Cosmology based on Finsler geometry

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GR21 session A3 - 12. July 2016
Outline

1. Motivation
2. Finsler cosmology
3. Tensorial Finsler cosmologies
4. Conclusion
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1. Motivation
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4. Conclusion
Why cosmology based on Finsler geometry?

- Modify spacetime geometry to address open problems:
  - Origin of dark matter and dark energy.
  - Homogeneity of the cosmic microwave background and inflation.
  - Fly-by anomaly in the solar system.

- Choose geometry which keeps well-known notions:
  - Divide tangent spaces into space-, time-, lightlike vectors.
  - Provide notions of future and past.
  - Distinguish curves corresponding to physical trajectories.
  - Define proper time along physical trajectories.
  - Determine trajectories of freely falling test masses.

- Geometry is determined by matter distribution.

Finsler spacetime geometry provides all these notions:
- Finsler length functional measures length of curves.
- Finsler metric has Lorentz signature.
- Orientability allows to distinguish future and past.
- Previously mentioned notions define future timelike curves.
- Finsler geodesics determine notion of free fall.
- Gravity theory on Finsler spacetimes exists.
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Finsler gravity action and field equations

- Spacetime manifold $M$ with Finsler function $F : TM \rightarrow \mathbb{R}^+$. 

Finsler gravity action:

$$S_G = \int_{\Sigma} R \cdot \text{Vol}(G|\Sigma).$$

$\Sigma$: Unit tangent bundle $TM|F = 1$.

$G$: Sasaki metric on $TM$.

$R$: Scalar curvature of Cartan non-linear connection.

Gravitational field equations by variation with respect to $F$:

$$-\frac{1}{F^2} \left\{ 6R + G_{ab}[\nabla_v v^a \nabla_v v^b R + 2F^2 J_{ca} \nabla_h b^c / S^c + 2\nabla_v v^a (S^c \nabla_h c^b / S^b) \right\} = T^a.$$ 

$a, b, c, \ldots$: Coordinate indices $0, \ldots, 7$ on $TM$.

$J_{ab}$: Tangent structure.

$S^a$: Geodesic spray.

$/S^a$: Landsberg covector.

$\nabla_h, \nabla_v$: Horizontal and vertical Berwald derivative.

$T^a$: Energy-momentum scalar.
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Finsler length function

- Finsler function $F : TM \to \mathbb{R}^+$. 
- Finsler length functional for $\gamma : \mathbb{R} \to M$:
  \[ \ell_{t_1}^{t_2}[\gamma] = \int_{t_1}^{t_2} F(\gamma(t), \dot{\gamma}(t)) \, dt. \]
- Parametrization invariance requires homogeneity:
  \[ F(x, \lambda y) = \lambda F(x, y) \quad \forall \lambda > 0. \]
Geometry of Finsler spacetimes

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Connection and curvature

- Split \( TTM = HTM \oplus VTM \) of the double tangent bundle.
- Projectors \( h, v \) onto subbundles.
- Curvature tensor \( R = -N_h \) defined via Nijenhuis tensor.
- Curvature scalar defined from curvature tensor.
Cosmological symmetry

Cosmological coordinates on $TM$ [MH ’15]

- Spherical coordinates $t, r, \vartheta, \varphi$ on $M$.
- Coordinates $y, u, v, w$ on each $T_x M$:

$$y \partial_t + w \left[ \cos u \sqrt{1 - kr^2} \partial_r + \frac{\sin u}{r} \left( \cos v \partial_\vartheta + \frac{\sin v}{\sin \vartheta} \partial_\varphi \right) \right] \in T_x M.$$
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Cosmologically symmetric Finsler spacetime

- Symmetry under rotations and translations (six vector fields).
- Most general Finsler function: $F(t, y, w)$.
- Homogeneity condition: $F(t, \lambda y, \lambda w) = \lambda F(t, y, w)$.
- Express Finsler function as $F(t, y, w) = y \tilde{F}(t, w/y)$. 

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Observer space coordinates

Observer trajectories

- Tangent vectors are future unit timelike vectors: $F = 1$.
- Future unit timelike vectors form shell in each $T_x M$.
- Introduce suitable coordinates on these shells.
Observer space coordinates

Observer trajectories

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Observer space coordinates [MH '15]

- Introduce coordinates:

\[
T = t, \quad R = r, \quad \Theta = \vartheta, \quad \Phi = \varphi, \quad Y = y \tilde{F} \left( t, \frac{w}{y} \right), \quad U = u, \quad V = v, \quad W = w/y.
\]

\[\Rightarrow \quad \text{Unit tangent bundle has } Y = 1.\]
\[\Rightarrow \quad \text{Light cone has } Y = 0.\]
Geodesics on cosmological background

- \( \gamma : \mathbb{R} \to M \) minimizes Finsler length functional.
- \( \iff \) \( \gamma \) satisfies second order ODE.
- \( \iff \) \( \dot{\gamma} : \mathbb{R} \to TM \) satisfies first order ODE.
- \( \iff \) \( \dot{\gamma} \) is integral curve of vector field \( \mathbf{S} \) called geodesic spray:

\[
\mathbf{S} = \frac{Y}{\tilde{F}} \left( \partial_T + W \cos U \sqrt{1 - kR^2} \partial_R 
+ \frac{W \sin U \cos V}{R} \partial_\Theta 
+ \frac{W \sin U \sin V}{R \sin \Theta} \partial_\Phi 
- \frac{W \sin U \sqrt{1 - kR^2}}{R \tan \Theta} \partial_V 
- \frac{\partial_T \partial_W \tilde{F}}{\partial_W \partial_W \tilde{F}} \partial_W \right).
\]

- Coordinate \( Y \) is constant along Finsler geodesics.
Geodesics on cosmological background

- \( \gamma : \mathbb{R} \rightarrow M \) minimizes Finsler length functional.
- \( \iff \gamma \) satisfies second order ODE.
- \( \iff \dot{\gamma} : \mathbb{R} \rightarrow TM \) satisfies first order ODE.
- \( \iff \dot{\gamma} \) is integral curve of vector field \( S \) called geodesic spray.

Coordinate \( Y \) is constant along Finsler geodesics.

Radial geodesic given by \( U = 0 \):

\[
S|_{U=0} = \frac{Y}{\tilde{F}} \left( \partial_T + W \sqrt{1 - kR^2} \partial_R - \frac{\partial_T \partial_W \tilde{F}}{\partial_W \partial_W \tilde{F}} \partial_W \right).
\]

Co-moving geodesic given by \( W = 0 \):

\[
S|_{W=0} = \frac{Y}{\tilde{F}} \left( \partial_T - \frac{\partial_T \partial_W \tilde{F}}{\partial_W \partial_W \tilde{F}} \partial_W \right).
\]
Kinetic theory of fluids \cite{Ehlers71, SarbachZannias13}

- Consider fluid as constituted by point particles.
- Particles follow piecewise geodesics between collisions.
- Continuum limit described by density $\phi : TM|_{Y=1} \rightarrow \mathbb{R}^+$. 
- Collisionless fluid satisfies Liouville equation $\mathcal{L}_S\phi = 0$. 

Fluid dynamics with cosmological symmetry

Kinetic theory of fluids [Ehlers '71], [Sarbach, Zannias '13]

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Cosmologically symmetric Finsler fluids [MH '15]

- Most general cosmologically symmetric fluid: $\phi = \phi(T, W)$.
- Liouville equation: $\partial_T \phi \partial_W \partial_W \tilde{F} = \partial_W \phi \partial_T \partial_W \tilde{F}$. 

Gravitational dynamics

Finsler gravity [Pfeifer, Wohlfarth '11]

- **Action:**

\[ S_G = \int_{\Sigma} R \, \text{Vol}(G|_{\Sigma}) . \]

- **Field equations:**

\[- \frac{1}{F^2} \left\{ 6R + G^{ab} \left[ \nabla^v_a \nabla^v_b R + 2F^2 J^c_a \nabla^h_b S_c + 2\nabla^v_a \left( S^c \nabla^h_c S_b \right) \right] \right\} = T.\]
**Gravitational dynamics**

**Finsler gravity** [Pfeifer, Wohlfarth '11]

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**Cosmological dynamics**

- **Structure of cosmological equations:** \( G[\tilde{F}](\mathcal{T}, W) = \mathcal{T}[\tilde{F}, \phi](\mathcal{T}, W) \).

- **Difficulties:**
  - Geometry scalar \( G \) is complicated even for cosmology.
  - No “standard construction” for \( \mathcal{T} \) of non-metric kinetic fluid.
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Metric spacetime

Geometry

- Tensor field: metric $g_{\mu\nu}$.
- Cosmology: FLRW metric $g = -dt \otimes dt + a^2(t)\gamma_{ij}[\kappa]dx^i \otimes dx^j$.
- Finsler function:

\[
F(x, y) = \sqrt{|g_{\mu\nu}y^\mu y^\nu|} = Y\sqrt{|1 - a^2(T)W^2|}
\]
Metric spacetime

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Gravitational dynamics

- Geometry scalar:

$$\mathcal{G} = \frac{6}{a^2(1 - W^2 a^2)} \left( a\ddot{a} - 2\dot{a}^2 - 2\kappa + W^2 a^3 \dddot{a} \right).$$

$\Rightarrow$ Reproduce structure of Friedmann equations.
## Length measure with one-forms

### Ingredients
- Tensor fields: metric $g_{\mu \nu}$, one-form $A_\mu$.
- Cosmology:
  - FLRW metric $g = -dt \otimes dt + a^2(t)\gamma_{ij}[\kappa]dx^i \otimes dx^j$.
  - Hypersurface normal $A = b(t)dt$.
**Length measure with one-forms**

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**Randers length measure** [Randers '41]

$F(x, y) = \sqrt{|g_{\mu\nu} y^\mu y^\nu| + A_\mu y^\mu} = Y \sqrt{|1 - a^2(T)W^2|} + Yb(T)$

**Bogoslovsky length measure** [Bogoslovsky '77]

$F(x, y) = (A_\mu y^\mu)^q \left(\sqrt{|g_{\mu\nu} y^\mu y^\nu|}\right)^{1-q} = Yb^q(T) \left(\sqrt{|1 - a^2(T)W^2|}\right)^{1-q}$
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Summary

- **Finsler spacetimes:**
  - Based on Finsler length function.
  - Make use of tensors on the tangent bundle.
  - Generalize standard notions of causality, observers and gravity.

- **Finsler cosmology:**
  - Geometry defined by function $\tilde{F}(T, W)$.
  - Simple form of geodesic equation.
  - Simple equation of motion for fluid dynamics.
  - Gravitational field equations are rather complicated.
  - Simplified models can be derived from tensorial geometries.
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Outlook

- Construct energy-momentum scalar for kinetic Finsler fluid.
- Find non-metric solutions for Finsler cosmology.
- Calculate cosmological parameters from Finsler geometry.
Conclusion

- Kinetic theory on the tangent bundle:

- Finsler spacetimes:

- Finsler fluids and cosmology: