Non-metric fluid dynamics and cosmology on Finsler spacetimes

Manuel Hohmann

Laboratory of Theoretical Physics
Institute of Physics
University of Tartu

23. June 2015
Motivation

- So far unexplained cosmological observations:
  - Accelerating expansion of the universe
  - Homogeneity of cosmic microwave background
So far unexplained cosmological observations:
- Accelerating expansion of the universe
- Homogeneity of cosmic microwave background

Models for explaining these observations:
- ΛCDM model / dark energy
- Inflation
Motivation

- So far unexplained cosmological observations:
  - Accelerating expansion of the universe
  - Homogeneity of cosmic microwave background
- Models for explaining these observations:
  - $\Lambda$CDM model / dark energy
  - Inflation
- Physical mechanisms are not understood:
  - Unknown type of matter?
  - Modification of the laws of gravity?
  - Scalar field in addition to metric mediating gravity?
  - Quantum gravity effects?
Motivation

- So far unexplained cosmological observations:
  - Accelerating expansion of the universe
  - Homogeneity of cosmic microwave background

- Models for explaining these observations:
  - $\Lambda$CDM model / dark energy
  - Inflation

- Physical mechanisms are not understood:
  - Unknown type of matter?
  - Modification of the laws of gravity?
  - Scalar field in addition to metric mediating gravity?
  - Quantum gravity effects?

- Idea here: modification of the geometrical structure of spacetime!
  - Replace metric spacetime geometry by Finsler geometry.
  - Similarly: replacing flat spacetime by curved spacetime led to GR.
Finsler spacetimes

- Finsler geometry of space widely used in physics:
  - Approaches to quantum gravity
  - Electrodynamics in anisotropic media
  - Modeling of astronomical data

Finsler geometry generalizes Riemannian geometry:
- Geometry described by Finsler function on the tangent bundle.
- Finsler function measures length of tangent vectors.
- Well-defined notions of connections, curvature, parallel transport.

Finsler spacetimes are suitable backgrounds for:
- Gravity
- Electrodynamics
- Other matter field theories

Possible explanations of yet unexplained phenomena:
- Fly-by anomaly
- Galaxy rotation curves
- Accelerating expansion of the universe
- Inflation
Finsler spacetimes

- Finsler geometry of space widely used in physics:
  - Approaches to quantum gravity
  - Electrodynamics in anisotropic media
  - Modeling of astronomical data
- Finsler geometry generalizes Riemannian geometry:
  - Geometry described by Finsler function on the tangent bundle.
  - Finsler function measures length of tangent vectors.
  - Well-defined notions of connections, curvature, parallel transport...
Finsler spacetimes

- Finsler geometry of space widely used in physics:
  - Approaches to quantum gravity
  - Electrodynamics in anisotropic media
  - Modeling of astronomical data

- Finsler geometry generalizes Riemannian geometry:
  - Geometry described by Finsler function on the tangent bundle.
  - Finsler function measures length of tangent vectors.
  - Well-defined notions of connections, curvature, parallel transport...

- Finsler spacetimes are suitable backgrounds for:
  - Gravity
  - Electrodynamics
  - Other matter field theories
Finsler spacetimes

- Finsler geometry of space widely used in physics:
  - Approaches to quantum gravity
  - Electrodynamics in anisotropic media
  - Modeling of astronomical data

- Finsler geometry generalizes Riemannian geometry:
  - Geometry described by Finsler function on the tangent bundle.
  - Finsler function measures length of tangent vectors.
  - Well-defined notions of connections, curvature, parallel transport.

- Finsler spacetimes are suitable backgrounds for:
  - Gravity
  - Electrodynamics
  - Other matter field theories

- Possible explanations of yet unexplained phenomena:
  - Fly-by anomaly
  - Galaxy rotation curves
  - Accelerating expansion of the universe
  - Inflation
Proper time along a curve in Lorentzian spacetime:

\[ \tau = \int_{t_1}^{t_2} \sqrt{-g_{ab}(x(t))\dot{x}^a(t)\dot{x}^b(t)} \, dt. \]
The clock postulate

- Proper time along a curve in Lorentzian spacetime:
  \[ \tau = \int_{t_1}^{t_2} \sqrt{-g_{ab}(x(t))\dot{x}^a(t)\dot{x}^b(t)} \, dt. \]

- Finsler geometry: use a more general length functional:
  \[ \tau = \int_{t_1}^{t_2} F(x(t), \dot{x}(t)) \, dt. \]

- Finsler function \( F : TM \to \mathbb{R}^+ \).
- Parametrization invariance requires homogeneity:
  \[ F(x, \lambda y) = \lambda F(x, y) \quad \forall \lambda > 0. \]
Finsler spacetimes

- Finsler geometries suitable for spacetimes exist. [C. Pfeifer, M. Wohlfarth '11]

⇒ Finsler metric with Lorentz signature:

\[ g^F_{ab}(x, y) = \frac{1}{2} \frac{\partial}{\partial y^a} \frac{\partial}{\partial y^b} F^2(x, y). \]

⇒ Notion of timelike, lightlike, spacelike tangent vectors.

Ω contains a closed connected component \( S_x \subseteq \Omega_x \).

⇒ Causality: \( S_x \) corresponds to physical observers.
Finsler spacetimes

- Finsler geometries suitable for spacetimes exist. [C. Pfeifer, M. Wohlfarth ’11]
  - Finsler metric with Lorentz signature:
    \[ g^F_{ab}(x, y) = \frac{1}{2} \frac{\partial}{\partial y^a} \frac{\partial}{\partial y^b} F^2(x, y). \]
  - Notion of timelike, lightlike, spacelike tangent vectors.
  - Unit vectors \( y \in T_xM \) defined by
    \[ F^2(x, y) = g^F_{ab}(x, y)y^ay^b = 1. \]
  - Set \( \Omega_x \subset T_xM \) of unit timelike vectors at \( x \in M \).
Finsler spacetimes

- Finsler geometries suitable for spacetimes exist. [C. Pfeifer, M. Wohlfarth ‘11]

⇒ Finsler metric with Lorentz signature:

\[
g_{ab}^F(x, y) = \frac{1}{2} \frac{\partial}{\partial y^a} \frac{\partial}{\partial y^b} F^2(x, y).
\]

⇒ Notion of timelike, lightlike, spacelike tangent vectors.

- Unit vectors \( y \in T_x M \) defined by

\[
F^2(x, y) = g_{ab}^F(x, y)y^a y^b = 1.
\]

⇒ Set \( \Omega_x \subset T_x M \) of unit timelike vectors at \( x \in M \).

- \( \Omega_x \) contains a closed connected component \( S_x \subseteq \Omega_x \).

⇝ Causality: \( S_x \) corresponds to physical observers.
Gravitational dynamics

- Gravitational action:

\[
S_G = \frac{1}{\kappa} \int_{\Sigma} \text{Vol} \tilde{G} R^{ab} y^b.
\]
Gravitational dynamics

- Gravitational action:

\[ S_G = \frac{1}{\kappa} \int_{\Sigma} \text{Vol}_G R^a_{\ ab} y^b. \]

- Gravitational field equations:

\[
\left[ g^{F\ ab} \bar{\partial}_a \bar{\partial}_b (R^c_{\ cd} y^d) - 6 \frac{R^a_{\ ab} y^b}{F^2} \\
+ 2g^{F\ ab} (\nabla_a S_b + S_a S_b + \bar{\partial}_a (y^c \delta_c S_b - N^c_{\ b} S_c)) \right] \bigg|_\Sigma = \kappa T|_\Sigma
\]
Gravitational dynamics

- Gravitational action:

\[ S_G = \frac{1}{\kappa} \int_{\Sigma} \text{Vol} \tilde{G} R^a_{\ ab} y^b. \]

- Gravitational field equations:

\[
\left[ g^F^{\ ab} \bar{\partial}_a \bar{\partial}_b (R^c_{\ cd} y^d) - 6 \frac{R^a_{\ ab} y^b}{F^2} + 2 g^F^{\ ab} (\nabla_a S_b + S_a S_b + \bar{\partial}_a (y^c \delta_c S_b - N^c_{\ b} S_c)) \right] \bigg|_{\Sigma} = \kappa T|_{\Sigma}
\]

- Geometry side obtained by variation of \( S_G \) with respect to \( F \).
Gravitational dynamics

- Gravitational action:
  \[ S_G = \frac{1}{\kappa} \int_{\Sigma} \text{Vol}_{\tilde{G}} R^a_{\; ab y^b}. \]

- Gravitational field equations:
  \[
  \left. \left[ g^F_{\; ab} \bar{\partial}_a \bar{\partial}_b (R^c_{\; cd y^d}) - 6 \frac{R^a_{\; ab y^b}}{F^2} \right. 
  \right. 
  \left. + 2g^F_{\; ab} (\nabla_a S_b + S_a S_b + \bar{\partial}_a (y^c \delta_c S_b - N^c_{\; b} S_c)) \right|_{\Sigma} = \kappa T|_{\Sigma}
  \]

- Geometry side obtained by variation of \( S_G \) with respect to \( F \).
- Variation of matter action yields energy-momentum scalar \( T \).
Point masses on Finsler spacetimes

- Point masses follow Finsler geodesics.
- Geodesic equation for curve $x(\tau)$ on spacetime $M$:
  \[
  \ddot{x}^a + N^a_{\ b}(x, \dot{x})\dot{x}^b = 0.
  \]
Point masses on Finsler spacetimes

- Point masses follow Finsler geodesics.
- Geodesic equation for curve $x(\tau)$ on spacetime $M$:
  \[
  \ddot{x}^a + N^a_{\ b}(x, \dot{x})\dot{x}^b = 0.
  \]
- Canonical lift of curve to tangent bundle $TM$:
  \[
  x, \quad y = \dot{x} \in O = \bigcup_{x \in M} S_x \subset TM.
  \]
- Lift of geodesic equation:
  \[
  \dot{x}^a = y^a, \quad \dot{y}^a = -N^a_{\ b}(x, y)y^b.
  \]
Point masses on Finsler spacetimes

- Point masses follow Finsler geodesics.
- Geodesic equation for curve $x(\tau)$ on spacetime $M$:
  \[
  \ddot{x}^a + N^a_{\ b}(x, \dot{x})\dot{x}^b = 0.
  \]
- Canonical lift of curve to tangent bundle $TM$:
  \[
  x, \quad y = \dot{x} \in O = \bigcup_{x \in M} S_x \subset TM.
  \]
- Lift of geodesic equation:
  \[
  \dot{x}^a = y^a, \quad \dot{y}^a = -N^a_{\ b}(x, y)y^b.
  \]
  ⇒ Solutions are integral curves of vector field on $O$:
  \[
  y^a \partial_a - y^b N^a_{\ b} \bar{\partial}_a = r.
  \]
  ⇒ Point mass trajectories modeled by integral curves of $r$ on $O$. 

Manuel Hohmann (University of Tartu)
Fluids on Finsler spacetimes

- **Single-component fluid:**
  - Constituted by classical, relativistic particles.
  - Particles have equal properties (mass, charge, ...).
  - Particles follow piecewise geodesic curves.
  - Endpoints of geodesics are interactions with other particles.

Manuel Hohmann (University of Tartu)

Finsler cosmology

23. June 2015
Fluids on Finsler spacetimes

- Single-component fluid:
  - Constituted by classical, relativistic particles.
  - Particles have equal properties (mass, charge, ...).
  - Particles follow piecewise geodesic curves.
  - Endpoints of geodesics are interactions with other particles.

- Continuum limit:
  - Phase space $O$ is filled with particles.
  - Particle density function $\phi : O \rightarrow \mathbb{R}^+$. 
Fluids on Finsler spacetimes

- Single-component fluid:
  - Constituted by classical, relativistic particles.
  - Particles have equal properties (mass, charge, ...).
  - Particles follow piecewise geodesic curves.
  - Endpoints of geodesics are interactions with other particles.

- Continuum limit:
  - Phase space $O$ is filled with particles.
  - Particle density function $\phi : O \rightarrow \mathbb{R}^+$. 

- Collisionless fluid:
  - Particles do not interact with other particles.
    $\Rightarrow$ Particles follow geodesics.
    $\Rightarrow$ Continuum dynamics given by Liouville equation:
    $$\mathcal{L}_r \phi = 0.$$
Example: collisionless dust fluid

- Variables describing a classical dust fluid:
  - Mass density \( \rho : M \to \mathbb{R}^+ \).
  - Velocity \( u : M \to O \).
Example: collisionless dust fluid

- Variables describing a classical dust fluid:
  - Mass density $\rho : M \rightarrow \mathbb{R}^+$.  
  - Velocity $u : M \rightarrow O$.
- Particle density function:

$$\phi(x, y) \sim \rho(x)\delta_{S_x}(y, u(x)).$$
Example: collisionless dust fluid

- Variables describing a classical dust fluid:
  - Mass density $\rho : \mathbb{M} \to \mathbb{R}^+$.  
  - Velocity $u : \mathbb{M} \to \mathbb{O}$.

- Particle density function:

$$
\phi(x, y) \sim \rho(x) \delta_{S_x}(y, u(x)).
$$

- Apply Liouville equation:

$$
0 = \nabla u^a = u^b \partial_b u^a + u^b N^a_b,
$$

$$
0 = \nabla_{\delta_a}(\rho u^a) = \partial_a(\rho u^a) + \frac{1}{2} \rho u^a g^F_{bc} \left( \partial_a g^F_{bc} - N^d_a \tilde{\partial}_d g^F_{bc} \right).
$$
Example: collisionless dust fluid

Variables describing a classical dust fluid:

- Mass density $\rho : M \to \mathbb{R}^+$. 
- Velocity $u : M \to O$.

Particle density function:

$$\phi(x, y) \sim \rho(x) \delta_{S_x}(y, u(x)) .$$

Apply Liouville equation:

$$0 = \nabla u^a = u^b \partial_b u^a + u^b N^a_b ,$$

$$0 = \nabla_{\delta_a}(\rho u^a) = \partial_a(\rho u^a) + \frac{1}{2} \rho u^a g^F_{bc} \left( \partial_a g^F_{bc} - N^d_a \bar{\partial}_d g^F_{bc} \right) .$$

Metric limit $F^2(x, y) = |g_{ab}(x)y^ay^b|$ yields Euler equations:

$$u^b \nabla_b u^a = 0 , \quad \nabla_a(\rho u^a) = 0 .$$
Fluid energy-momentum

- Energy-momentum functional $T[\phi]$?

Known result for metric perfect fluid:
Density $\rho$.
Pressure $p$.
Velocity $u^a$.

$$T \rho, p, u^a(x, y) = (1 - 6 g^{ab}(x) u_a(x) y^b) \rho(x) + 3 (1 - 2 g^{ab}(x) u_a(x) y^b)^2 p(x).$$

Generalize to Finsler fluid:
Consider dust: $p = 0$.
Consider superposition of dust with different velocities.
Integrate over contributions from each velocity.
Generalize $g^{ab} u_a v_b$ to Finsler angle.

$$T \phi(x, v) = m \int_S x d^3v' \sqrt{\det h(x, v')} \phi(x, v' \cos^2 \angle(v, v')).$$
Energy-momentum functional $T[\phi]$?

Known result for metric perfect fluid:
- Density $\rho$.
- Pressure $p$.
- Velocity $u^a$.

$$T_{\rho,p,u}(x, y) = (1 - 6(g_{ab}(x)u^a(x)y^b)^2)\rho(x) + 3(1 - 2(g_{ab}(x)u^a(x)y^b)^2)p(x).$$
Fluid energy-momentum

- Energy-momentum functional $T[\phi]$?
- Known result for metric perfect fluid:
  - Density $\rho$.
  - Pressure $p$.
  - Velocity $u^a$.

$$T_{\rho, p, u}(x, y) = (1 - 6(g_{ab}(x)u^a(x)y^b)^2)\rho(x) + 3(1 - 2(g_{ab}(x)u^a(x)y^b)^2)p(x).$$

- Generalize to Finsler fluid:
  - Consider dust: $p = 0$.
  - Consider superposition of dust with different velocities.
  - Integrate over contributions from each velocity.
  - Generalize $g_{ab}u^av^b$ to Finsler angle.

$$T_{\phi}(x, v) = m \int_{S_x} d^3v' \sqrt{\det h(x, v')}\phi(x, v')(1 - 6\cos^2\angle(v, v')).$$
Introduce suitable coordinates on $TM$: $t, r, \theta, \varphi, y^t, y^r, y^\theta, y^\varphi$. 

Most general Finsler function obeying cosmological symmetry:

$$F = F(t, y^t, w), w^2 = (y^r)^2 - kr^2 + r^2(y^\theta)^2 + \sin^2 \theta (y^\varphi)^2.$$ 

Homogeneity of Finsler function $F(t, y^t, w) = y^t \tilde{F}(t, w/y^t)$.

Introduce new coordinates: $	ilde{y} = y^t \tilde{F}(t, w/y^t)$, $	ilde{w} = w/y^t$.

$\Rightarrow$ Coordinates on observer space $O$ with $\tilde{y} \equiv 1$.

$\Rightarrow$ Geometry function $\tilde{F}(t, \tilde{w})$ on $O$. 

Manuel Hohmann (University of Tartu)

Finsler cosmology

23. June 2015 11 / 15
Introduce suitable coordinates on $TM$: 
\[ t, r, \theta, \varphi, y^t, y^r, y^\theta, y^\varphi. \]

Most general Finsler function obeying cosmological symmetry:
\[ F = F(t, y^t, w), \quad w^2 = \frac{(y^r)^2}{1 - kr^2} + r^2 \left( (y^\theta)^2 + \sin^2 \theta (y^\varphi)^2 \right). \]

Homogeneity of Finsler function $F(t, y^t, w) = y^t \tilde{F}(t, w/y^t)$. 

Cosmological symmetry

- Introduce suitable coordinates on $TM$:

$$t, r, \theta, \varphi, y^t, y^r, y^\theta, y^\varphi.$$ 

- Most general Finsler function obeying cosmological symmetry:

$$F = F(t, y^t, w), \quad w^2 = \frac{(y^r)^2}{1 - kr^2} + r^2 \left((y^\theta)^2 + \sin^2 \theta (y^\varphi)^2\right).$$

- Homogeneity of Finsler function $F(t, y^t, w) = y^t \tilde{F}(t, w/y^t)$.

- Introduce new coordinates: $\tilde{y} = y^t \tilde{F}(t, w/y^t)$, $\tilde{w} = w/y^t$.

$\Rightarrow$ Coordinates on observer space $O$ with $\tilde{y} \equiv 1$.

$\Rightarrow$ Geometry function $\tilde{F}(t, \tilde{w})$ on $O$. 

Manuel Hohmann (University of Tartu)
Most general fluid obeying cosmological symmetry:

\[ \phi = \phi(t, \tilde{\nu}) . \]
Cosmological fluid dynamics

- Most general fluid obeying cosmological symmetry:
  \[ \phi = \phi(t, \tilde{\nu}). \]

- Collisionless fluid satisfies Liouville equation:
  \[ 0 = \mathcal{L}_r \phi = \frac{1}{\tilde{F}} \left( \partial_t \phi - \frac{\partial_t \tilde{\nu} \tilde{F}}{\partial \tilde{\nu} \tilde{F}} \partial_{\tilde{\nu}} \phi \right). \]
Cosmological fluid dynamics

- Most general fluid obeying cosmological symmetry:

\[
\phi = \phi(t, \tilde{\nu}) .
\]

- Collisionless fluid satisfies Liouville equation:

\[
0 = \mathcal{L}_r \phi = \frac{1}{\tilde{F}} \left( \partial_t \phi \ - \ \frac{\partial_t \partial_{\tilde{\nu}} \tilde{F}}{\partial_{\tilde{\nu}} \tilde{F}} \right) .
\]

- Example: collisionless dust fluid \( \phi(x, y) \sim \rho(x) \delta_{S_x}(y, u(x)) \):

\[
u(t) = \frac{1}{\tilde{F}(t, 0)} \partial_t , \quad \partial_t \left( \rho(t) \sqrt{g^F(t, 0)} \right) = 0 .
\]
Start from gravitational field equations:

\[
\begin{align*}
& \left[ g^{F \, ab} \bar{\partial}_a \bar{\partial}_b (R^c_{\, cd} y^d) - 6 \frac{R^a_{\, ab} y^b}{F^2} \\
+ 2 g^{F \, ab} (\nabla_a S_b + S_a S_b + \bar{\partial}_a (y^c \delta_c S_b - \mathcal{N}^c_{\, b} S_c)) \right] \bigg|_\Sigma = \kappa T|_\Sigma
\end{align*}
\]
Start from gravitational field equations:

\[
\left[ g^{F \ ab} \bar{\partial}_a \bar{\partial}_b (R^c_{\ cd} y^d) - 6 \frac{R^a_{\ ab} y^b}{F^2} + 2g^{F \ ab} (\nabla_a S_b + S_a S_b + \bar{\partial}_a (y^c \delta_c S_b - N^c_{\ b} S_c)) \right]_{\Sigma} = \kappa T_{\Sigma}
\]

Some terms simplify for cosmological symmetry: \( R^a_{\ ab} y^b \).
Cosmological gravitational dynamics

- Start from gravitational field equations:

\[
\left[ g^{F \, ab} \partial_a \partial_b (R^c_{\, cd} y^d) - 6 \frac{R^a_{\, ab} y^b}{F^2} \right.
+ 2 g^{F \, ab} \left( \nabla_a S_b + S_a S_b + \partial_a (y^c \delta_c S_b - N^c_{\, b} S_c) \right) \bigg|_\Sigma = \kappa T|_\Sigma
\]

- Some terms simplify for cosmological symmetry: \( R^a_{\, ab} y^b \).
- Some terms don’t simplify at all: \( N^a_{\, b} \), \( \nabla_a S_b \).
Cosmological gravitational dynamics

- Start from gravitational field equations:

\[
\left[ g^F_{\,ab} \bar{\partial}_a \bar{\partial}_b (R^c_{\,cd} y^d) - 6 \frac{R^a_{\,ab} y^b}{F^2} \\
+ 2 g^F_{\,ab} (\nabla_a S_b + S_a S_b + \bar{\partial}_a (y^c \delta_c S_b - N^c_{\,b} S_c)) \right]_{\Sigma} = \kappa T |_{\Sigma}
\]

- Some terms simplify for cosmological symmetry: \( R^a_{\,ab} y^b \).
- Some terms don’t simplify at all: \( N^a_{\,b}, \nabla_a S_b \).
- Simplify the problem:
  - Finsler perturbation of metric geometry.
  - Finsler function using higher rank tensors: \( H_{a_1 \ldots a_n} y^{a_1} \ldots y^{a_n} \).
Summary

- **Finsler spacetimes:**
  - Define geometry by length functional.
  - Observer space $O$ of physical four-velocities.
  - Geodesics are integral curves of vector field on $O$.
  - Dynamics given by gravitational action.
Finsler spacetimes:
- Define geometry by length functional.
- Observer space $O$ of physical four-velocities.
- Geodesics are integral curves of vector field on $O$.
- Dynamics given by gravitational action.

Fluid dynamics:
- Model fluids by point mass trajectories.
- Define fluid density on observer space.
- Collisionless fluid satisfies Liouville equation.
Finsler spacetimes:
- Define geometry by length functional.
- Observer space $O$ of physical four-velocities.
- Geodesics are integral curves of vector field on $O$.
- Dynamics given by gravitational action.

Fluid dynamics:
- Model fluids by point mass trajectories.
- Define fluid density on observer space.
- Collisionless fluid satisfies Liouville equation.

Cosmology:
- All quantities depend on only two coordinates $t, \tilde{w}$.
- Simple equation of motion for cosmological fluid matter.
- Gravitational field equation becomes involved.
Derivation of Finsler-Friedmann equations:
- Finsler perturbation of metric background.
- Simple model Finsler geometry from higher rank tensors.
- Fully general Finsler-Friedmann equations?
Derivation of Finsler-Friedmann equations:
- Finsler perturbation of metric background.
- Simple model Finsler geometry from higher rank tensors.
- Fully general Finsler-Friedmann equations?

Solving for cosmological dynamics
- Dark energy?
- Inflation?