Finsler spacetimes, observer space and Cartan geometry

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Investing
in your future

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Outline

Motivation

- Pseudo-Riemannian geometry of spacetime has multiple roles:
  - Causality
  - Observers, observables and observations
  - Gravity

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  - Loop quantum gravity
  - Spin foam networks
  - Causal dynamical triangulations

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- How to serve the same roles as pseudo-Riemannian geometry?

Why Finsler geometry of spacetimes?

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? Possible explanations of yet unexplained phenomena:

  ? Galaxy rotation curves

  ? Accelerating expansion of the universe

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- Solution:
  - Consider space O of all allowed observers.
  - Describe experiments on observer space instead of spacetime.

  ⇒ Observer dependence of physical quantities follows naturally.

  ⇒ No preferred observers.
  - Geometry of observer space modeled by Cartan geometry.

Geometrical structures

Metric geometry

Manifold M

Lorentzian metric g

Orientation

Time orientation

Finsler geometry

Tangent bundle TM

Geometry function $L : TM \to \mathbb{R}$

Finsler function $F : TM \to \mathbb{R}$

Finsler metric $g^F(x, y)$

Cartan non-linear connection $N^a{}_b$

Cartan linear connection ∇

Cartan geometry

Lie group $G = \mathrm{ISO}_0(3, 1)$

Closed subgroup $K = \mathrm{SO}(3)$

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From metric to Finsler

Coordinates (x^a) on M

Coordinates (x^a, y^a) on TM

Define $L(x, y) = g_{ab}(x)y^a y^b$

From Finsler to Cartan

Space O of observer 4-velocities

Space P of observer frames

Define A from connection ∇

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Pseudo-Riemannian spacetime geometry

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$\Rightarrow$ Arc length for curves $t \mapsto \gamma(t) \in M$ defined by the metric:

$$\tau_2 - \tau_1 = \int_{t_1}^{t_2} \sqrt{|g_{ab}(\gamma(t))\dot{\gamma}^a(t)\dot{\gamma}^b(t)|}dt\,.$$

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- Observables are components of tensor fields.
- Tensor components must be expressed in suitable basis.

⇒ Metric provides notion of orthonormal frames:

$$g_{ab}f_i^a f_j^b = \eta_{ij}\,.$$

⇒ Orthogonal frame bundle $\tilde{\pi} : P \to M$ is principal $\mathrm{SO}(1,3)$-bundle.

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Basics of Finsler spacetimes

- Finsler geometry defined by length functional for curve γ:

$$\tau_2 - \tau_1 = \int_{t_1}^{t_2} F(\gamma(t), \dot{\gamma}(t))dt$$

- Finsler function $F : TM \to \mathbb{R}^+$.
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- Introduce manifold-induced coordinates (x^a, y^a) on TM:
  - Coordinates x^a on M.
  - Define coordinates y^a for $y^a \frac{\partial}{\partial x^a} \in T_x M$.
  - Tangent bundle TTM spanned by $\left\{\partial_a = \frac{\partial}{\partial x^a}, \bar{\partial}_a = \frac{\partial}{\partial y^a}\right\}$.

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  - Tangent bundle TTM spanned by $\left\{\partial_a = \frac{\partial}{\partial x^a}, \bar{\partial}_a = \frac{\partial}{\partial y^a}\right\}$.
- n-homogeneous functions on TM: $f(x, \lambda y) = \lambda^n f(x, y)$.
  - n-homogeneous smooth geometry function $L : TM \to \mathbb{R}$.
  - $\Rightarrow$ 1-homogeneous Finsler function $F = |L|^{\frac{1}{n}}$.

$\Rightarrow$ Finsler metric with Lorentz signature:

$$g^F_{ab}(x, y) = \frac{1}{2}\bar{\partial}_a\bar{\partial}_b F^2(x, y)\,.$$

Connections on Finsler spacetimes

- Cartan non-linear connection:

$$N^a{}_b = \frac{1}{4}\bar{\partial}_b \left[g^{F\,ac}(y^d \partial_d \bar{\partial}_c F^2 - \partial_c F^2) \right] .$$

$\Rightarrow$ Berwald basis of TTM:

$$\{\delta_a = \partial_a - N^b{}_a \bar{\partial}_b, \bar{\partial}_a\} .$$

$\Rightarrow$ Dual Berwald basis of T^*TM:

$$\{dx^a, \delta y^a = dy^a + N^a{}_b dx^b\} .$$

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- Cartan linear connection:

$$\nabla_{\delta_a}\delta_b = F^c{}_{ab}\delta_c\,,\ \nabla_{\delta_a}\bar{\partial}_b = F^c{}_{ab}\bar{\partial}_c\,,\ \nabla_{\bar{\partial}_a}\delta_b = C^c{}_{ab}\delta_c\,,\ \nabla_{\bar{\partial}_a}\bar{\partial}_b = C^c{}_{ab}\bar{\partial}_c\,,$$

$$F^c{}_{ab} = \frac{1}{2}g^{F\,cd}(\delta_a g^F_{bd} + \delta_b g^F_{ad} - \delta_d g^F_{ab})\,,$$

$$C^c{}_{ab} = \frac{1}{2}g^{F\,cd}(\bar{\partial}_a g^F_{bd} + \bar{\partial}_b g^F_{ad} - \bar{\partial}_d g^F_{ab})\,.$$

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Introduction to Cartan geometry

- Cartan geometry modeled on Klein geometry G/H:
  - G and H are Lie groups.
  - $H \subset G$ is a closed subgroup of G.
  - $\Rightarrow$ Coset space G/H is a homogeneous space acted upon by G.

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  1. For each $p \in P$, $A_p = A|_{T_pP} : T_pP \to \mathfrak{g}$ is a linear isomorphism.
  2. A is H-equivariant: $(R_h)^*A = \mathrm{Ad}(h^{-1}) \circ A$ for all $h \in H$.
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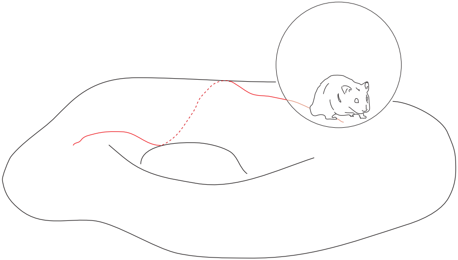
  - Dimension of the total space: $\dim P = \dim G$.
  - $\Rightarrow$ Dimension of the base manifold: $\dim M = \dim G - \dim H = \dim G/H$.

Toy model for Cartan geometry: The hamster ball



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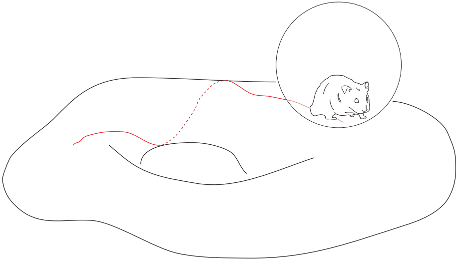

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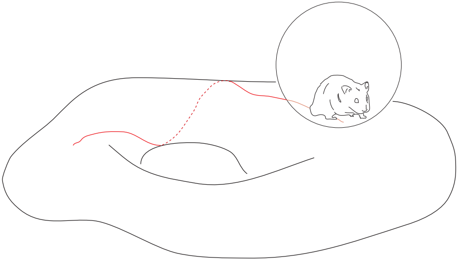

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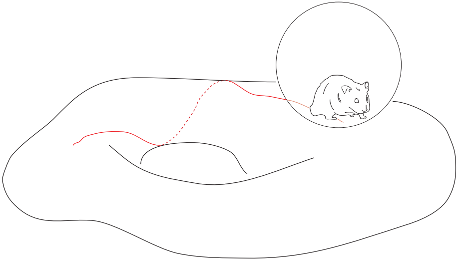

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⇒ Surface M "traced" by $S^2 \cong \mathrm{SO}(3)/\mathrm{SO}(2) = G/H$.

⇒ Geometry of M fully described by Hamster ball motion.

Klein geometries for spacetime and observer space

- Consider groups $G \supset H \supset K$:
  - "Inhomogeneous group" - symmetry group of homogeneous space:

$$G_\Lambda = \begin{cases} \mathrm{SO}_0(4,1) & \Lambda = 1 \\ \mathrm{ISO}_0(3,1) & \Lambda = 0 \\ \mathrm{SO}_0(3,2) & \Lambda = -1 \end{cases} .$$

  - "Homogeneous group" $H = \mathrm{SO}_0(3,1)$ - stabilizer of a point.
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- Induced split of Lie algebra $\mathfrak{g}$ via Ad:
  - Irreducible representations of $H \subset G$ on $\mathfrak{g}$:

$$\mathfrak{g} = \underbrace{\mathfrak{h}}_{\text{Lorentz transformations}} \oplus \underbrace{\mathfrak{z}}_{\text{translations}} .$$

  - Irreducible representations of $K \subset G$ on $\mathfrak{g}$:

$$\mathfrak{h} = \underbrace{\mathfrak{k}}_{\text{rotations}} \oplus \underbrace{\mathfrak{y}}_{\text{boosts}}, \quad \mathfrak{z} = \underbrace{\vec{\mathfrak{z}}}_{\text{spatial translations}} \oplus \underbrace{\mathfrak{z}^0}_{\text{temporal translations}} .$$

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- *Fundamental vector fields* $\underline{A} : \mathfrak{g} \to \Gamma(TP)$ as "inverse" of A.

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- *Cartan connection* $A = \omega + e \in \Omega^1(P, \mathfrak{g})$.
- *Fundamental vector fields* $\underline{A} : \mathfrak{g} \to \Gamma(TP)$ as “inverse” of A.

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$$\begin{array}{ccccccccc} T_pP & = & R_pP & + & B_pP & + & \vec{H}_pP & + & H^0_pP \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathfrak{g} & = & \mathfrak{k} & + & \mathfrak{y} & + & \vec{\mathfrak{z}} & + & \mathfrak{z}^0 \end{array}$$

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Outline

From pseudo-Riemannian to Finsler

- Metric-induced 2-homogeneous geometry function:

$$L(x,y) = g_{ab}(x)y^a y^b\,.$$

$\Rightarrow$ Finsler function $F(x,y) = \sqrt{|L(x,y)|}$.

$\Rightarrow$ Finsler metric

$$g^F(x,y) = \begin{cases} -g(x,y) & \text{for } y \text{ timelike,} \\ g(x,y) & \text{for } y \text{ spacelike.} \end{cases}$$

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⇒ Cartan non-linear connection:

$$N^a{}_b = \Gamma^a{}_{bc} y^c \,.$$

⇒ Cartan linear connection:

$$F^a{}_{bc} = \Gamma^a{}_{bc} \,, \quad C^a{}_{bc} = 0 \,.$$

From Finsler to Cartan

- Need to construct $A \in \Omega^1(P, \mathfrak{g})$.
- Recall that

$$\begin{array}{ccccc} \mathfrak{g} & = & \mathfrak{h} & \oplus & \mathfrak{z} \\ A & = & \omega & + & e \end{array}$$

- Definition of e: Use the *solder form*:

$$e^i = f^{-1}{}^i{}_a dx^a \,.$$

- Definition of ω: Use the *Cartan linear connection*:

$$\omega^i{}_j = f^{-1}{}^i{}_a \left[df^a_j + f^b_j \left(dx^c F^a{}_{bc} + (dx^d N^c{}_d + df^c_0) C^a{}_{bc}\right)\right] \,.$$

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- Let $a = z^i \mathcal{Z}_i + \frac{1}{2} h^i{}_j \mathcal{H}_i{}^j \in \mathfrak{g}$.
- Fundamental vector fields:

$$\underline{A}(a) = z^i f^a_i \left(\partial_a - f^b_j F^c{}_{ab} \bar{\partial}^j_c \right) + \left(h^i{}_j f^a_i - h^i{}_0 f^b_i f^c_j C^a{}_{bc} \right) \bar{\partial}^j_a \,.$$

Outline

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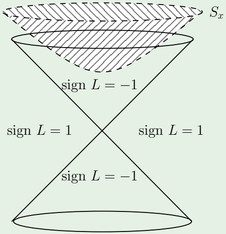
Causal structure

Metric geometry

Geometry function:

$$L = g_{ab}y^a y^b$$

y^a timelike for $L < 0$.



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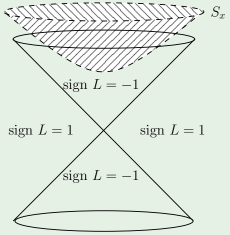


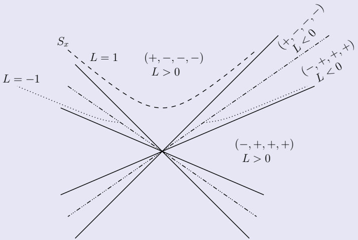
Finsler geometry

Fundamental geometry function L

Hessian:

$$g^L_{ab}(x, y) = \frac{1}{2}\bar{\partial}_a\bar{\partial}_b L(x, y)$$

Use sign of L and signature of g^L.



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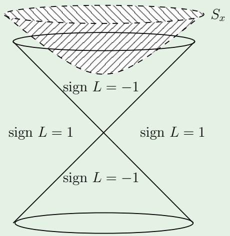

Cartan geometry

Observer space:

$$O = \bigcup_{x \in M} S_x$$

O contains only future unit timelike vectors.

Finsler geometry

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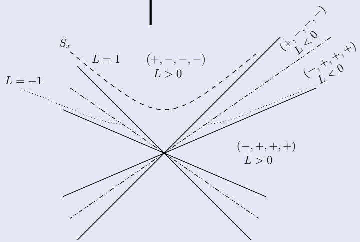

Causality of Finsler spacetimes

- "Unit timelike condition" required for Finsler spacetimes:
  For all $x \in M$ the set

$$\Omega_x = \left\{ y \in T_xM \,\middle|\, |L(x,y)| = 1, \operatorname{sig} \bar{\partial}_a\bar{\partial}_b L(x,y) = (\epsilon, -\epsilon, -\epsilon, -\epsilon) \right\}$$

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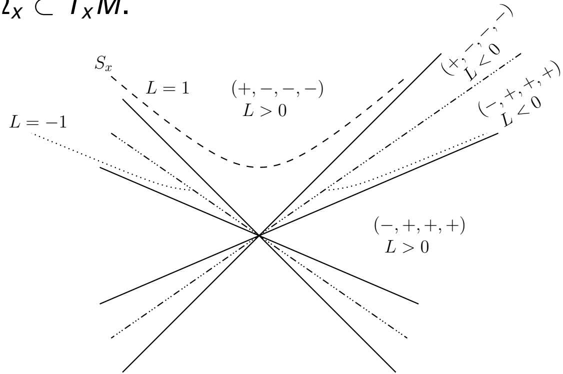
$\Rightarrow$ S_x contains physical observers.

$\Rightarrow$ $\mathbb{R}^+ S_x$ is convex cone.



The observer frame bundle

- Observer space of a Finsler spacetime:
  - Consider all allowed observer tangent vectors:

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  - $\Rightarrow$ Complete $y = f_0$ to a frame f_i with $g^F_{ab}(x, y) f_i^a f_j^b = -\eta_{ij}$.
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  - Natural projection $\pi : P \to O$ discards spatial frame components.
- Group action on the frame bundle:
  - SO(3) acts on spatial frame components by rotations.
  - Action is free and transitive on fibers of $\pi : P \to O$.
  - $\Rightarrow$ $\pi : P \to O$ is principal K-bundle.

Outline

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Timelike curve γ:

$$\begin{array}{cccc} \gamma & : & \mathbb{R} & \rightarrow & M \\ & & \tau & \mapsto & \gamma(\tau) \end{array}$$

$$g_{ab}\dot{\gamma}^a\dot{\gamma}^b = -1$$

Orthonormal frame f:

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$$\tilde{e}^i\dot{\Gamma}(\tau) = \delta_0^i$$

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$$f \in \pi^{-1}(\Gamma(\tau)) \subset P$$

Inertial observers

Metric geometry

Minimize arc length integral:

$$\int_{t_1}^{t_2} \sqrt{|g_{ab}(\gamma(t))\dot{\gamma}^a(t)\dot{\gamma}^b(t)|}dt$$

Geodesic equation:

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- Observer trajectories and canonical lifts:
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  - Lift γ to a curve $\Gamma = (\gamma, \dot{\gamma})$ in TM.
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$\Rightarrow$ Γ is integral curve of geodesic spray:

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Observers on Cartan observer space

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- Boost component of the tangent vector:
  - Measures acceleration in observer's frame.
  - Inertial observers are non-accelerating: $b^\alpha\dot{\Gamma} = 0$.
  - $\Rightarrow$ Inertial observers follow integral curves of time translation: $\dot{\Gamma} = \underline{e}_0$.

Observers from Finsler to Cartan

- Generating vector field on Finsler spacetimes:
  - ◦ Geodesic spray **S** preserves Finsler function: $\mathbf{S}F = 0$.
  - ⇒ Geodesic spray **S** is tangent to observer space O (level set).
  - ⇝ Define Reeb vector field $\mathbf{r} = \mathbf{S}|_O$.
  - ◦ Coordinate expression: $\mathbf{r} = y^a(\partial_a - N^b{}_a\bar{\partial}_b)$.

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⇒ Cartan trajectories correspond to Finslerian parallel transport.

Outline

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$$S_{\mathrm{EH}} = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\,R$$

Gravity

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Using linear connection:

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Cartan geometry

Using horizontal vector fields:

$$S_{\text{H}} = \int_O \tilde{b}^\alpha([\underline{\tilde{e}}_\alpha, \underline{\tilde{e}}_0])\,\text{Vol}_O$$

Using Cartan curvature:

$$S_{\text{C}} = \int_O \kappa_{\mathfrak{h}}(\tilde{F}_{\mathfrak{h}} \wedge \tilde{F}_{\mathfrak{h}}) \wedge \text{Vol}_S$$

Gravity

Metric geometry

Einstein-Hilbert action:

$$S_{\text{EH}} = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\,R$$

Finsler geometry

Using non-linear connection:

$$S_{\text{N}} = \frac{1}{\kappa}\int_\Sigma \mathrm{Vol}_{\tilde{G}}\, R^a{}_{ab}y^b$$

Using linear connection:

$$S_{\text{L}} = \frac{1}{\kappa}\int_\Sigma \mathrm{Vol}_{\tilde{G}}\, g^{F\,ab}R^c{}_{acb}$$

$\Leftrightarrow$

$\Leftrightarrow$

Cartan geometry

Using horizontal vector fields:

$$S_{\text{H}} = \int_O \tilde{b}^\alpha([\underline{\tilde{e}}_\alpha, \underline{\tilde{e}}_0])\,\mathrm{Vol}_O$$

Using Cartan curvature:

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- MacDowell-Mansouri gravity on observer space: [S. Gielen, D. Wise '12]

$$S_G = \int_O \epsilon_{\alpha\beta\gamma}\,\mathrm{tr}_{\mathfrak{h}}(F_{\mathfrak{h}} \wedge \star F_{\mathfrak{h}}) \wedge b^\alpha \wedge b^\beta \wedge b^\gamma$$

  - Hodge operator $\star$ on $\mathfrak{h}$.
  - Non-degenerate H-invariant inner product $\mathrm{tr}_{\mathfrak{h}}$ on $\mathfrak{h}$.
  - Boost part $b \in \Omega_1(P, \mathfrak{y})$ of the Cartan connection.

Gravity from Cartan to Finsler

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- Translate terms into Finsler language (with $R = d\omega + \frac{1}{2}[\omega, \omega]$):
  - Curvature scalar:

$$[e, e] \wedge \star R \rightsquigarrow g^{F\,ab} R^c{}_{acb}\, dV\,.$$

  - Cosmological constant:

$$[e, e] \wedge \star [e, e] \rightsquigarrow dV\,.$$

  - Gauss-Bonnet term:

$$R \wedge \star R \rightsquigarrow \epsilon^{abcd} \epsilon^{efgh} R_{abef} R_{cdgh}\, dV\,.$$

$\Rightarrow$ Gravity theory on Finsler spacetime.

Gravity from Finsler to Cartan

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$$R^a{}_{ab} y^b = b^\alpha[\underline{A}(\mathcal{Z}_\alpha), \underline{A}(\mathcal{Z}_0)] \,.$$

$\Rightarrow$ Gravity theory on observer space.

Outline

Summary

- Finsler spacetimes
  - Generalization of pseudo-Riemannian spacetimes.
  - Geometry defined by function L on TM.
  - Lengths measured by Finsler function $F = |L|^{\frac{1}{n}}$.
  - Metric generalized by Finsler metric g^F_{ab}.

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- Different geometries provide compatible definitions of:
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  - Observers
  - Observables
  - Gravity

Caveats and outlook

- Observer space not most suitable for Lagrange theory:
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  - Critical sections: solutions of Euler-Lagrange equations.
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References

M. Hohmann,
"Extensions of Lorentzian spacetime geometry: From Finsler to Cartan and vice versa,"
Phys. Rev. D **87** (2013) no.12, 124034 [arXiv:1304.5430 [gr-qc]].

M. Hohmann,
"Observer dependent geometries,"
in: "Mathematical Structures of the Universe", Copernicus Center Press, Krakow, 2014 [arXiv:1403.4005 [math-ph]].

M. Hohmann,
"Spacetime and observer space symmetries in the language of Cartan geometry,"
J. Math. Phys. **57** (2016) no.8, 082502 [arXiv:1505.07809 [math-ph]].