Cartan geometric structures in gravity and their symmetries

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Motivation: problems in gravity

- So far unexplained cosmological observations:
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- Idea here: modification of the geometric structure of spacetime!
  - Study classical gravity theories based on modified geometry.
  - Consider geometries as effective models of quantum gravity.
  - Derive observable effects to test modified geometry.

Outline

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Cartan geometry

- Ingredients of a Cartan geometry:
  - A Lie group G with a closed subgroup $H \subset G$.
  - A principal H-bundle $\pi : P \to M$.
  - A 1-form $A \in \Omega^1(P, \mathfrak{g})$ on P with values in $\mathfrak{g}$.

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- Conditions on the Cartan connection A:
  - For each $p \in P$, $A_p : T_pP \to \mathfrak{g}$ is a linear isomorphism.
  - A is right-equivariant: $(R_h)^*A = \mathrm{Ad}(h^{-1}) \circ A \quad \forall h \in H$.
  - A restricts to the Maurer-Cartan form of H on $\ker \pi_*$.

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- Fundamental vector fields:
  - ⇒ A has an “inverse” $\underline{A} : \mathfrak{g} \to \Gamma(TP)$.
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  - M “locally looks like” homogeneous space G/H.
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- Curvature of the Cartan connection:
  - Curvature defined by $F = \mathrm{d}A + \frac{1}{2}[A \wedge A] \in \Omega^2_H(P, \mathfrak{g})$.
  - Curvature measures deviation between M and G/H.

First-order reductive models

- First-order Cartan geometry:
  - Adjoint representations of $H \subset G$ on $\mathfrak{g}$ and $\mathfrak{h}$.
  - Quotient representation of H on $\mathfrak{g}/\mathfrak{h}$ is faithful.

$\Rightarrow$ "Fake tangent bundle" $\mathcal{T} = \mathcal{P} \times_H \mathfrak{g}/\mathfrak{h}$.

$\Rightarrow$ $\mathcal{P}$ is "fake frame bundle": "admissible" frames $\mathfrak{g}/\mathfrak{h} \to \mathcal{T}_x$ for $x \in M$.

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- Reductive Cartan geometry:
  - Direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ of vector spaces.
  - $\mathfrak{h}$ and $\mathfrak{z}$ are subrepresentations of $\mathrm{Ad}\,H$ on $\mathfrak{g}$.

$\Rightarrow$ Cartan connection $A = \omega + e$ splits: $\omega \in \Omega^1(\mathcal{P}, \mathfrak{h})$ and $e \in \Omega^1(\mathcal{P}, \mathfrak{z})$.

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$\Rightarrow$ Cartan geometry $(\tilde{\pi} : P \to M, \tilde{A})$ with $\tilde{A} = \tilde{\omega} + \tilde{e}$.

- $\tilde{e}$: solder form on $P \subset FM$.
- Drop tilde and consider Cartan geometries on $\mathcal{P} \equiv P \subset FM$.

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Cartan geometry of pseudo-Riemannian spacetime

- Choose Lie groups:
  - Let

$$G = \begin{cases} \mathrm{SO}_0(4,1) & \Lambda > 0 \\ \mathrm{ISO}_0(3,1) & \Lambda = 0 \\ \mathrm{SO}_0(3,2) & \Lambda < 0 \end{cases}, \quad H = \mathrm{SO}_0(3,1)\,.$$

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$\Rightarrow$ Spacetime (M, g) can be reconstructed from Cartan geometry.

Curvature decomposition

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⇒ Decomposition of Cartan curvature:

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⇒ Use $A = \omega + e$:

$$F_{\mathfrak{h}} = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega] + \frac{1}{2}[e \wedge e]\,, \quad F_{\mathfrak{z}} = \mathrm{d}e + [\omega \wedge e]\,.$$

MacDowell-Mansouri gravity in Cartan geometry

- MacDowell-Mansouri gravity in terms of Cartan geometry: [D. Wise '06]

$$S_G = \int_M \mathrm{tr}_{\mathfrak{h}}(F_{\mathfrak{h}} \wedge \star F_{\mathfrak{h}})\,.$$

  - Hodge operator $\star$ on $\mathfrak{h}$.
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  - Curvature scalar:

$$[e \wedge e] \wedge \star R \rightsquigarrow g^{ab} R^c{}_{acb} \, \mathrm{d}V \, .$$

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$\Rightarrow$ Gravity theory formulated through Cartan geometry.

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The clock postulate

- Proper time along a curve $\gamma : \mathbb{R} \to M$ in Lorentzian spacetime:

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  - Define coordinates y^a for $y^a \frac{\partial}{\partial x^a} \in T_xM$.
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  - Tangent bundle TTM spanned by $\left\{\partial_a = \frac{\partial}{\partial x^a}, \bar{\partial}_a = \frac{\partial}{\partial y^a}\right\}$.
- Parametrization invariance requires homogeneity:

$$F(x, \lambda y) = \lambda F(x, y) \quad \forall \lambda > 0\,.$$

Definition of Finsler spacetimes

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$$g^F_{ab}(x,y) = \frac{1}{2}\frac{\partial}{\partial y^a}\frac{\partial}{\partial y^b}F^2(x,y)\,.$$

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$\Rightarrow$ Set $\Omega_x \subset T_xM$ of unit timelike vectors at $x \in M$.

- Ω_x contains a closed connected component $S_x \subseteq \Omega_x$.
- Causality: S_x corresponds to physical observers.

Connections on Finsler spacetimes

- Cartan non-linear connection:

$$N^a{}_b = \frac{1}{4}\bar{\partial}_b \left[g^{F\,ac}(y^d \partial_d \bar{\partial}_c F^2 - \partial_c F^2) \right] .$$

$\Rightarrow$ Berwald basis of TTM:

$$\{\delta_a = \partial_a - N^b{}_a \bar{\partial}_b, \bar{\partial}_a\} .$$

$\Rightarrow$ Dual Berwald basis of T^*TM:

$$\{\mathrm{d}x^a, \delta y^a = \mathrm{d}y^a + N^a{}_b \mathrm{d}x^b\} .$$

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$$\nabla_{\delta_a}\delta_b = F^c{}_{ab}\delta_c\,,\ \nabla_{\delta_a}\bar{\partial}_b = F^c{}_{ab}\bar{\partial}_c\,,\ \nabla_{\bar{\partial}_a}\delta_b = C^c{}_{ab}\delta_c\,,\ \nabla_{\bar{\partial}_a}\bar{\partial}_b = C^c{}_{ab}\bar{\partial}_c\,,$$

$$F^c{}_{ab} = \frac{1}{2}g^{F\,cd}(\delta_a g^F_{bd} + \delta_b g^F_{ad} - \delta_d g^F_{ab})\,,$$

$$C^c{}_{ab} = \frac{1}{2}g^{F\,cd}(\bar{\partial}_a g^F_{bd} + \bar{\partial}_b g^F_{ad} - \bar{\partial}_d g^F_{ab})\,.$$

Outline

Observer space

- Recall from the definition of Finsler spacetimes:
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- Complete $y = f_0$ to a frame f_i with $g^F_{ab}(x,y)f_i^a f_j^b = -\eta_{ij}$.
- Let P be the space of all observer frames.

$\Rightarrow$ $\pi : P \to O$ is a principal SO(3)-bundle.

- In general no principal $\mathrm{SO}_0(3,1)$-bundle $\tilde{\pi} : P \to M$.

Cartan connection - translational part

- Need to construct $A \in \Omega^1(P, \mathfrak{g})$.
- Recall that

$$\begin{array}{ccccc} \mathfrak{g} & = & \mathfrak{h} & \oplus & \mathfrak{z} \\ A & = & \omega & + & e \end{array}$$

$\Rightarrow$ Need to construct $\omega \in \Omega^1(P, \mathfrak{h})$ and $e \in \Omega^1(P, \mathfrak{z})$.

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- Definition of e: Use the *solder form*.
  - Let $w \in T_{(x,f)}P$ be a tangent vector.
  - Differential of the projection $\tilde{\pi} : P \to M$ yields $\tilde{\pi}_*(w) \in T_xM$.
  - View frame f as a linear isometry $f : \mathfrak{z} \to T_xM$.
  - Solder form given by $e(w) = f^{-1}(\tilde{\pi}_*(w))$.

Cartan connection - boost / rotational part

- Definition of ω:
  - Frames (x, f) and (x, f') related by generalized Lorentz transform. [C. Pfeifer, M. Wohlfarth '11]
  - Relation between f and f' defined by parallel transport on O.
  - Tangent vector $w \in T_{(x,f)}P$ "shifts" frame f by small amount.
  - Compare shifted frame with parallely transported frame.
  - Measure the difference using the original frame:

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$\Rightarrow$ Frames f_i^a and $f_i^a + \Delta f_i^a$ are orthonormal wrt the same metric.

$\Rightarrow$ $\omega(w) \in \mathfrak{h}$ is an infinitesimal Lorentz transform.

Cartan connection and fundamental vector fields

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$\Rightarrow$ $A_p : T_pP \to \mathfrak{g}$ and $\underline{A}_p : \mathfrak{g} \to T_pP$ complement each other.

Split of the tangent bundle TP

- Consider adjoint representation $\mathtt{Ad} : K \subset G \to \mathtt{Aut}(\mathfrak{g})$ of K on $\mathfrak{g}$.
- $\mathfrak{g}$ splits into irreducible subrepresentations of $\mathtt{Ad}$.

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- $\mathfrak{g}$ splits into irreducible subrepresentations of Ad.
- Induced decompositions of A and TP:

$$
\begin{array}{ccccccccc}
\mathfrak{g} & = & \mathfrak{k} & \oplus & \mathfrak{h} & \oplus & \vec{\mathfrak{z}} & \oplus & \mathfrak{z}_0 \\
\uparrow A & = & \uparrow \Omega & + & \uparrow b & + & \uparrow \vec{e} & + & \uparrow e^0 \\
T_pP & = & R_pP & \oplus & B_pP & \oplus & \vec{H}_pP & \oplus & H^0_pP \\
 & & \Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow \\
 & & \text{rotations} & & \text{boosts} & & \text{spatial translations} & & \text{temporal translations}
\end{array}
$$

- Subbundles of TP spanned by fundamental vector fields $\underline{A}$.

Time translation

- Consider the fundamental vector field

$$\mathbf{t} = \underline{A}(\mathcal{Z}_0) = f_0^a \partial_a - f_j^a N^b{}_a \bar{\partial}_b^j \qquad \Leftrightarrow \qquad \omega^i{}_j(\mathbf{t}) = 0\,, \quad e^i(\mathbf{t}) = \delta_0^i\,.$$

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- From $\omega^\alpha{}_\beta(\mathbf{t}) = 0$ follows:

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$\Rightarrow$ Frame f is parallelly transported.

Curvature of the Cartan connection

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- $R^d{}_{cab}$, $P^d{}_{cab}$, $S^d{}_{cab}$: curvature of Cartan linear connection.

Outline

Gravity from Cartan to Finsler

- MacDowell-Mansouri gravity on observer space: [S. Gielen, D. Wise '12]

$$S_G = \int_O \epsilon_{\alpha\beta\gamma} \operatorname{tr}_{\mathfrak{h}}(F_{\mathfrak{h}} \wedge \star F_{\mathfrak{h}}) \wedge b^\alpha \wedge b^\beta \wedge b^\gamma$$

  - Hodge operator $\star$ on $\mathfrak{h}$.
  - Non-degenerate H-invariant inner product $\mathrm{tr}_{\mathfrak{h}}$ on $\mathfrak{h}$.

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- Translate terms into Finsler language (with $R = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega]$):
  - Curvature scalar:

$$[e \wedge e] \wedge \star R \rightsquigarrow g^{F\,ab} R^c{}_{acb}\,\mathrm{d}V\,.$$

  - Cosmological constant:

$$[e \wedge e] \wedge \star [e \wedge e] \rightsquigarrow \mathrm{d}V\,.$$

  - Gauss-Bonnet term:

$$R \wedge \star R \rightsquigarrow \epsilon^{abcd} \epsilon^{efgh} R_{abef} R_{cdgh}\,\mathrm{d}V\,.$$

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$\Rightarrow$ Gravity theory on Finsler spacetime.

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$$S_G = \int_O \mathrm{d}^4x \, \mathrm{d}^3y \, \sqrt{-\tilde{G}} R^a{}_{ab} y^b \,.$$

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⇒ Gravity theory on observer space.

Outline

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Symmetries of first-order reductive Cartan geometry

- Frame bundle lift of a vector field $\xi^a\partial_a \in \mathrm{Vect}(M)$ to $\mathrm{GL}(M)$:

$$\bar{\xi} = \xi^a \frac{\partial}{\partial x^a} + f_i{}^a \partial_a \xi^b \frac{\partial}{\partial f_i{}^b} \in \mathrm{Vect}(\mathrm{GL}(M))\,.$$

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- Symmetry condition is invariance of Cartan connection:

$$\mathcal{L}_{\bar{\xi}} \omega = 0\,.$$

Riemann-Cartan, Riemann & Weizenböck

- Riemann-Cartan spacetime:
  - Metric g and torsion T determine connection

$$\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc} - g_{be}T^e{}_{cd} - g_{ce}T^e{}_{bd}) + \frac{1}{2}T^a{}_{cb}\,.$$

$\Rightarrow$ Cartan geometry with Cartan curvature $F = \mathrm{d}A + A \wedge A \in \Omega^2(P, \mathfrak{g})$.

$\Rightarrow$ Symmetry of Cartan geometry $\Leftrightarrow \mathcal{L}_\xi g = 0$, $\mathcal{L}_\xi T = 0$.

Riemann-Cartan, Riemann & Weizenböck

- Riemann-Cartan spacetime:
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Riemann-Cartan, Riemann & Weizenböck

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- Weizenböck spacetime:
  - Vielbein h determines Weizenböck connection

$$\Gamma^a{}_{bc} = h_i^a \partial_c h_b^i \, .$$

$\Rightarrow$ Cartan geometry with Cartan curvature $F = \mathrm{d}A + A \wedge A \in \Omega^2(P, \mathfrak{z})$.
$\Rightarrow$ Symmetry of Cartan geometry $\Leftrightarrow \mathcal{L}_\xi h = \lambda h, \lambda \in \mathfrak{h}$.

Outline

Symmetries of observer space

- Structures induced by Cartan geometry ($\pi : P \to O, A$):
  - Tangent bundle split $TO = VO \oplus \vec{H}O \oplus H^0O$.
  - Projectors $P_V, P_{\vec{H}}, P_{H^0}, P_H = P_{\vec{H}} + P_{H^0}$ onto subbundles.
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- $\Xi \in \mathrm{Vect}(O)$ called "spatio-temporal" if:
  - Boost component of Ξ is time derivative of spatial translation:

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- Symmetry of Cartan geometry:
  - $\underline{\underline{\Xi}}$ is tangent to $P \subset FO = \mathrm{GL}(O)$.
  - A is invariant under $\underline{\underline{\Xi}}$, i.e., $\mathcal{L}_{\underline{\underline{\Xi}}} A = 0$.

Finsler spacetime symmetries

- Tangent bundle lift of a vector field $\xi^a\partial_a \in \mathrm{Vect}(M)$ to TM:

$$\hat{\xi} = \xi^a \frac{\partial}{\partial x^a} + y^a \partial_a \xi^b \frac{\partial}{\partial y^b} \in \mathrm{Vect}(TM)\,.$$

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- One-to-one correspondence between:
  1. Symmetry vector fields $\xi \in \mathrm{Vect}(M)$ of Finsler spacetime.
  2. Symmetry vector fields $\Xi \in \mathrm{Vect}(O)$ on Finsler observer space.

  $\Rightarrow$ Vector field Ξ is spatio-temporal.

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Summary

- First-order reductive Cartan geometry:
  - Split $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{z}$ of Lie algebra induced by ad.
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- Spacetime and observer space symmetries:
    - Notion of symmetry for first-order reductive Cartan geometry.
    - Derive notions of symmetry for spacetime model geometries.
    - Observer space model: notion of “spatio-temporal” symmetry.
    - Equivalent definition of symmetry of Finsler spacetime.

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