Cosmology based on Finsler geometry

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GR21 session A3 - 12. July 2016

Outline

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Why cosmology based on Finsler geometry?

- Modify spacetime geometry to address open problems:
  - Origin of dark matter and dark energy.
  - Homogeneity of the cosmic microwave background and inflation.
  - Fly-by anomaly in the solar system.

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  - Divide tangent spaces into space-, time-, lightlike vectors.
  - Provide notions of future and past.
  - Distinguish curves corresponding to physical trajectories.
  - Define proper time along physical trajectories.
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  - Finsler geodesics determine notion of free fall.
  - Gravity theory on Finsler spacetimes exists.

Finsler gravity action and field equations

- Spacetime manifold M with Finsler function $F : TM \to \mathbb{R}^+$.

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- Finsler gravity action:

$$S_G = \int_\Sigma \mathcal{R}\,\mathrm{Vol}(G|_\Sigma)\,.$$

  - Σ: Unit tangent bundle $TM|_{F=1}$.
  - G: Sasaki metric on TM.
  - $\mathcal{R}$: Scalar curvature of Cartan non-linear connection.

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- Gravitational field equations by variation with respect to F:

$$-\frac{1}{F^2}\left\{6\mathcal{R} + G^{ab}\left[\nabla^v_a\nabla^v_b\mathcal{R} + 2F^2 J^c{}_a\nabla^h_b\$_c + 2\nabla^v_a\left(\mathbf{S}^c\nabla^h_c\$_b\right)\right]\right\} = \mathcal{T}\,.$$

  - $a, b, c, \ldots$: Coordinate indices $0, \ldots, 7$ on TM.
  - $J^a{}_b$: Tangent structure.
  - $\mathbf{S}^a$: Geodesic spray.
  - $\$_a$: Landsberg covector.
  - ∇^h, ∇^v: Horizontal and vertical Berwald derivative.
  - $\mathcal{T}$: Energy-momentum scalar.

Outline

Geometry of Finsler spacetimes

Finsler length function

- Finsler function $F : TM \to \mathbb{R}^+$.
- Finsler length functional for $\gamma : \mathbb{R} \to M$:

$$\ell_{t_1}^{t_2}[\gamma] = \int_{t_1}^{t_2} F(\gamma(t), \dot{\gamma}(t))dt\,.$$

- Parametrization invariance requires homogeneity:

$$F(x, \lambda y) = \lambda F(x, y) \quad \forall \lambda > 0\,.$$

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Connection and curvature

- Split $TTM = HTM \oplus VTM$ of the double tangent bundle.
- Projectors **h**, **v** onto subbundles.
- Curvature tensor $R = -N_{\mathbf{h}}$ defined via Nijenhuis tensor.
- Curvature scalar defined from curvature tensor.

Cosmological symmetry

Cosmological coordinates on TM [MH '15]

- Spherical coordinates t, r, ϑ, φ on M.
- Coordinates y, u, v, w on each T_xM:

$$y\partial_t + w\left[\cos u\sqrt{1-kr^2}\partial_r + \frac{\sin u}{r}\left(\cos v\partial_\vartheta + \frac{\sin v}{\sin\vartheta}\partial_\varphi\right)\right] \in T_xM\,.$$

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Cosmologically symmetric Finsler spacetime

- Symmetry under rotations and translations (six vector fields).
- Most general Finsler function: $F(t, y, w)$.
- Homogeneity condition: $F(t, \lambda y, \lambda w) = \lambda F(t, y, w)$.
- Express Finsler function as $F(t, y, w) = y\tilde{F}(t, w/y)$.

Observer space coordinates

Observer trajectories

- Tangent vectors are future unit timelike vectors: $F = 1$.
- Future unit timelike vectors form shell in each T_xM.
- Introduce suitable coordinates on these shells.

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- Introduce suitable coordinates on these shells.

Observer space coordinates [MH '15]

- Introduce coordinates:

$$T = t, R = r, \Theta = \vartheta, \Phi = \varphi, Y = y\tilde{F}\left(t, \frac{w}{y}\right), U = u, V = v, W = w/y\,.$$

$\Rightarrow$ Unit tangent bundle has $Y = 1$.

$\Rightarrow$ Light cone has $Y = 0$.

Geodesics on cosmological background

- $\gamma : \mathbb{R} \to M$ minimizes Finsler length functional.

$\Leftrightarrow$ γ satisfies second order ODE.

$\Leftrightarrow$ $\dot{\gamma} : \mathbb{R} \to TM$ satisfies first order ODE.

$\Leftrightarrow$ $\dot{\gamma}$ is integral curve of vector field **S** called geodesic spray:

$$\mathbf{S} = \frac{Y}{\tilde{F}}\left(\partial_T + W\cos U\sqrt{1-kR^2}\partial_R + \frac{W\sin U\cos V}{R}\partial_\Theta + \frac{W\sin U\sin V}{R\sin\Theta}\partial_\Phi - \frac{W\sin U\sqrt{1-kR^2}}{R}\partial_U - \frac{W\sin U\sin V}{R\tan\Theta}\partial_V - \frac{\partial_T\partial_W\tilde{F}}{\partial_W\partial_W\tilde{F}}\partial_W\right).$$

- Coordinate Y is constant along Finsler geodesics.

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- Coordinate Y is constant along Finsler geodesics.
- Radial geodesic given by $U = 0$:

$$\mathbf{S}|_{U=0} = \frac{Y}{\tilde{F}}\left(\partial_T + W\sqrt{1-kR^2}\partial_R - \frac{\partial_T\partial_W\tilde{F}}{\partial_W\partial_W\tilde{F}}\partial_W\right).$$

- Co-moving geodesic given by $W = 0$:

$$\mathbf{S}|_{W=0} = \frac{Y}{\tilde{F}}\left(\partial_T - \frac{\partial_T\partial_W\tilde{F}}{\partial_W\partial_W\tilde{F}}\partial_W\right).$$

Fluid dynamics with cosmological symmetry

Kinetic theory of fluids [Ehlers '71], [Sarbach, Zannias '13]

- Consider fluid as constituted by point particles.
- Particles follow piecewise geodesics between collisions.
- Continuum limit described by density $\phi : TM|_{Y=1} \to \mathbb{R}^+$.
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Cosmologically symmetric Finsler fluids [MH '15]

- Most general cosmologically symmetric fluid: $\phi = \phi(T, W)$.
- Liouville equation: $\partial_T\phi\partial_W\partial_W\tilde{F} = \partial_W\phi\partial_T\partial_W\tilde{F}$.

Gravitational dynamics

Finsler gravity [Pfeifer, Wohlfarth '11]

- Action:

$$S_G = \int_\Sigma \mathcal{R}\,\mathrm{Vol}(G|_\Sigma)\,.$$

- Field equations:

$$-\frac{1}{F^2}\left\{6\mathcal{R} + G^{ab}\left[\nabla^v_a\nabla^v_b\mathcal{R} + 2F^2 J^c{}_a\nabla^h_b \$_c + 2\nabla^v_a\left(\mathbf{S}^c\nabla^h_c \$_b\right)\right]\right\} = \mathcal{T}\,.$$

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Cosmological dynamics

- Structure of cosmological equations: $\mathcal{G}[\tilde{F}](T,W) = \mathcal{T}[\tilde{F},\phi](T,W)$.
- Difficulties:
  - Geometry scalar $\mathcal{G}$ is complicated even for cosmology.
  - No "standard construction" for $\mathcal{T}$ of non-metric kinetic fluid.

Outline

Metric spacetime

Geometry

- Tensor field: metric $g_{\mu\nu}$.
- Cosmology: FLRW metric $g = -dt \otimes dt + a^2(t)\gamma_{ij}[\kappa]dx^i \otimes dx^j$.
- Finsler function:

$$F(x, y) = \sqrt{|g_{\mu\nu}y^\mu y^\nu|} = Y\sqrt{\left|1 - a^2(T)W^2\right|}$$

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Gravitational dynamics

- Geometry scalar:

$$\mathcal{G} = \frac{6}{a^2(1 - W^2 a^2)}\left(a\ddot{a} - 2\dot{a}^2 - 2\kappa + W^2 a^3 \ddot{a}\right) .$$

$\Rightarrow$ Reproduce structure of Friedmann equations.

Length measure with one-forms

Ingredients

- Tensor fields: metric $g_{\mu\nu}$, one-form A_μ.
- Cosmology:
  - FLRW metric $g = -dt \otimes dt + a^2(t)\gamma_{ij}[\kappa]dx^i \otimes dx^j$.
  - Hypersurface normal $A = b(t)dt$.

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Randers length measure [Randers '41]

$$F(x,y) = \sqrt{|g_{\mu\nu}y^\mu y^\nu|} + A_\mu y^\mu = Y\sqrt{\left|1 - a^2(T)W^2\right|} + Yb(T)$$

Bogoslovsky length measure [Bogoslovsky '77]

$$F(x,y) = (A_\mu y^\mu)^q \left(\sqrt{|g_{\mu\nu}y^\mu y^\nu|}\right)^{1-q} = Yb^q(T)\left(\sqrt{\left|1 - a^2(T)W^2\right|}\right)^{1-q}$$

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Conclusion

Summary

- Finsler spacetimes:
  - Based on Finsler length function.
  - Make use of tensors on the tangent bundle.
  - Generalize standard notions of causality, observers and gravity.
- Finsler cosmology:
  - Geometry defined by function $\tilde{F}(T, W)$.
  - Simple form of geodesic equation.
  - Simple equation of motion for fluid dynamics.
  - Gravitational field equations are rather complicated.
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Outlook

- Construct energy-momentum scalar for kinetic Finsler fluid.
- Find non-metric solutions for Finsler cosmology.
- Calculate cosmological parameters from Finsler geometry.

Conclusion

- Kinetic theory on the tangent bundle:
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