Spacetime symmetries in Cartan language

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DPG-Tagung Berlin – Session MP 16
19. März 2015

Motivation

- Spacetime symmetries important in physics / gravity:
  - Planar symmetry for gravitational waves.
  - Spherical symmetry for stellar objects.
  - Axial symmetry for rotating systems.
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- Framework can be generalized to observer space:
  - All measurements are performed by observers.
  - Measurements depend on observer's frame (velocity).
  - Quantum gravity: possible non-tensorial velocity dependence.
  - Observer space: space of all physical velocities.
  - Geometry of observer space naturally given by Cartan geometry.

Complete lifts of vector fields

- Tangent bundle lift:
  - Diffeomorphism group $\varphi : \mathbb{R} \times M \to M$ induces $\hat{\varphi} : \mathbb{R} \times TM \to TM$:
$$\hat{\varphi}_t = \varphi_{t*}\,.$$
  - $\hat{\varphi}$ generated by vector field $\hat{\xi} \in \mathrm{Vect}(TM)$.
  - In coordinates (x^a, y^a) on TM:
$$\hat{\xi} = \xi^a\partial_a + y^b\partial_b\xi^a\bar{\partial}_a\,.$$

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- Frame bundle lift:
  - Frame bundle $FM = \mathrm{GL}(M)$: linear maps $f \to T_xM, x \in M$.
  - Diffeomorphism group $\varphi : \mathbb{R} \times M \to M$ induces $\bar{\varphi} : \mathbb{R} \times FM \to FM$:

$$\bar{\varphi}_t(f) = \varphi_{t*} \circ f\,.$$

  - $\bar{\varphi}$ generated by vector field $\bar{\xi} \in \mathrm{Vect}(FM)$.
  - In coordinates (x^a, f_i^a) on TM:

$$\bar{\xi} = \xi^a \partial_a + f_i^b \partial_b \xi^a \bar{\partial}_a^i\,.$$

Cartan geometry

- Klein geometry: Lie group G with closed subgroup $H \subset G$.
- Cartan geometry $(\pi : \mathcal{P} \to M, A)$ modeled on G/H:
  - Principal H-bundle $\pi : \mathcal{P} \to M$
  - 1-form $A \in \Omega^1(\mathcal{P}, \mathfrak{g})$

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- Conditions on Cartan connection $A \in \Omega^1(\mathcal{P}, \mathfrak{g})$:
  - For each $p \in \mathcal{P}$, $A_p : T_p\mathcal{P} \to \mathfrak{g}$ is linear isomorphism.
  - Equivariance: $(R_h)^*A = \mathrm{Ad}(h^{-1}) \circ A \quad \forall h \in H$.
  - A restricts to Maurer-Cartan form of H on $\ker \pi_*$.

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- Equivalent: fundamental vector fields $\underline{A} : \mathfrak{g} \to \mathrm{Vect}(\mathcal{P})$:
  - For each $p \in \mathcal{P}$, $\underline{A}_p : \mathfrak{g} \to T_p\mathcal{P}$ is linear isomorphism.
  - Equivariance: $R_{h*} \circ \underline{A} = \underline{A} \circ \mathrm{Ad}(h^{-1}) \quad \forall h \in H$.
  - $\underline{A}$ restricts to canonical vector fields on $\mathfrak{h}$.

First order reductive models

- First order Cartan geometry:
  - Adjoint representations of $H \subset G$ on $\mathfrak{g}$ and $\mathfrak{h}$.
  - Quotient representation of H on $\mathfrak{g}/\mathfrak{h}$ is faithful.

$\Rightarrow$ "Fake tangent bundle" $\mathcal{T} = \mathcal{P} \times_H \mathfrak{g}/\mathfrak{h}$.

$\Rightarrow$ $\mathcal{P}$ is "fake frame bundle": "admissible" frames $\mathfrak{g}/\mathfrak{h} \to \mathcal{T}_x$ for $x \in M$.

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- Reductive Cartan geometry:
  - Direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ of vector spaces.
  - $\mathfrak{h}$ and $\mathfrak{z}$ are subrepresentations of Ad H on $\mathfrak{g}$.

$\Rightarrow$ Cartan connection $A = \omega + e$ splits: $\omega \in \Omega^1(\mathcal{P}, \mathfrak{h})$ and $e \in \Omega^1(\mathcal{P}, \mathfrak{z})$.

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$\Rightarrow$ Cartan geometry $(\tilde{\pi} : P \to M, \tilde{A})$ with $\tilde{A} = \tilde{\omega} + \tilde{e}$.

- $\tilde{e}$: solder form on $P \subset FM$.
- Drop tilde and consider Cartan geometries on $\mathcal{P} \equiv P \subset FM$.

Symmetries in Cartan language

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- Solder form $e \in \Omega^1(P, \mathfrak{z})$:
  - For $v \in T_pP$ defined by $e(v) = p^{-1}(\pi_*(v))$.
  - Satisfies $\bar{\varphi}^* e = e$ for any diffeomorphism $\varphi : M \to M$.

  $\Rightarrow$ $\mathcal{L}_{\bar{\xi}} e = 0$ for any $\xi \in \mathrm{Vect}(M)$.

  $\Rightarrow$ Only need to check $\mathcal{L}_{\bar{\xi}} \omega = 0$.

The orthogonal model geometry

- Model geometry for $3 + 1$-dimensional spacetime:

$$G = \begin{cases} \mathrm{SO}_0(4,1) & \Lambda > 0 \\ \mathrm{ISO}_0(3,1) & \Lambda = 0 \\ \mathrm{SO}_0(3,2) & \Lambda < 0 \end{cases}, \qquad H = \mathrm{SO}_0(3,1)\,.$$

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$\Rightarrow$ Cartan geometry $(\pi : P \to M, A) \leftrightsquigarrow$ metric spacetime:

- Metric g derived from solder form e.
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- Symmetry of Cartan connection under vector field $\xi \in \mathrm{Vect}(M)$:
  - $\bar{\xi}$ is tangent to $P \Leftrightarrow \mathcal{L}_\xi g = 0$.
  - $\mathcal{L}_{\bar{\xi}}\omega = 0 \Leftrightarrow \mathcal{L}_\xi \Gamma = 0$.

Riemann-Cartan, Riemann & Weizenböck

- Riemann-Cartan spacetime:
  - Metric g and torsion T determine connection

$$\Gamma^{a}{}_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc} - g_{be}T^{e}{}_{cd} - g_{ce}T^{e}{}_{bd}) + \frac{1}{2}T^{a}{}_{cb}\,.$$

$\Rightarrow$ Cartan geometry with Cartan curvature $F = dA + A \wedge A \in \Omega^2(P, \mathfrak{g})$.

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- Weizenböck spacetime:
  - Vielbein h determines Weizenböck connection

$$\Gamma^a{}_{bc} = h^a_i \partial_c h^i_b \,.$$

$\Rightarrow$ Cartan geometry with Cartan curvature $F = dA + A \wedge A \in \Omega^2(P, \mathfrak{z})$.
$\Rightarrow$ Symmetry of Cartan geometry $\Leftrightarrow \mathcal{L}_\xi h = \lambda h, \lambda \in \mathfrak{h}$.

The observer space model

- Model geometry for $3 + 3 + 1$-dimensional observer space:

$$G = \begin{cases} \mathrm{SO}_0(4,1) & \Lambda > 0 \\ \mathrm{ISO}_0(3,1) & \Lambda = 0 \\ \mathrm{SO}_0(3,2) & \Lambda < 0 \end{cases}, \quad H = \mathrm{SO}_0(3,1), \quad K = \mathrm{SO}(3).$$

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$\Rightarrow$ Split $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{y} \oplus \vec{\mathfrak{z}} \oplus \mathfrak{z}^0$ of the Poincaré algebra:

- $\mathfrak{k}$: spatial rotations.
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- Cartan geometry $(\pi : P \to O, A)$ modeled on G/K with $P \subset FO$:

  $\Rightarrow$ Split $A = \Omega + b + \vec{e} + e^0$ of the Cartan connection.

  $\Rightarrow$ Induces split $TP = RP \oplus BP \oplus \vec{H}P \oplus H^0P$.

Symmetries of observer space

- Structures induced by Cartan geometry ($\pi : P \to O, A$):
  - Tangent bundle split $TO = VO \oplus \vec{H}O \oplus H^0O$.
  - Projectors $P_V, P_{\vec{H}}, P_{H^0}, P_H = P_{\vec{H}} + P_{H^0}$ onto subbundles.
  - Vector bundle isomorphism $\Theta : VO \to \vec{H}O$.
  - “Time translation” vector field $\mathbf{r} \in \Gamma(H^0O)$.

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- $\Xi \in \mathrm{Vect}(O)$ generates "spacetime" diffeomorphism if:
  - Boost component of Ξ is time derivative of spatial translation:

$$P_H \circ \mathcal{L}_{\mathbf{r}}(P_H \circ \Xi) = \Theta \circ P_V \circ \Xi\,.$$

  - Ξ does not depend on vertical directions:

$$P_H \circ \mathcal{L}_{\Upsilon}(P_H \circ \Xi) = 0 \quad \text{for} \quad \Upsilon \in \Gamma(VO)\,.$$

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- Symmetry of spacetime *M*:
  - Generated by vector field $\xi \in \mathrm{Vect}(M)$.
  - Induces lifted vector fields $\hat{\xi} \in \mathrm{Vect}(TM)$ and $\bar{\xi} \in \mathrm{Vect}(FM)$.

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- Further research topics:
  - Construct observer spaces with particular symmetries.
  - Local Lorentz invariance of teleparallel theories?