Harmonic d-tensors

A tool for calculating symmetric Finsler spacetimes

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Outline

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Pullback bundle vs. fibered product

- Definition of a pullback bundle:
  - Smooth manifolds M, N.
  - Fiber bundle $\pi : E \to M$.
  - Smooth map $\phi : N \to M$.
  - Pullback bundle $\phi^*\pi : \phi^*E \to N$, where
    - total space: $\phi^*E = \{(p, e) \in N \times E, \phi(p) = \pi(e)\}$,
    - projection: $\phi^*\pi(p, e) = p$.
  - Isomorphisms between fibers $F \cong (\phi^*E)_p \cong E_{\phi(p)}$.
  - Fiber bundle structure of E induces fiber bundle structure on ϕ^*E:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\psi} & U \times F \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_1} & \\ U & & \end{array} \quad \Rightarrow \quad \begin{array}{ccc} (\phi^*\pi)^{-1}(\phi^{-1}(U)) & \xrightarrow{\tilde{\psi}} & \phi^{-1}(U) \times F \\ {\scriptstyle \phi^*\pi}\downarrow & \swarrow {\scriptstyle \mathrm{pr}_1} & \\ U & & \end{array}$$

where U trivializes E around $\phi(p)$ and $\tilde{\psi}(p, e) = (p, \mathrm{pr}_2(\psi(e)))$.

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    where U trivializes E around $\phi(p)$ and $\tilde{\psi}(p, e) = (p, \mathrm{pr}_2(\psi(e)))$.
- For $N = E$ and $\phi = \pi$: $\phi^*E = E \times_M E$.

D-tensors

- Definition of d-tensors:
  - Tangent bundle: $\tau : TM \to M$.
  - Pullback bundle: $\pi = \tau^*\tau : TM \times_M TM \to TM$.
  - Tensor bundles: $\mathcal{T}^r_s(\pi) \cong (TM \times_M TM)^{\otimes r} \otimes (TM \times_M T^*M)^{\otimes s}$.
  - (r, s)-d-tensor field: section of $\mathcal{T}^r_s(\pi)$.

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  - (r, s)-d-tensor field: section of $\mathcal{T}^r_s(\pi)$.
- Relation to the double tangent bundle $\varpi : TTM \to TM$:
  - Canonical injective strong bundle map:

$$\begin{array}{rcccl} \mathbf{i} & : & TM \times_M TM & \to & TTM \\ & & (v, w) & \mapsto & \frac{d}{dt}(v + tw)\Big|_{t=0} \end{array}$$

  - Canonical surjective strong bundle map:

$$\begin{array}{rcccl} \mathbf{j} & : & TTM & \to & TM \times_M TM \\ & & \xi & \mapsto & (\varpi(\xi), \tau_*(\xi)) \end{array}$$

  - Exact sequence:

$$0 \to TM \times_M TM \xrightarrow{\mathbf{i}} TTM \xrightarrow{\mathbf{j}} TM \times_M TM \to 0$$

  - Vertical tangent bundle: $VTM = \operatorname{im}\mathbf{i} = \ker\mathbf{j}$.

Diffeomorphisms acting on d-tensors

- Lift of diffeomorphisms to d-tensors:
  - Diffeomorphism $\varphi : M \to M$.
  - $\Rightarrow$ Lift to the tangent bundle: $\varphi_* : TM \to TM$.
  - $\Rightarrow$ Lift to the pullback bundle: $\varphi_* \times_M \varphi_* : TM \times_M TM \to TM \times_M TM$.
  - $\Rightarrow$ Lift to $\mathcal{T}^r_s(\pi)$ via pushforward / pullback.

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- Infinitesimal diffeomorphisms:
  - Vector field $X = X^a\partial_a \in \mathfrak{X}(M)$.
  - ⇒ Complete lift $\hat{X} = X^a\partial_a + y^a\partial_a X^b\bar{\partial}_b \in \mathfrak{X}(TM)$.
  - ⇒ Action on d-tensor $T \in \mathcal{T}^r_s(\pi)$ in coordinate basis of TM:

$$
\begin{aligned}
(\mathcal{L}_{\hat{X}}T)^{a_1\cdots a_r}{}_{b_s} = {} & X^c\partial_c T^{a_1\cdots a_r}{}_{b_1\cdots b_s} + y^d\partial_d X^c\bar{\partial}_c T^{a_1\cdots a_r}{}_{b_1\cdots b_s} \\
& - \partial_c X^{a_1} T^{ca_2\cdots a_r}{}_{b_1\cdots b_s} - \ldots - \partial_c X^{a_r} T^{a_1\cdots a_{r-1}c}{}_{b_1\cdots b_s} \\
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- Introduce short notation:
  - Vector field $\mathbf{X} \in \mathfrak{X}(TM)$.
  - Corresponding Lie derivative $\mathcal{X} = i\mathcal{L}_{\mathbf{X}}$.

Outline

Co-rotated cylindrical coordinates on $T\mathbb{R}^3$

- Coordinates x^1, x^2, x^3 on $M = \mathbb{R}^3$.
- Induced coordinates $x^1, x^2, x^3, y^1, y^2, y^3$ on TM.
- Co-rotated cylindrical coordinates $r, \bar{\rho}, \bar{z}, \beta, \theta, \phi$ on TM:

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$$

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \bar{\rho}\cos\beta \\ \bar{\rho}\sin\beta \\ \bar{z} \end{pmatrix}$$

Generating vector fields of SO(3)

- Generating vector fields $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \in \mathfrak{X}(M)$ of SO(3):

$$\mathbf{r}_1 = \sin\phi\partial_\theta + \frac{\cos\phi}{\tan\theta}\partial_\phi\,,$$
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- Canonical lifts to vector fields $\mathbf{R}_1, \mathbf{R}_2, \mathbf{R}_3 \in \mathfrak{X}(TM)$:

$$\mathbf{R}_1 = \sin\phi\,\partial_\theta + \frac{\cos\phi}{\tan\theta}\partial_\phi - \frac{\cos\phi}{\sin\theta}\partial_\beta\,,$$

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- Operators $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3 \in \mathfrak{A}(\mathcal{T}(\pi))$:

$$\mathcal{R}_k Y = i\mathcal{L}_{\mathbf{R}_k} Y\,.$$

Auxiliary vector fields and co-rotations

- Auxiliary vector fields $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3 \in \mathfrak{X}(TM)$:

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- Important remarks:
  - Vector fields $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3 \in \mathfrak{X}(TM)$ are not canonical lifts.
  - $\Rightarrow$ Operators $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \in \mathfrak{A}(\mathcal{C}^\infty(TM))$ do not act on d-tensors.

Further operators and algebra relations

- Rotation algebra:

$$[\mathcal{R}_i, \mathcal{R}_j] = i\epsilon_{ijk}\mathcal{R}_k\,, \quad [\mathcal{B}_i, \mathcal{B}_j] = i\epsilon_{ijk}\mathcal{B}_k\,, \quad [\mathcal{B}_i, \mathcal{R}_j] = 0\,.$$

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- Introduce ladder operators and Casimir:

$$\mathcal{R}_\pm = \mathcal{R}_1 \pm i\mathcal{R}_2\,, \quad \mathcal{R}_z = \mathcal{R}_3\,, \quad \mathcal{B}_\pm = \mathcal{B}_1 \pm i\mathcal{B}_2\,, \quad \mathcal{B}_z = \mathcal{B}_3\,,$$
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$$[\mathcal{R}_z, \mathcal{R}_\pm] = \pm\mathcal{R}_\pm\,, \quad [\mathcal{R}_+, \mathcal{R}_-] = 2\mathcal{R}_z\,, \quad [\mathcal{R}_\pm, \mathcal{R}^2] = [\mathcal{R}_z, \mathcal{R}^2] = 0\,,$$
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- Operators constitute algebra of a rigid rotor.

⇒ $\mathcal{R}^2, \mathcal{R}_z, \mathcal{B}_z$ mutually commute.

Scalar representations of rotation algebra

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- Azimuthal parts $\mathcal{R}_z F = mF$ and $\mathcal{B}_z F = nF$:

$$\mathcal{R}_z F = -i\partial_\phi F \Rightarrow \Phi(\phi) = e^{im\phi}\,, \quad m \in \mathbb{Z}\,,$$
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- Zenith part $\mathcal{R}^2 F = l(l+1)F$:

$$\Theta''(\theta) + \frac{\Theta'(\theta)}{\tan\theta} + \left(\frac{2mn\cos\theta - m^2 - n^2}{\sin^2\theta} + l(l+1)\right)\Theta(\theta) = 0\,.$$

Definition of scalar spherical harmonics

- General formula:

$$\mathcal{Y}_{l,m,n}(\theta,\phi,\beta) = N_{l,m,n} e^{im\phi} e^{in\beta} \cos^{m+n}\frac{\theta}{2} \sin^{|m-n|}\frac{\theta}{2} \cdot {}_2F_1\left(\max(m,n) - l, \max(m,n) + l + 1; |m-n| + 1; \sin^2\frac{\theta}{2}\right)$$

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- Normalization constants:

$$N_{l,m,n} = (-1)^{\max(m,n)} \frac{\sqrt{(2l+1)}}{|m-n|!} \sqrt{\frac{(l-\min(m,n))!(l+\max(m,n))!}{(l-\max(m,n))!(l+\min(m,n))!}}\,.$$

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- Related to Wigner D-matrices up to constant factors.

Properties of scalar spherical harmonics

- Eigenvalue relations:

$$\mathcal{R}^2\mathcal{Y}_{l,m,n} = l(l+1)\mathcal{Y}_{l,m,n}\,, \quad \mathcal{R}_z\mathcal{Y}_{l,m,n} = m\mathcal{Y}_{l,m,n}\,, \quad \mathcal{B}_z\mathcal{Y}_{l,m,n} = n\mathcal{Y}_{l,m,n}\,.$$

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- Ladder operators:

$$\mathcal{R}_\pm\mathcal{Y}_{l,m,n} = \sqrt{(l \mp m)(l \pm m + 1)}\mathcal{Y}_{l,m\pm1,n}\,,$$
$$\mathcal{B}_\pm\mathcal{Y}_{l,m,n} = \sqrt{(l \mp n)(l \pm n + 1)}\mathcal{Y}_{l,m,n\pm1}\,.$$

Properties of scalar spherical harmonics

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$$\mathcal{R}^2\mathcal{Y}_{l,m,n} = l(l+1)\mathcal{Y}_{l,m,n}\,, \quad \mathcal{R}_z\mathcal{Y}_{l,m,n} = m\mathcal{Y}_{l,m,n}\,, \quad \mathcal{B}_z\mathcal{Y}_{l,m,n} = n\mathcal{Y}_{l,m,n}\,.$$

- Ladder operators:

$$\mathcal{R}_\pm\mathcal{Y}_{l,m,n} = \sqrt{(l\mp m)(l\pm m+1)}\mathcal{Y}_{l,m\pm 1,n}\,,$$
$$\mathcal{B}_\pm\mathcal{Y}_{l,m,n} = \sqrt{(l\mp n)(l\pm n+1)}\mathcal{Y}_{l,m,n\pm 1}\,.$$

- Orthogonality and normalization:

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi}\mathcal{Y}_{l,m,n}(\theta,\phi,\beta)\overline{\mathcal{Y}_{l',m',n'}}(\theta,\phi,\beta)\sin\theta\,d\theta\,d\phi\,d\beta = 8\pi^2\delta_{ll'}\delta_{mm'}\delta_{nn'}\,.$$

Definition of spherical harmonic d-tensors

- Basis $\mathbf{e}_{-1}, \mathbf{e}_0, \mathbf{e}_1$ of $\mathcal{T}^0_1(\pi)$ such that

$$\mathcal{R}^2\mathbf{e}_m = 2\mathbf{e}_m\,, \quad \mathcal{R}_z\mathbf{e}_m = m\mathbf{e}_m\,, \quad \mathcal{R}_\pm\mathbf{e}_m = \sqrt{(1 \mp m)(2 \pm m)}\mathbf{e}_{m\pm 1}$$

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$$\overset{m}{\underset{n}{\mathbf{Y}}}{}_{l_0 l_1 \cdots l_k} = (-1)^{l_k - m}\sqrt{2l_k + 1}\sum_{m',\mu}\begin{pmatrix} l_k & l_{k-1} & 1 \\ m & -m' & -\mu \end{pmatrix}\overset{m'}{\underset{n}{\mathbf{Y}}}{}_{l_0 l_1 \cdots l_{k-1}} \otimes \mathbf{e}_\mu$$

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- Analogue construction for dual basis and mixed tensors.

Properties of spherical harmonic d-tensors

- Eigenvalue relations:

$$\mathcal{R}^2 \overset{m}{\underset{n}{\mathbf{Y}}}{}_{l_0 l_1 \cdots l_k} = l_k(l_k + 1) \overset{m}{\underset{n}{\mathbf{Y}}}{}_{l_0 l_1 \cdots l_k} \,,$$

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Further properties and examples of formulas

- Transpose of tensors of rank 2:

$$\left(\underset{n}{\overset{m}{\mathbf{Y}}}{}_{l_0 l_1 l_2}\right)^t = \sum_l (-1)^{l+l_1}\sqrt{2l+1}\sqrt{2l_1+1}\begin{Bmatrix} l_0 & l_1 & 1 \\ l_2 & l & 1 \end{Bmatrix}\underset{n}{\overset{m}{\mathbf{Y}}}{}_{l_0 l l_2}\,.$$

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- Vertical gradient operator for $f = f(r,\bar{\rho},\bar{z})$:

$$\nabla^v\left(f\overset{m}{\underset{n}{\mathbf{Y}}}{}_{l_0 l_1\cdots l_k}\right) = \left[\frac{1}{\sqrt{2}}\left(n\frac{f}{\bar{\rho}} - f_{\bar{\rho}}\right)\overset{0}{\underset{1}{\mathbf{Y}}}{}_1{}^0 + \frac{1}{\sqrt{2}}\left(n\frac{f}{\bar{\rho}} + f_{\bar{\rho}}\right)\overset{0}{\underset{-1}{\mathbf{Y}}}{}_1{}^0 - f_{\bar{z}}\overset{0}{\underset{0}{\mathbf{Y}}}{}_1{}^0\right]\otimes\overset{m}{\underset{n}{\mathbf{Y}}}{}_{l_0 l_1\cdots l_k}\,.$$

Application example: Finsler metric

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$$p = \frac{1}{2}\nabla^v L = -\frac{1}{2} L_{\bar{z}} \overset{0}{\underset{0}{\mathbf{Y}}}{}_1{}^0 - \frac{1}{2\sqrt{2}} L_{\bar{\rho}} \left(\overset{0}{\underset{1}{\mathbf{Y}}}{}_1{}^0 - \overset{0}{\underset{-1}{\mathbf{Y}}}{}_1{}^0 \right) .$$

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- Second derivative:

$$\begin{aligned} g &= \frac{1}{2}\nabla^v\nabla^v L \\ &= -\frac{1}{2\sqrt{3}}\left(\frac{L_{\bar{\rho}}}{\bar{\rho}} + L_{\bar{\rho}\bar{\rho}} + L_{\bar{z}\bar{z}}\right)\underset{0}{\overset{0}{\mathbf{Y}}}{}_0{}^{1\,0} - \frac{1}{2\sqrt{6}}\left(\frac{L_{\bar{\rho}}}{\bar{\rho}} + L_{\bar{\rho}\bar{\rho}} - 2L_{\bar{z}\bar{z}}\right)\underset{0}{\overset{0}{\mathbf{Y}}}{}_2{}^{1\,0} \\ &\quad + \frac{1}{2}L_{\bar{\rho}\bar{z}}\left(\underset{1}{\overset{0}{\mathbf{Y}}}{}_2{}^{1\,0} - \underset{-1}{\overset{0}{\mathbf{Y}}}{}_2{}^{1\,0}\right) + \frac{1}{4}\left(L_{\bar{\rho}\bar{\rho}} - \frac{L_{\bar{\rho}}}{\bar{\rho}}\right)\left(\underset{2}{\overset{0}{\mathbf{Y}}}{}_2{}^{1\,0} + \underset{-2}{\overset{0}{\mathbf{Y}}}{}_2{}^{1\,0}\right). \end{aligned}$$

Application example: Finsler metric

- Inverse Finsler metric:

$$g^{-1} = \frac{2}{\sqrt{3}} \frac{L_{\bar{\rho}}\left(L_{\bar{\rho}\bar{\rho}} + L_{\bar{z}\bar{z}}\right) - \bar{\rho}\left(L_{\bar{\rho}\bar{z}}^2 - L_{\bar{\rho}\bar{\rho}}L_{\bar{z}\bar{z}}\right)}{L_{\bar{\rho}}\left(L_{\bar{\rho}\bar{z}}^2 - L_{\bar{\rho}\bar{\rho}}L_{\bar{z}\bar{z}}\right)} \overset{0}{\underset{0}{\mathbf{Y}}}{}^{0}{}_{1\,0}$$
$$- \frac{\sqrt{2}}{\sqrt{3}} \frac{L_{\bar{\rho}}\left(2L_{\bar{\rho}\bar{\rho}} - L_{\bar{z}\bar{z}}\right) + \bar{\rho}\left(L_{\bar{\rho}\bar{z}}^2 - L_{\bar{\rho}\bar{\rho}}L_{\bar{z}\bar{z}}\right)}{L_{\bar{\rho}}\left(L_{\bar{\rho}\bar{z}}^2 - L_{\bar{\rho}\bar{\rho}}L_{\bar{z}\bar{z}}\right)} \overset{0}{\underset{0}{\mathbf{Y}}}{}^{2}{}_{1\,0}$$
$$+ \frac{2L_{\bar{\rho}\bar{z}}}{L_{\bar{\rho}\bar{z}}^2 - L_{\bar{\rho}\bar{\rho}}L_{\bar{z}\bar{z}}} \left(\overset{0}{\underset{1}{\mathbf{Y}}}{}^{2}{}_{1\,0} - \overset{0}{\underset{-1}{\mathbf{Y}}}{}^{2}{}_{1\,0}\right)$$
$$- \frac{L_{\bar{\rho}}L_{\bar{z}\bar{z}} + \bar{\rho}\left(L_{\bar{\rho}\bar{z}}^2 - L_{\bar{\rho}\bar{\rho}}L_{\bar{z}\bar{z}}\right)}{L_{\bar{\rho}}\left(L_{\bar{\rho}\bar{z}}^2 - L_{\bar{\rho}\bar{\rho}}L_{\bar{z}\bar{z}}\right)} \left(\overset{0}{\underset{2}{\mathbf{Y}}}{}^{2}{}_{1\,0} + \overset{0}{\underset{-2}{\mathbf{Y}}}{}^{2}{}_{1\,0}\right) .$$

Outline

Adapted coordinates on $T\mathbb{R}^4$

- Coordinates on TM for $M = \mathbb{R}^4$: $r, w, \alpha, \beta, \theta^+, \theta^-, \phi^+, \phi^-$

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- Coordinates on TM for $M = \mathbb{R}^4$: $r, w, \alpha, \beta, \theta^+, \theta^-, \phi^+, \phi^-$
- Relation to Cartesian induced coordinates:

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} = M \cdot \begin{pmatrix} 0 \\ 0 \\ r\sin\frac{\beta}{2} \\ r\cos\frac{\beta}{2} \end{pmatrix}, \quad \begin{pmatrix} y^1 \\ y^2 \\ y^3 \\ y^4 \end{pmatrix} = M \cdot \begin{pmatrix} 0 \\ 0 \\ w\cos\frac{\alpha+\beta}{2} \\ -w\sin\frac{\alpha+\beta}{2} \end{pmatrix},$$

$$M = \begin{pmatrix} \sin\frac{\phi^-}{2} & \cos\frac{\phi^-}{2} & 0 & 0 \\ -\cos\frac{\phi^-}{2} & \sin\frac{\phi^-}{2} & 0 & 0 \\ 0 & 0 & \sin\frac{\phi^-}{2} & \cos\frac{\phi^-}{2} \\ 0 & 0 & -\cos\frac{\phi^-}{2} & \sin\frac{\phi^-}{2} \end{pmatrix} \cdot \begin{pmatrix} \cos\frac{\phi^+}{2} & -\sin\frac{\phi^+}{2} & 0 & 0 \\ \sin\frac{\phi^+}{2} & \cos\frac{\phi^+}{2} & 0 & 0 \\ 0 & 0 & \cos\frac{\phi^+}{2} & \sin\frac{\phi^+}{2} \\ 0 & 0 & -\sin\frac{\phi^+}{2} & \cos\frac{\phi^+}{2} \end{pmatrix}$$

$$\cdot \begin{pmatrix} \sin\frac{\theta^-}{2} & 0 & \cos\frac{\theta^-}{2} & 0 \\ 0 & \sin\frac{\theta^-}{2} & 0 & -\cos\frac{\theta^-}{2} \\ -\cos\frac{\theta^-}{2} & 0 & \sin\frac{\theta^-}{2} & 0 \\ 0 & \cos\frac{\theta^-}{2} & 0 & \sin\frac{\theta^-}{2} \end{pmatrix} \cdot \begin{pmatrix} \cos\frac{\theta^+}{2} & 0 & \sin\frac{\theta^+}{2} & 0 \\ 0 & \cos\frac{\theta^+}{2} & 0 & \sin\frac{\theta^+}{2} \\ -\sin\frac{\theta^+}{2} & 0 & \cos\frac{\theta^+}{2} & 0 \\ 0 & -\sin\frac{\theta^+}{2} & 0 & \cos\frac{\theta^+}{2} \end{pmatrix}.$$

Generating vector fields of SO(4)

- Generating vector fields of $SO(4) \cong SO(3) \times SO(3)/\mathbb{Z}_2$ on $M = \mathbb{R}^3$:

$$\mathbf{r}_1 = -x^2\partial_3 + x^3\partial_2\,, \qquad \mathbf{r}_2 = -x^3\partial_1 + x^1\partial_3\,, \qquad \mathbf{r}_3 = -x^1\partial_2 + x^2\partial_1\,,$$
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- Canonical lifts $\mathbf{J}^{\pm}_k$ of vector fields $\mathbf{j}^{\pm}_k = \mathbf{r}_k \pm \mathbf{t}_k$ in cosmological coordinates:

$$\mathbf{J}^{\pm}_1 = \sin\phi^{\pm}\partial_{\theta^{\pm}} + \frac{\cos\phi^{\pm}}{\tan\theta^{\pm}}\partial_{\phi^{\pm}} - \frac{\cos\phi^{\pm}}{\sin\theta^{\pm}}\partial_{\beta}\,,$$
$$\mathbf{J}^{\pm}_2 = -\cos\phi^{\pm}\partial_{\theta^{\pm}} + \frac{\sin\phi^{\pm}}{\tan\theta^{\pm}}\partial_{\phi^{\pm}} - \frac{\sin\phi^{\pm}}{\sin\theta^{\pm}}\partial_{\beta}\,,$$
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$$\mathbf{J}_3^{\pm} = -\partial_{\phi^{\pm}}\,.$$

- Operators $\mathcal{J}_k^{\pm} \in \mathfrak{A}(\mathcal{T}(\pi))$:

$$\mathcal{J}_k^{\pm} Y = i\mathcal{L}_{\mathbf{J}_k^{\pm}} Y\,.$$

Symmetry algebra for SO(4)

- Auxiliary vector field $\mathbf{B} \in \mathfrak{X}(TM)$ and operator $\mathcal{B} \in \mathfrak{A}(\mathcal{C}^\infty(TM))$:

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⇒ $(\mathcal{J}^\pm)^2, \mathcal{J}_z^\pm, \mathcal{B}$ mutually commute.

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$$\mathcal{B}F = -i\partial_\beta F \Rightarrow B(\beta) = e^{in\beta}, \quad n \in \mathbb{Z}\,.$$

- Zenith part $(\mathcal{J}^\pm)^2 F = l^\pm(l^\pm + 1)F$:

$$(\Theta^\pm)''(\theta^\pm) + \frac{(\Theta^\pm)'(\theta^\pm)}{\tan\theta^\pm} + \left(\frac{2m^\pm n\cos\theta^\pm - (m^\pm)^2 - n^2}{\sin^2\theta^\pm} + l^\pm(l^\pm + 1)\right)\Theta^\pm(\theta^\pm) = 0\,.$$

Definition of scalar cosmological harmonics on *TM*

- Definition of cosmological scalar harmonics:

$$
\begin{aligned}
\mathcal{Z}_{l^+,l^-,m^+,m^-,n}(\theta^+,\theta^-,\phi^+,\phi^-,\beta) = N_{l^+,l^-,m^+,m^-,n} e^{im^+\phi^+} e^{im^-\phi^-} e^{in\beta} \\
\cdot {}_2F_1\left(\max(m^+,n) - l, \max(m^+,n) + l + 1; |m^+ - n| + 1; \sin^2\frac{\theta^+}{2}\right) \\
\cdot {}_2F_1\left(\max(m^-,n) - l, \max(m^-,n) + l + 1; |m^- - n| + 1; \sin^2\frac{\theta^-}{2}\right) \\
\cdot \cos^{m^+ + n}\frac{\theta^+}{2} \sin^{|m^+ - n|}\frac{\theta^+}{2} \cos^{m^- + n}\frac{\theta^-}{2} \sin^{|m^- - n|}\frac{\theta^-}{2}
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- Conditions on parameters:
  - l^+, l^-, m^+, m^-, n either all integers or all half-integers.
  - $|n| \leq \min(l^+, l^-)$ and $|m^\pm| \leq l^\pm$.

Properties of scalar cosmological harmonics on *TM*

- Eigenvalue relations:

$$(\mathcal{J}^{\pm})^2\mathcal{Z} = l^{\pm}(l^{\pm}+1)\mathcal{Z}\,, \quad \mathcal{J}_z^{\pm}\mathcal{Z} = m^{\pm}\mathcal{Z}\,, \quad \mathcal{B}\mathcal{Z} = n\mathcal{Z}$$

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$$\mathcal{J}^{+}_{\pm}\mathcal{Z}_{l^+,l^-,m^+,m^-,n} = \sqrt{(l^+ \mp m^+)(l^+ \pm m^+ + 1)}\mathcal{Z}_{l^+,l^-,m^+\pm 1,m^-,n}\,,$$
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- Orthogonality and normalization:

$$\int_0^{4\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\pi}\mathcal{Z}_{l^+,l^-,m^+,m^-,n}(\theta^+,\theta^-,\phi^+,\phi^-,\beta)\overline{\mathcal{Z}_{\tilde{l}^+,\tilde{l}^-,\tilde{m}^+,\tilde{m}^-,\tilde{n}}}(\theta^+,\theta^-,\phi^+,\phi^-,\beta)$$
$$\sin\theta^+\,\sin\theta^-\,d\theta^+\,d\theta^-\,d\phi^+\,d\phi^-\,d\beta = 32\pi^3\delta_{l^+\tilde{l}^+}\delta_{l^-\tilde{l}^-}\delta_{m^+\tilde{m}^+}\delta_{m^-\tilde{m}^-}\delta_{n\tilde{n}}\,.$$

Cosmological d-tensor basis

- Introduce basis of $\mathcal{T}^0_1$:

$$\mathbf{e}_{\frac{1}{2},\frac{1}{2}}\,, \quad \mathbf{e}_{-\frac{1}{2},\frac{1}{2}}\,, \quad \mathbf{e}_{\frac{1}{2},-\frac{1}{2}}\,, \quad \mathbf{e}_{-\frac{1}{2},-\frac{1}{2}}$$

- Operator relations:

$$(\mathcal{J}^{\pm})^2\mathbf{e}_{m^+,m^-} = \frac{3}{4}\mathbf{e}_{m^+,m^-}\,, \quad \mathcal{J}^{\pm}_z\mathbf{e}_{m^+,m^-} = m^{\pm}\mathbf{e}_{m^+,m^-}\,,$$

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Cosmological d-tensor basis

- Introduce basis of $\mathcal{T}^1_0$:

$$\mathbf{e}^{\frac{1}{2},\frac{1}{2}}\,, \quad \mathbf{e}^{-\frac{1}{2},\frac{1}{2}}\,, \quad \mathbf{e}^{\frac{1}{2},-\frac{1}{2}}\,, \quad \mathbf{e}^{-\frac{1}{2},-\frac{1}{2}}$$

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- Analogue construction for dual basis.

Recursive construction of cosmological d-tensors

- Zeroth order tensors in $\mathcal{T}_0^0(\pi)$:

$$\overset{m^+,m^-}{\underset{n}{\mathbf{Z}}}{}_{\{l^+,l^-\}} = \mathcal{Z}_{l^+,l^-,m^+,m^-,n}$$

- Recursive definition in $\mathcal{T}_k^0(\pi)$:

$$\overset{m^+,m^-}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}} = (-1)^{l_k^+ + l_k^- - m^+ - m^-}\sqrt{2l_k^+ + 1}\sqrt{2l_k^- + 1}$$

$$\cdot \sum_{m^{+\prime},m^{-\prime},\mu^+,\mu^-} \begin{pmatrix} l_k^+ & l_{k-1}^+, & \frac{1}{2} \\ m^+ & -m^{+\prime} & -\mu^+ \end{pmatrix} \begin{pmatrix} l_k^- & l_{k-1}^-, & \frac{1}{2} \\ m^- & -m^{-\prime} & -\mu^- \end{pmatrix}$$

$$\cdot \overset{m^{+\prime},m^{-\prime}}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_{k-1}^+,l_{k-1}^-\}} \otimes \mathbf{e}_{\mu^+,\mu^-}\,,$$

Cosmological operator relations

- Eigenvalue relations:

$$(\mathcal{J}^{\pm})^2 \overset{m^+,m^-}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}} = l_k^{\pm}(l_k^{\pm}+1) \overset{m^+,m^-}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}}\,,$$

$$\mathcal{J}_z^{\pm} \overset{m^+,m^-}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}} = m^{\pm} \overset{m^+,m^-}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}}\,.$$

- Ladder operators:

$$\mathcal{J}_{\pm}^{+} \overset{m^+,m^-}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}} = \sqrt{(l_k^+ \mp m^+)(l_k^+ \pm m^+ + 1)} \overset{m^+\pm 1,m^-}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}}\,,$$

$$\mathcal{J}_{\pm}^{-} \overset{m^+,m^-}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}} = \sqrt{(l_k^- \mp m^-)(l_k^- \pm m^- + 1)} \overset{m^+,m^-\pm 1}{\underset{n}{\mathbf{Z}}}{}_{\{l_0^+,l_0^-\}\{l_1^+,l_1^-\}\cdots\{l_k^+,l_k^-\}}\,.$$

Outline

Summary

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  - Defining objects of Finsler geometry.
  - Sections of tensor bundle over pullback bundle $TM \times_M TM$.
  - Geometric interpretation via double tangent bundle TTM.

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- Harmonic d-tensors:
  - D-tensor representations of $\mathrm{SO}(3), \mathrm{SO}(4), \ldots$
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  - Simplify calculation of d-tensors in Finsler gravity.

Outlook

- Construction of harmonic d-tensors:
  - Construct further helpful formulas for harmonic d-tensors.
  - Generalize construction to other symmetry groups.
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