes:
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# Physical motivation

- A simple experiment: light propagation in spacetime $(M, g)$.
  - A supernova occurs at some “beacon” event $x_0 \in M$.
  - Light from the supernova follows a null geodesic $\gamma$ in $M$.
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$\Rightarrow$ Consider observer frames as more fundamental than spacetime.

$\Rightarrow$ Spacetime emerges from equivalence classes of observer frames.

- Geometric theory based on this assumption?

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- Problems:
  - Breaking of Copernican principle.
  - No observation of (strongly) broken symmetry.
- Solution:
  - Consider space $O$ of all allowed observers.
  - Describe experiments on observer space instead of spacetime.
  - $\Rightarrow$ Observer dependence of physical quantities follows naturally.
  - $\Rightarrow$ No preferred observers.
  - Geometry of observer space modeled by Cartan geometry.

# Why Finsler geometry of spacetimes?

- Finsler geometry of space widely used in physics:
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- Possible explanations of yet unexplained phenomena:
  - Fly-by anomaly
  - Galaxy rotation curves
  - Accelerating expansion of the universe

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  - Local infinitesimal Lorentz transforms:
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    - Leave base point $x = \pi(p)$ invariant.
    
    $\Rightarrow$ Tangent vectors to fibers $\pi^{-1}(x)$.
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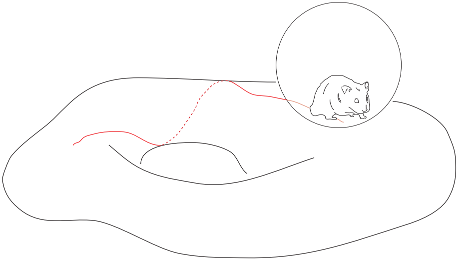
$\Rightarrow$ Geometry encoded in mapping between $T_pP$ and $\mathfrak{g}$.

# The hamster ball



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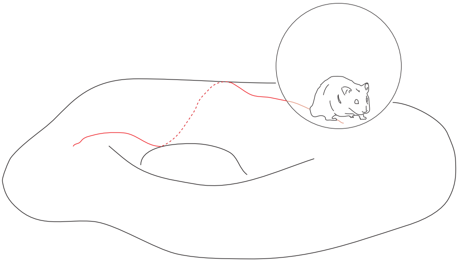

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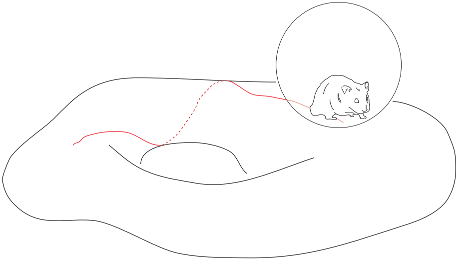

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  - Rotations around its position $x = \pi(p)$: subalgebra $\mathfrak{so}(2)$.
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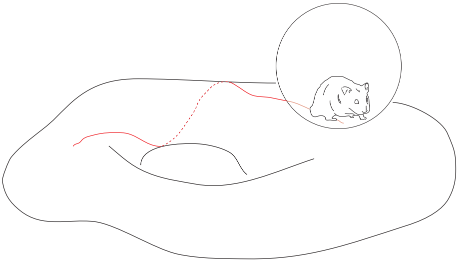

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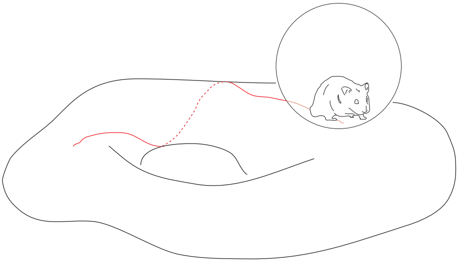

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$\Rightarrow$ Surface $M$ modeled by homogeneous space $\mathrm{SO}(3)/\,\mathrm{SO}(2) \cong S^2$.

$\Rightarrow$ Geometry of $M$ encoded in Hamster ball motion.

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$$\begin{array}{ccc} T_pP & \hookleftarrow & T_p\pi^{-1}(x) \\ \downarrow & & \downarrow \\ \mathfrak{g} & \hookleftarrow & \mathfrak{h} \end{array}$$

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⇒ Geometry of $M$ encoded in $A$ resp. $\underline{A}$.

# Example: Cartan geometry of spacetime

- Choose Lie groups:
  - Let

$$G = \begin{cases} \mathrm{SO}_0(4,1) & \Lambda > 0 \\ \mathrm{ISO}_0(3,1) & \Lambda = 0 \\ \mathrm{SO}_0(3,2) & \Lambda < 0 \end{cases}, \quad H = \mathrm{SO}_0(3,1)\,.$$

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- Choose principal $H$-bundle:
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- Choose Cartan connection:
  - $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ splits into direct sum.
  - Let $e \in \Omega^1(P, \mathfrak{z})$ be the solder form of $\tilde{\pi} : P \to M$.
  - Let $\omega \in \Omega^1(P, \mathfrak{h})$ be the Levi-Civita connection.

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# Example: Cartan geometry of observer space

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  $\Rightarrow$ Coset spaces $G/K$ are the maximally symmetric observer spaces.
- Choose principal $K$-bundle:
  - Let $(M, g)$ be a Lorentzian manifold.
  - Let $O$ be the future unit timelike vectors on $M$.
  - Let $P$ be the oriented time-oriented orthonormal frames on $M$.
  $\Rightarrow$ $\pi : P \to O$ is a principal $K$-bundle.
- Choose Cartan connection:
  - $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ splits into direct sum.
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# The clock postulate

- Proper time along a curve in Lorentzian spacetime:

$$\tau = \int_{t_1}^{t_2} \sqrt{-g_{ab}(x(t))\dot{x}^a(t)\dot{x}^b(t)}dt\,.$$

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- Finsler geometry: use a more general length functional:

$$\tau = \int_{t_1}^{t_2} F(x(t), \dot{x}(t))dt\,.$$

- Finsler function $F : TM \to \mathbb{R}^+$.
- Parametrization invariance requires homogeneity:

$$F(x, \lambda y) = \lambda F(x, y) \quad \forall \lambda > 0\,.$$

# Definition of Finsler spacetimes

- Finsler geometries suitable for spacetimes exist. [C. Pfeifer, M. Wohlfarth '11]

$\Rightarrow$ Finsler metric with Lorentz signature:

$$g^F_{ab}(x, y) = \frac{1}{2} \frac{\partial}{\partial y^a} \frac{\partial}{\partial y^b} F^2(x, y) \,.$$

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- Unit vectors $y \in T_xM$ defined by

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- $\Omega_x$ contains a closed connected component $S_x \subseteq \Omega_x$.

$\rightsquigarrow$ Causality: $S_x$ corresponds to physical observers.

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# Observer space

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- Let $P$ be the space of all observer frames.

$\Rightarrow$ $\pi : P \to O$ is a principal SO(3)-bundle.

- In general no principal $SO_0(3, 1)$-bundle $\tilde{\pi} : P \to M$.

# Construction of the Cartan connection

- Need to construct $A \in \Omega^1(P, \mathfrak{g})$.
- Recall that

$$\begin{array}{ccccc} \mathfrak{g} & = & \mathfrak{h} & \oplus & \mathfrak{z} \\ A & = & \omega & + & e \end{array}$$

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- Definition of $e$: Use the *solder form*.
  - Let $w \in T_{(x,f)}P$ be a tangent vector.
  - Differential of the projection $\tilde{\pi} : P \to M$ yields $\tilde{\pi}_*(w) \in T_xM$.
  - View frame $f$ as a linear isometry $f : \mathfrak{z} \to T_xM$.
  - Solder form given by

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$$e(w) = f^{-1}(\tilde{\pi}_*(w)) \,.$$

- Definition of $\omega$: Use the *Cartan linear connection*.
  - Tangent vector $w \in T_{(x,f)}P$ “shifts” frame $f$ by small amount.
  - Compare shifted frame with parallely transported frame.
  - Both frames differ by Lorentz transform. [C. Pfeifer, M. Wohlfarth '11]
  - Measure the difference using the original frame:

$$\Delta f_i^a = \epsilon f_j^a \omega^j{}_i(w) \,.$$

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- Coefficients of Cartan linear connection:

$$N^a{}_b = \frac{1}{4}\bar{\partial}_b \left[g^{F\,aq}\left(y^p \partial_p \bar{\partial}_q F^2 - \partial_q F^2\right)\right] \,,$$

$$F^a{}_{bc} = \frac{1}{2} g^{F\,ap}\left(\delta_b g^F_{pc} + \delta_c g^F_{bp} - \delta_p g^F_{bc}\right) \,,$$

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$\Rightarrow$ $A = \omega + e$ is a Cartan connection on $\pi : P \to O$.

# Fundamental vector fields

- Let $a = z^i \mathcal{Z}_i + \frac{1}{2} h^i{}_j \mathcal{H}_i{}^j \in \mathfrak{g}$.
- Define the vector field

$$\underline{A}(a) = z^i f_i^a \left( \partial_a - f_j^b F^c{}_{ab} \bar{\partial}_c^j \right) + \left( h^i{}_j f_i^a - h^i{}_0 f_i^b f_j^c C^a{}_{bc} \right) \bar{\partial}_a^j \,.$$

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$\Rightarrow$ For all $p \in P$ we find

$$A(\underline{A}(a)(p)) = a \,.$$

$\Rightarrow$ For all $w \in T_pP$ we find

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- Horizontal vector fields $\underline{A}(\mathfrak{z})$: translations.
- Vertical vector fields $\underline{A}(\mathfrak{h})$: Lorentz transforms.

# Time translation

- Consider the fundamental vector field of the time translator $\mathcal{Z}_0$,

$$\mathbf{t} = \underline{A}(\mathcal{Z}_0) = f_0^a \partial_a - f_j^a N^b{}_a \bar{\partial}_b^j \qquad \Leftrightarrow \qquad \omega^i{}_j(\mathbf{t}) = 0\,, \quad e^i(\mathbf{t}) = \delta_0^i\,.$$

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$\Rightarrow$ Frame $f$ is parallely transported.

$\Rightarrow$ Integral curves of $\mathbf{t}$ define freely falling observers.

# Curvature of the Cartan connection

- Curvature $F = F_{\mathfrak{h}} + F_{\mathfrak{z}} \in \Omega^2(P, \mathfrak{g})$ defined by

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- $R^d{}_{cab}$, $P^d{}_{cab}$, $S^d{}_{cab}$: curvature of Cartan linear connection.

$\Rightarrow$ Cartan geometry reproduces well-known Finsler objects.

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# Spacetime

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- $\Rightarrow$ $\underline{A}(\mathfrak{h})$ can be integrated to a foliation $\mathcal{F}$ with leaf space $M$.
- ↯ *Condition 2*: foliation $\mathcal{F}$ must be strictly simple.
- $\Rightarrow$ Leaf space $M$ is a smooth manifold.
- $\Rightarrow$ Canonical projection $\tilde{\pi} : P \to M$ is a submersion.
- Canonical projections $\tilde{\pi} = \pi' \circ \pi$:

$$P \xrightarrow{\pi} O \xrightarrow{\pi'} M, \qquad P \xrightarrow{\tilde{\pi}} M$$

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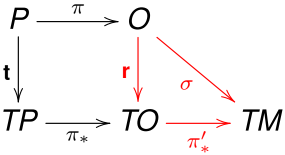


- Integral curves $\lambda \mapsto o(\lambda) \in O$ of $\mathbf{r}$ must be canonical lifts:

$$\sigma(o(\lambda)) = \frac{d}{d\lambda}\pi'(o(\lambda)) = \pi'_*(\dot{o}(\lambda)) = \pi'_*(\mathbf{r}(o(\lambda)))\,.$$

$\Rightarrow$ Uniquely defined map $\sigma = \pi'_* \circ \mathbf{r}$.

↯ *Condition 3*: $\sigma$ must be an embedding.

# Finsler geometry

- Finsler function must be positively homogeneous of degree one:

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⇒ Finsler spacetime geometry on $\mathbb{R}\sigma(O)$.

- No Finsler geometry on $TM \setminus \mathbb{R}\sigma(O)$.
- Cartan geometry describes only geometry visible to observers.

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# Reconstruction of a given Finsler spacetime

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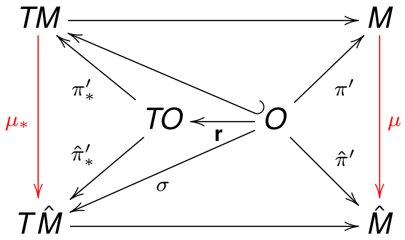

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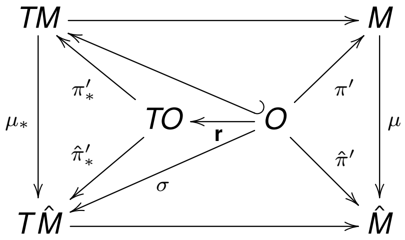


- $\mu$ preserves the Finsler function on timelike vectors.

$\Rightarrow$ Reconstruction of the original Finsler geometry.

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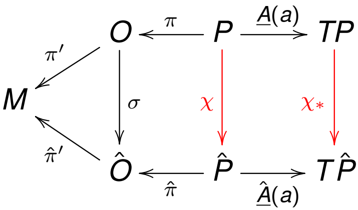

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  - Start from a Cartan observer space $(\pi : P \to O, A)$.
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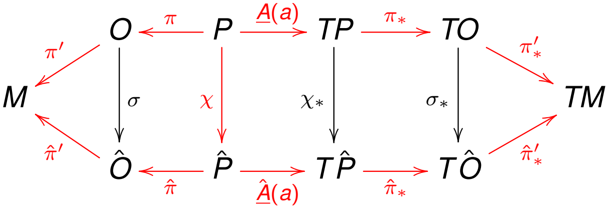
  - Construct a new Cartan observer space $(\hat{\pi} : \hat{P} \to \hat{O}, \hat{A})$.
- Equivalence of $(\pi : P \to O, A)$ and $(\hat{\pi} : \hat{P} \to \hat{O}, \hat{A})$?
- Only if a “Cartan morphism” $\chi$ exists:




- Every Cartan morphism $\chi = (x, f)$ takes the form

$$x(p) = \pi'(\pi(p)), \quad f_i(p) = \pi'_*(\pi_*(\underline{A}(\mathcal{Z}_i)(p)))$$

$\Rightarrow$ Simple test for equivalence of $(\pi : P \to O, A)$ and $(\hat{\pi} : \hat{P} \to \hat{O}, \hat{A})$.

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# Gravity from Cartan to Finsler

- MacDowell-Mansouri gravity on observer space: [S. Gielen, D. Wise '12]

$$S_G = \int_O \epsilon_{\alpha\beta\gamma} \operatorname{tr}_{\mathfrak{h}}(F_{\mathfrak{h}} \wedge \star F_{\mathfrak{h}}) \wedge b^\alpha \wedge b^\beta \wedge b^\gamma$$

  - Hodge operator $\star$ on $\mathfrak{h}$.
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- Translate terms into Finsler language (with $R = d\omega + \frac{1}{2}[\omega, \omega]$):
  - Curvature scalar:

$$[e, e] \wedge \star R \rightsquigarrow g^{F\,ab} R^c{}_{acb}\, dV\,.$$

  - Cosmological constant:

$$[e, e] \wedge \star [e, e] \rightsquigarrow dV\,.$$

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- Future projects:
  - Consistent matter coupling.
  - Study of exact solutions.
  - Effects of deviations from metric geometry?
  - Geometrodynamics of Finsler spacetimes.
  - ...