Finsler and Cartan geometric physical backgrounds
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Moduli Operads Dynamics II
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Outline

1. Introduction
2. Causality
3. Observers
4. Gravity
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Motivation

- **Metric geometry** of spacetime serves multiple roles:
  - Causality
  - Observers, observables and observations
  - Gravity

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- **Geometry imposes several conditions:**
  - Local Lorentz invariance
  - General covariance

Theories of quantum gravity may break these conditions:
- Loop quantum gravity
- Spin foam networks
- Causal dynamical triangulations
  \[ \Rightarrow \]
  Possible stronger, non-tensorial dependence of physical quantities on observer's motion.

\[ \Rightarrow \]
More general, non-tensorial, "observer dependent" geometries:
- Finsler spacetimes
- Cartan geometry on observer space
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Finsler and Cartan geometry
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  - **Finsler spacetimes**
  - **Cartan geometry on observer space**

- How to serve the same roles as metric geometry?
Why Finsler geometry of spacetimes?

- Finsler geometry of space widely used in physics:
  - Approaches to quantum gravity
  - Electrodynamics in anisotropic media
  - Modeling of astronomical data

Finsler spacetimes are suitable backgrounds for:
- Gravity
- Electrodynamics
- Other matter field theories

Possible explanations of yet unexplained phenomena:
- Fly-by anomaly
- Galaxy rotation curves
- Accelerating expansion of the universe
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  - Clock postulate: proper time equals arc length along trajectories.
  - Geometry described by Finsler metric.
  - Well-defined notions of connections, curvature, parallel transport...
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- Quantum gravity suggests breaking of...
  - ...local Lorentz invariance.
  - ...general covariance.

Problems:
- Breaking of Copernican principle for observers.
- No observation of (strongly) broken symmetry.

Solution:
- Consider space $O$ of all allowed observers.
- Describe experiments on observer space instead of spacetime.

$\Rightarrow$ Observer dependence of physical quantities follows naturally.

$\Rightarrow$ No preferred observers.

Geometry of observer space modeled by Cartan geometry.
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**Geometrical structures**

**Metric geometry**
- Manifold $M$
- Lorentzian metric $g$
- Orientation
- Time orientation

**Finsler geometry**
- Tangent bundle $TM$
- Geometry function $L : TM \rightarrow \mathbb{R}$
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- Lie group $G = \text{ISO}_0(3, 1)$
- Closed subgroup $K = \text{SO}(3)$
- Principal $K$-bundle $\pi : P \rightarrow O$
- Cartan connection $A \in \Omega^1(P, g)$

**From metric to Finsler**
- Coordinates $(x^a)$ on $M$
- Coordinates $(x^a, y^a)$ on $TM$
- Define $L(x, y) = g_{ab}(x)y^ay^b$

**From Finsler to Cartan**
- Space $O$ of observer 4-velocities
- Space $P$ of observer frames
- Define $A$ from connection $\nabla$
Ingredients of metric spacetime geometry:
- 4-dimensional spacetime manifold $M$.
- Metric $g_{ab}$ of Lorentzian signature $(-, +, +, +)$.
- Orientation and time orientation of frames.
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Clock postulate: proper time measured by arc length.

Arc length for curves $t \mapsto \gamma(t) \in M$ defined by the metric:

$$
\tau_2 - \tau_1 = \int_{t_1}^{t_2} \sqrt{|g_{ab}(\gamma(t))\dot{\gamma}^a(t)\dot{\gamma}^b(t)|} \, dt.
$$
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$$\tau_2 - \tau_1 = \int_{t_1}^{t_2} \sqrt{|g_{ab}(\gamma(t))\dot{\gamma}^a(t)\dot{\gamma}^b(t)|} \, dt.$$ 

Observables are components of tensor fields.

Tensor components must be expressed in suitable basis.

Metric provides notion of orthonormal frames:

$$g_{ab} f^a_i f^b_j = \eta_{ij}.$$
Basics of Finsler spacetimes

- Finsler geometry defined by length functional for curve $\gamma$:
  \[
  \tau_2 - \tau_1 = \int_{t_1}^{t_2} F(\gamma(t), \dot{\gamma}(t)) dt
  \]

- Finsler function $F : TM \rightarrow \mathbb{R}^+$.  

- Finsler geometries suitable for spacetimes exist. [C. Pfeifer, M. Wohlfarth '11]
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- Introduce manifold-induced coordinates $(x^a, y^a)$ on $TM$:
  - Coordinates $x^a$ on $M$.
  - Define coordinates $y^a$ for $y^a \frac{\partial}{\partial x^a} \in T_xM$.
  - Tangent bundle $TTM$ spanned by $\left\{ \partial_a = \frac{\partial}{\partial x^a}, \bar{\partial} a = \frac{\partial}{\partial y^a} \right\}$. 
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- Finsler geometry defined by length functional for curve $\gamma$:
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  - Tangent bundle $TTM$ spanned by $\{ \partial_a = \frac{\partial}{\partial x^a}, \bar{\partial}_a = \frac{\partial}{\partial y^a} \}$.
- $n$-homogeneous functions on $TM$: $f(x, \lambda y) = \lambda^n f(x, y)$.
  - $n$-homogeneous smooth geometry function $L : TM \to \mathbb{R}$.
  \[ \Rightarrow \text{1-homogeneous Finsler function } F = |L|^\frac{1}{n}. \]
  \[ \Rightarrow \text{Finsler metric with Lorentz signature:} \]
  \[ g_{ab}^F(x, y) = \frac{1}{2} \bar{\partial}_a \bar{\partial}_b F^2(x, y). \]
Connections on Finsler spacetimes

- Cartan non-linear connection:
  
  \[ N^a_b = \frac{1}{4} \bar{\partial}_b \left[ g^{F \, ac} (y^d \partial_d \bar{\partial}_c F^2 - \partial_c F^2) \right] . \]

⇒ Berwald basis of \( TTM \):

\[ \{ \delta_a = \partial_a - N^b_a \bar{\partial}_b, \bar{\partial}_a \} . \]

⇒ Dual Berwald basis of \( T^* TM \):

\[ \{ dx^a, \delta y^a = dy^a + N^a_b dx^b \} . \]

⇒ Splits \( TTM = HTM \oplus VTM \) and \( T^* TM = H^* TM \oplus V^* TM \).
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\[ \Rightarrow \text{Splits } TTM = HTM \oplus VTM \text{ and } T^*TM = H^*TM \oplus V^*TM. \]

- Cartan linear connection:

\[ \nabla_{\delta_a} \delta_b = F^c_{ab} \delta_c, \quad \nabla_{\delta_a} \bar{\partial}_b = F^c_{ab} \bar{\partial}_c, \quad \nabla_{\bar{\partial}_a} \delta_b = C^c_{ab} \delta_c, \quad \nabla_{\bar{\partial}_a} \bar{\partial}_b = C^c_{ab} \bar{\partial}_c, \]

\[ F^c_{ab} = \frac{1}{2} g^F cd (\delta_a g^F_{bd} + \delta_b g^F_{ad} - \delta_d g^F_{ab}), \]

\[ C^c_{ab} = \frac{1}{2} g^F cd (\bar{\partial}_a g^F_{bd} + \bar{\partial}_b g^F_{ad} - \bar{\partial}_d g^F_{ab}). \]
Consider a hamster ball on a two-dimensional surface:
- Two-dimensional Riemannian manifold \((M, g)\).
- Orthonormal frame bundle \(\pi : P \to M\) is principal \(\text{SO}(2)\)-bundle.
- Hamster position and orientation marks frame \(p \in P\).
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Hamster’s degrees of freedom \(\in T_pP\):
- Rotations around its position \(x = \pi(p)\).
- “Rolling without slipping” over \(M\).
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Hamster’s degrees of freedom \(\in T_p P \sim \text{ball motions} \in g = \mathfrak{so}(3)\):
- Rotations around its position \(x = \pi(p)\): subalgebra \(\mathfrak{h} = \mathfrak{so}(2)\).
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$\Rightarrow$ Surface $M$ “traced” by $S^2 \cong \text{SO}(3)/\text{SO}(2) = G/H$.
$\Rightarrow$ Geometry of $M$ fully described by Hamster ball motion.
Consider Lorentzian manifold \((M, g)\).

Orthonormal frame bundle \(\tilde{\pi} : P \to M\).
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Orthonormal frame bundle \(\tilde{\pi}: P \to M\).
Split of the tangent spaces \(T_pP\):
\[
T_pP = V_pP + H_pP
\]
- Infinitesimal Lorentz transforms \(\in V_pP\).
- Infinitesimal translations \(\in H_pP\).
Consider Lorentzian manifold \((M, g)\).

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Split of the tangent spaces \(T_pP \cong g\):

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- Infinitesimal Lorentz transforms \(\in V_pP \cong \mathfrak{h}\).
- Infinitesimal translations \(\in H_pP \cong \mathfrak{z}\).

Corresponding split of Poincaré algebra \(g\):
- Lorentz algebra \(\mathfrak{h}\).
- Translations \(\mathfrak{z}\).
Consider Lorentzian manifold \((M, g)\).

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Split of the tangent spaces \(T_pP \cong g\):

\[
T_pP = V_pP + H_pP
\]

\[
A_p = \omega_p + e_p
\]

\[
g = h + \mathfrak{z}
\]

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- Infinitesimal translations \(\in H_pP \cong \mathfrak{z}\).

Corresponding split of Poincaré algebra \(g\):
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Cartan connection \(A = \omega + e \in \Omega^1(P, g)\).
Consider Lorentzian manifold \((M, g)\).

Orthonormal frame bundle \(\tilde{\pi} : P \rightarrow M\) is principal \(H\)-bundle.

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**Cartan connection** \(A = \omega + e \in \Omega^1(P, \mathfrak{g})\).

**Fundamental vector fields** \(\underline{A} : \mathfrak{g} \rightarrow \Gamma(TP)\) as “inverse” of \(A\).
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Fundamental vector fields \( A : \mathfrak{g} \to \Gamma(TP) \) as “inverse” of \( A \).

\( \Rightarrow \) Geometry of \( M \) encoded in \( A \) resp. \( A \).
Consider Lorentzian manifold \((M, g)\).
Future unit timelike vectors \(O \subset TM\).
Orthonormal frame bundle \(\pi : P \to O\).
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Split of the tangent spaces \(T_p P \cong g\):

\[
T_p P = R_p P + B_p P + \tilde{H}_p P + H^0_p P
\]

- Infinitesimal rotations \(\in R_p P \cong \mathfrak{k}\).
- Infinitesimal Lorentz boosts \(\in B_p P \cong \eta\).
- Infinitesimal spatial translations \(\in \tilde{H}_p P \cong \tilde{\mathfrak{j}}\).
- Infinitesimal temporal translations \(\in H^0_p P \cong \mathfrak{j}^0\).
Consider Lorentzian manifold $(M, g)$.
Future unit timelike vectors $O \subset TM$.
Orthonormal frame bundle $\pi : P \to O$ is principal $K$-bundle.
Split of the tangent spaces $T_P P \cong g$:

\[
\begin{align*}
T_P P &= R_P P + B_P P + \tilde{H}_P P + H^0_P P \\
A_p &= \Omega_p + b_p + \tilde{e}_p + e^0_p \\
g &= \mathfrak{k} + \mathfrak{h} + \mathfrak{z} + \mathfrak{z}^0
\end{align*}
\]

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**Cartan connection** $A = \Omega + b + \tilde{e} + e^0 \in \Omega^1(P, g)$.

**Fundamental vector fields** $\underline{A} : g \to \Gamma(TP)$ as “inverse” of $A$.

$\Rightarrow$ Geometry of $M$ encoded in $A$ resp. $\underline{A}$. [S. Gielen, D. Wise ’12]
From metric to Finsler

- Metric-induced 2-homogeneous geometry function:

\[ L(x, y) = g_{ab}(x) y^a y^b. \]

⇒ Finsler function \( F(x, y) = \sqrt{|L(x, y)|}. \)

⇒ Finsler metric

\[ g^F(x, y) = \begin{cases} 
-g(x, y) & \text{for } y \text{ timelike}, \\
 g(x, y) & \text{for } y \text{ spacelike}. 
\end{cases} \]
Metric-induced 2-homogeneous geometry function:

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⇒ Cartan non-linear connection:

\[ N^a_{\ b} = \Gamma^a_{\ bc}y^c. \]

⇒ Cartan linear connection:

\[ F^a_{\ bc} = \Gamma^a_{\ bc}, \quad C^a_{\ bc} = 0. \]
Need to construct \( A \in \Omega^1(P, \mathfrak{g}) \).

Recall that

\[
\begin{align*}
\mathfrak{g} &= \mathfrak{h} \oplus \mathfrak{z} \\
A &= \omega + e
\end{align*}
\]

Definition of \( e \): Use the solder form:

\[
e^i = f^{-1}i_a dx^a.
\]

Definition of \( \omega \): Use the Cartan linear connection:

\[
\omega^i_j = f^{-1}i_a \left[ df^a_j + f^b_j \left( dx^c F^a_{bc} + (dx^d N^c_d + df^c_0) C^a_{bc} \right) \right].
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Need to construct \( A \in \Omega^1(P, g) \).

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\[
\omega^i_j = f^{-1}^{i} a \left[ df^a_j + f^b_j \left( dx^c F^{ab}^c_{bc} + (dx^d N^c_d + df^0_c) C^{abc} \right) \right].
\]

Let \( a = z^i \mathcal{Z}_i + \frac{1}{2} h^i_j \mathcal{H}_{ij} \in g \).

Fundamental vector fields:

\[
A(a) = z^i f^a_i \left( \partial_a - f^b_j F^c_{ab} \bar{\partial}_c \right) + \left( h^i_j f^a_i - h^i_j f^b f^c_i C^{abc} \right) \bar{\partial}_a.
\]
Causal structure

Metric geometry

Geometry function:

\[ L = g_{ab} y^a y^b \]

\( y^a \) timelike for \( L < 0 \).
Causal structure

Metric geometry

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Finsler geometry

Fundamental geometry function \( L \)

Hessian:

\[ g^L_{ab}(x, y) = \frac{1}{2} \bar{\partial}_a \bar{\partial}_b L(x, y) \]

Use sign of \( L \) and signature of \( g^L \).
**Causal structure**

**Metric geometry**

Geometry function:

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Use sign of \( L \) and signature of \( g^L \).

**Cartan geometry**

Observer space:

\[ O = \bigcup_{x \in M} S_x \]

\( O \) contains only future unit timelike vectors.
“Unit timelike condition” required for Finsler spacetimes:
For all $x \in M$ the set

$$\Omega_x = \{ y \in T_xM \mid |L(x, y)| = 1, \text{sig} \bar{\partial}_a \bar{\partial}_b L(x, y) = (\epsilon, -\epsilon, -\epsilon, -\epsilon) \}$$

with $\epsilon = L(x, y)/|L(x, y)|$ contains a non-empty closed connected component $S_x \subseteq \Omega_x \subset T_xM$. 
"Unit timelike condition" required for Finsler spacetimes:
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with \( \epsilon = \frac{L(x, y)}{|L(x, y)|} \) contains a non-empty closed connected component \( S_x \subseteq \Omega_x \subset T_x M \).

\( \Rightarrow \) \( S_x \) contains physical observers.
\( \Rightarrow \) \( \mathbb{R}^+ S_x \) is convex cone.
Observer space of a Finsler spacetime:

- Consider all allowed observer tangent vectors:

\[ O = \bigcup_{x \in M} S_x. \]

- Tangent vectors \( y \in S_x \) satisfy \( g_{ab}^F(x, y)y^a y^b = 1. \)
The observer frame bundle

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- Construct orthonormal observer frames:
  \[ \Rightarrow \text{Complete } y = f_0 \text{ to a frame } f_i \text{ with } g^F_{ab}(x, y)f^af^b = -\eta_{ij}. \]
  - Let \( P \) be the space of all observer frames.
  - Natural projection \( \pi : P \to O \) discards spatial frame components.
The observer frame bundle

- Observer space of a Finsler spacetime:
  - Consider all allowed observer tangent vectors:
  
  \[
  O = \bigcup_{x \in M} S_x.
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  - Tangent vectors \( y \in S_x \) satisfy
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  - Construct orthonormal observer frames:
    \[\Rightarrow\]
    Complete \( y = f_0 \) to a frame \( f_i \) with
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    g^F_{ab}(x, y) f_i^a f_j^b = -\eta_{ij}.
    \]
    
    - Let \( P \) be the space of all observer frames.
    
    - Natural projection \( \pi : P \to O \) discards spatial frame components.

- Group action on the frame bundle:
  - \( \text{SO}(3) \) acts on spatial frame components by rotations.
  
  - Action is free and transitive on fibers of \( \pi : P \to O \).

  \[\Rightarrow\]
  \( \pi : P \to O \) is principal \( K \)-bundle.
Observers

**Metric geometry**

Timelike curve $\gamma$:

\[ \gamma : \mathbb{R} \rightarrow M \]
\[ \tau \mapsto \gamma(\tau) \]

\[ g_{ab} \dot{\gamma}^a \dot{\gamma}^b = -1 \]

Orthonormal frame $f$:

\[ f_0^a = \dot{\gamma}^a \]

\[ g_{ab} f_i^a f_j^b = \eta_{ij} \]
**Metric geometry**

Timelike curve $\gamma$:
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**Cartan geometry**

Observer curve $\Gamma$:

$$\Gamma : \mathbb{R} \to O \quad \tau \mapsto \Gamma(\tau)$$

Lift condition:

$$\tilde{e}^i \dot{\Gamma}(\tau) = \delta_0^i$$

Orthonormal frame $f$:

$$f \in \pi^{-1}(\Gamma(\tau)) \subset P$$
Metric geometry

Minimize arc length integral:

$$\int_{t_1}^{t_2} \sqrt{|g_{ab}(\gamma(t))\dot{\gamma}^a(t)\dot{\gamma}^b(t)|} \, dt$$

Geodesic equation:

$$\ddot{\gamma}^a + \Gamma^a_{bc} \dot{\gamma}^b \dot{\gamma}^c = 0$$
Inertial observers

**Metric geometry**

Minimize arc length integral:

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Geodesic equation:

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\ddot{\gamma}^a + \Gamma^a_{bc} \dot{\gamma}^b \dot{\gamma}^c = 0
\]

**Finsler geometry**

Minimize arc length integral:

\[
\int_{t_1}^{t_2} F(\gamma(t), \dot{\gamma}(t)) dt
\]

Geodesic equation:

\[
\ddot{\gamma}^a + \mathcal{N}^a_{bc} \dot{\gamma}^b \dot{\gamma}^c = 0
\]

Geodesic spray:

\[
\mathbf{S} = y^a(\partial_a - \mathcal{N}^b_{a}\bar{\partial}_b)
\]

Integral curves:

\[
\dot{\Gamma}(\tau) = \mathbf{S}(\Gamma(\tau))
\]
## Metric geometry

Minimize arc length integral:

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## Finsler geometry

Minimize arc length integral:

\[
\int_{t_1}^{t_2} F(\gamma(t), \dot{\gamma}(t)) \, dt
\]

Geodesic equation:

\[
\ddot{\gamma}^a + N^a_b \dot{\gamma}^b = 0
\]

## Cartan geometry

Geodesic condition:

\[
\tilde{b}^\alpha \dot{\Gamma}(\tau) = 0
\]

Integral curves:

\[
\dot{\Gamma}(\tau) = \tilde{e}_0(\Gamma(\tau))
\]

## Finsler and Cartan geometry

Geodesic spray:

\[
S = y^a (\partial_a - N^b_a \bar{\partial}_b)
\]

Integral curves:

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\dot{\Gamma}(\tau) = S(\Gamma(\tau))
\]
Observer trajectories:
- Observer trajectory $\gamma$ in $M$.
- $\dot{\gamma}$ must be timelike and future-directed.

Inertial observers:
Minimize arc-length functional:
\[
\int_{t_1}^{t_2} \sqrt{|g_{ab}(\gamma(t))\dot{\gamma}^a(t)\dot{\gamma}^b(t)|} \, dt.
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Geodesic equation:
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Observers on metric spacetimes

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  \ddot{\gamma}^a + \Gamma^a_{bc} \dot{\gamma}^b \dot{\gamma}^c = 0.
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Observer trajectories and canonical lifts:

- Observer trajectory $\gamma$ in $M$.
- Lift $\gamma$ to a curve $\Gamma = (\gamma, \dot{\gamma})$ in $TM$.
- Curves $\Gamma$ in $TM$ are canonical lifts if and only if

$$dx^a \dot{\Gamma} = y^a.$$ 

- Tangent vector $\dot{\gamma}(\tau) \in S_{\gamma(\tau)}$; $\Gamma$ is curve in $O \subset TM$. 


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Inertial observers:
- Minimize arc length functional:
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  $\Rightarrow$ Geodesic equation:
  \[ \ddot{\gamma}^a + N^a_b \dot{\gamma}^b = 0. \]

  $\Rightarrow$ $\Gamma$ is integral curve of geodesic spray:
  \[ \dot{\Gamma} = S = y^a \delta_a. \]
Observer curves:

Consider curve $\Gamma$ in $O$.

⇒ Tangent vector splits into translation and boost:

$$\dot{\Gamma} = \left(e^i \dot{\Gamma}\right) e_i + \left(b^\alpha \dot{\Gamma}\right) b_\alpha.$$
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$\Rightarrow$ Tangent vector splits into translation and boost:

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Translational component of the tangent vector:

Split into time and space components:

$$(e^i \dot{\Gamma}) e_i = (e^0 \dot{\Gamma}) e_0 + (e^\alpha \dot{\Gamma}) e_\alpha.$$ 

Components are relative to observer’s frame.

$\Rightarrow$ Physical observer: translation corresponds to time direction:

$$e^0 \dot{\Gamma} = 1 \land e^\alpha \dot{\Gamma} = 0 \iff e^i \dot{\Gamma} = \delta^i_0.$$
Observers on Cartan observer space

- Observer curves:
  - Consider curve $\Gamma$ in $O$.
  - Tangent vector splits into translation and boost:
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- Translational component of the tangent vector:
  - Split into time and space components:
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  - Components are relative to observer’s frame.
  - Physical observer: translation corresponds to time direction:
    \[
    e^0 \dot{\Gamma} = 1 \land e^\alpha \dot{\Gamma} = 0 \iff e^i \dot{\Gamma} = \delta^i_0 .
    \]

- Boost component of the tangent vector:
  - Measures acceleration in observer’s frame.
  - Inertial observers are non-accelerating: $b^\alpha \dot{\Gamma} = 0$.
  - Inertial observers follow integral curves of time translation: $\dot{\Gamma} = e^0_0$. 

Manuel Hohmann (University of Tartu)  Finsler and Cartan geometry  4 June 2014  24 / 31
1. Introduction
2. Causality
3. Observers
4. Gravity
5. Conclusion
Einstein-Hilbert action:

\[ S_{EH} = \frac{1}{2\kappa} \int_M d^4 x \sqrt{-g} \ R \]
Gravity

Metric geometry

Einstein-Hilbert action:

\[ S_{EH} = \frac{1}{2\kappa} \int_M d^4 x \sqrt{-g} R \]

Finsler geometry

Using non-linear connection:

\[ S_N = \frac{1}{\kappa} \int_\Sigma \text{Vol}_{\tilde{G}} R^a_{ab} y^b \]

Using linear connection:

\[ S_L = \frac{1}{\kappa} \int_\Sigma \text{Vol}_{\tilde{G}} g^F_{ab} R^c_{acb} \]
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**Cartan geometry**

Using horizontal vector fields:

\[ S_H = \int_O \tilde{b}^\alpha ([\tilde{e}_\alpha, \tilde{e}_0]) \text{Vol}_O \]

Using Cartan curvature:

\[ S_C = \int_O \kappa_h (\tilde{F}_h \wedge \tilde{F}_h) \wedge \text{Vol}_S \]
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MacDowell-Mansouri gravity on observer space: [S. Gielen, D. Wise ’12]

\[ S_G = \int_\mathcal{O} \epsilon_{\alpha\beta\gamma} \text{tr}_\mathcal{H}(F_\mathcal{H} \wedge \star F_\mathcal{H}) \wedge b^\alpha \wedge b^\beta \wedge b^\gamma \]

- Hodge operator \( \star \) on \( \mathcal{H} \).
- Non-degenerate \( H \)-invariant inner product \( \text{tr}_\mathcal{H} \) on \( \mathcal{H} \).
- Boost part \( b \in \Omega_1(\mathcal{H}, \eta) \) of the Cartan connection.
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Translate terms into Finsler language (with \( R = d\omega + \frac{1}{2} [\omega, \omega] \)):

- Curvature scalar:  
  \[ [e, e] \wedge \star R \sim g^{ab} R^c_{acb} dV. \]

- Cosmological constant:  
  \[ [e, e] \wedge \star [e, e] \sim dV. \]

- Gauss-Bonnet term:  
  \[ R \wedge \star R \sim \epsilon^{abcd} \epsilon^{efgh} R_{abef} R_{cdgh} dV. \]

\( \Rightarrow \) Gravity theory on Finsler spacetime.
Finsler gravity action: [C. Pfeifer, M. Wohlfarth '11]

\[ S_G = \int_O d^4x \, d^3y \, \sqrt{-\tilde{G}} R^a_{\, \, ab} y^b. \]

- Sasaki metric $\tilde{G}$ on $O$.
- Non-linear curvature $R^a_{\, \, ab}$.

⇒ Gravity theory on observer space.
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Translate terms into Cartan language:

\[
d^4x \, d^3y \, \sqrt{-\tilde{G}} = \epsilon_{ijkl} \epsilon_{\alpha\beta\gamma} \, e^i \wedge e^j \wedge e^k \wedge e^l \wedge b^\alpha \wedge b^\beta \wedge b^\gamma,
\]

\[
R^a_{\, \, ab} y^b = b^\alpha [A(z_\alpha), A(z_0)].
\]

⇒ Gravity theory on observer space.
Summary

- **Finsler spacetimes**
  - Generalization of *metric spacetimes*.
  - Geometry defined by function $L$ on $TM$.
  - Lengths measured by Finsler function $F = |L|^{\frac{1}{n}}$.
  - Metric generalized by Finsler metric $g^{F}_{ab}$. 

Cartan geometry on observer space can be obtained from Finsler spacetimes.
Geometry on principal $SO(3)$-bundle $P \rightarrow O$.
Space $O$ of physical observer four-velocities.
Space $P$ of physical observer frames.
Geometry defined by Cartan connection $A \in \Omega^{1}(P, g)$.
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Different geometries provide compatible definitions of:

- Causality
- Observers
- Observables
- Gravity
Open questions

- Experimental effects of non-tensorial structures?
- Properties of matter (gauge) theories on these backgrounds?
- Quantization of these structures?
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