Outline

1. Introduction
2. Gauge-invariant higher order perturbations
3. Parametrized post-Newtonian formalism
4. Gauge-invariant PPN formalism
5. Example: PPN limit of scalar-tensor gravity
6. Conclusion
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Motivation

- Experimental tests of modified gravity theories:
  - Cosmological observations (CMB, supernovae, ...).
  - Gravitational waves.
  - Direct observation of black holes.
  - Solar system, pulsars, ...
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  - Weak-field approximation of metric gravity theories.
  - Assumes particular coordinate system (“universe rest frame”).
  - Characterizes gravity theories by 10 (constant) parameters.
  - Parameters closely related to solar system observations.
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- Extensions of the PPN formalism:
  - More fundamental fields constituting the metric.
  - More than one dynamical metric.
  - Diffeomorphism invariant / purely geometric formalism.
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Use gauge-invariant higher order perturbation theory.
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2 Gauge-invariant higher order perturbations

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4 Gauge-invariant PPN formalism

5 Example: PPN limit of scalar-tensor gravity

6 Conclusion
Concept and use of gauge

- Reference spacetime:
  - Manifold $M_0$ with metric $g^{(0)}$ and coordinates $(x^\mu)$.
  - Usually some highly symmetric standard spacetime:
    - maximally symmetric spacetime: Minkowski, (anti-)de Sitter
    - cosmological (background) solution of a gravity theory
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- Physical spacetime:
  - Manifold $M$ with metric $g$.
  - No preferred / canonical choice of coordinates.
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\[ \text{\texttt{\textbackslash i}} \text{ No canonical relation between physical and reference spacetime.} \]
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\[\text{No canonical relation between physical and reference spacetime.}\]

1. No identification of points on $M$ and $M_0$: no coordinates on $M$. 
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\[\frac{\text{No canonical relation between physical and reference spacetime.}}{\text{}}\]

1. No identification of points on $M$ and $M_0$: no coordinates on $M$.
2. No possibility to compare $g$ and $g^{(0)}$: different manifolds.
Concept and use of gauge

• Reference spacetime:
  ○ Manifold $M_0$ with metric $g^{(0)}$ and coordinates $(x^\mu)$.
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• Physical spacetime:
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  ○ No preferred / canonical choice of coordinates.

\[ \n \text{No canonical relation between physical and reference spacetime.} \]
\[ \text{1. No identification of points on } M \text{ and } M_0: \text{ no coordinates on } M. \]
\[ \text{2. No possibility to compare } g \text{ and } g^{(0)}: \text{ different manifolds.} \]

\[ \n \text{Introduce a } \textit{gauge}: \text{ diffeomorphism } \chi : M_0 \to M. \]
\[ \text{1. Identification of (coordinated) points on } M \text{ and } M_0. \]
\[ \text{2. Comparison between reference metric } g^{(0)} \text{ and } \chi^* g = \chi^* g \text{ on } M_0. \]
Parameter dependent physical metric:

- Assume physical metric $g \equiv g_\epsilon$ depends on parameter $\epsilon \in \mathbb{R}$.
- Assume every $g_\epsilon$ is defined on its own $M_\epsilon$.
- Assume $g_0 = g^{(0)}$ is the reference metric on $M_0$.

$\Rightarrow$ Family of physical spacetimes $(M_\epsilon, g_\epsilon)$. 

Aim: perturbative expansion of $g_\epsilon$ in $\epsilon$ around $g^{(0)}$.

Metrics $g_\epsilon$ are defined on different manifolds for different $\epsilon$.

Use gauge to relate different manifolds:

- Family of diffeomorphisms $X_\epsilon : M_0 \to M_\epsilon$.
- Assume $X_0 = \text{id}_{M_0}$ on the reference spacetime.

Perturbative expansion:

- Pullback $X_\epsilon g_\epsilon = X_\epsilon^* g_\epsilon$ defined on $M_0$.
- Introduce series expansion in $\epsilon$:
  $$X_\epsilon g_\epsilon = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \partial X_\epsilon g_\epsilon |_{\epsilon=0} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} X_\epsilon g_\epsilon^{(k)}.$$ 
- Series coefficients $X_\epsilon g_\epsilon^{(k)}$ depend on gauge choice $X_\epsilon$. 
Gauge and perturbations

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$\therefore$ Metrics $g_\epsilon$ are defined on different manifolds for different $\epsilon$. 
Gauge and perturbations

- Parameter dependent physical metric:
  - Assume physical metric \( g \equiv g_\epsilon \) depends on parameter \( \epsilon \in \mathbb{R} \).
  - Assume every \( g_\epsilon \) is defined on its own \( M_\epsilon \).
  - Assume \( g_0 = g^{(0)} \) is the reference metric on \( M_0 \).
  \Rightarrow\text{Family of physical spacetimes } (M_\epsilon, g_\epsilon).

- Aim: perturbative expansion of \( g_\epsilon \) in \( \epsilon \) around \( g^{(0)} \).
  \text{Metrics } g_\epsilon \text{ are defined on different manifolds for different } \epsilon.
  \Rightarrow\text{Use gauge to relate different manifolds:}
  - Family of diffeomorphisms \( \chi_\epsilon : M_0 \to M_\epsilon \).
  - Assume \( \chi_0 = \text{id}_{M_0} \) on the reference spacetime.
Gauge and perturbations

- Parameter dependent physical metric:
  - Assume physical metric \( g \equiv g_\epsilon \) depends on parameter \( \epsilon \in \mathbb{R} \).
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    \( \Rightarrow \) Family of physical spacetimes \( (M_\epsilon, g_\epsilon) \).

- Aim: perturbative expansion of \( g_\epsilon \) in \( \epsilon \) around \( g^{(0)} \).

\( \Leftrightarrow \) Metrics \( g_\epsilon \) are defined on different manifolds for different \( \epsilon \).

\( \Rightarrow \) Use gauge to relate different manifolds:
  - Family of diffeomorphisms \( \mathcal{X}_\epsilon : M_0 \to M_\epsilon \).
  - Assume \( \mathcal{X}_0 = \text{id}_{M_0} \) on the reference spacetime.

- Perturbative expansion:
  - Pullback \( \mathcal{X}^* g_\epsilon = \mathcal{X}_\epsilon^* g_\epsilon \) defined on \( M_0 \).
Gauge and perturbations

- Parameter dependent physical metric:
  - Assume physical metric $g \equiv g_\epsilon$ depends on parameter $\epsilon \in \mathbb{R}$.
  - Assume every $g_\epsilon$ is defined on its own $M_\epsilon$.
  - Assume $g_0 = g^{(0)}$ is the reference metric on $M_0$.
  $\implies$ Family of physical spacetimes $(M_\epsilon, g_\epsilon)$.

- Aim: perturbative expansion of $g_\epsilon$ in $\epsilon$ around $g^{(0)}$.

‡ Metrics $g_\epsilon$ are defined on different manifolds for different $\epsilon$.

$\implies$ Use gauge to relate different manifolds:
  - Family of diffeomorphisms $\chi_\epsilon : M_0 \to M_\epsilon$.
  - Assume $\chi_0 = \text{id}_{M_0}$ on the reference spacetime.

- Perturbative expansion:
  - Pullback $\chi g_\epsilon = \chi_\epsilon^* g_\epsilon$ defined on $M_0$.
  - Introduce series expansion in $\epsilon$:

\[
\chi g_\epsilon = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \frac{\partial^k \chi g_\epsilon}{\partial \epsilon^k} \bigg|_{\epsilon=0} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \chi g^{(k)}.
\]
Gauge and perturbations

- Parameter dependent physical metric:
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  - Assume every \( g_{\epsilon} \) is defined on its own \( M_{\epsilon} \).
  - Assume \( g_0 = g^{(0)} \) is the reference metric on \( M_0 \).
    \[ \Rightarrow \] Family of physical spacetimes \((M_\epsilon, g_\epsilon)\).

- Aim: perturbative expansion of \( g_\epsilon \) in \( \epsilon \) around \( g^{(0)} \).

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\[ \Rightarrow \] Use gauge to relate different manifolds:
  - Family of diffeomorphisms \( \mathcal{X}_\epsilon : M_0 \to M_\epsilon \).
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- Perturbative expansion:
  - Pullback \( \mathcal{X} g_\epsilon = \mathcal{X}_\epsilon^* g_\epsilon \) defined on \( M_0 \).
  - Introduce series expansion in \( \epsilon \):
    \[
    \mathcal{X} g_\epsilon = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \frac{\partial^k \mathcal{X} g_\epsilon}{\partial \epsilon^k} \bigg|_{\epsilon=0} = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \mathcal{X} g^{(k)}. 
    \]
  - Series coefficients \( \mathcal{X} g^{(k)} \) depend on gauge choice \( \mathcal{X} \).
Gauge invariant perturbations

- Choose a fixed “distinguished” gauge $S_\epsilon : M_0 \rightarrow M_\epsilon$:
  - E.g., impose gauge conditions on the metric.
  - Examples: harmonic gauge used for GWs, standard PPN gauge.
Gauge invariant perturbations

- Choose a fixed “distinguished” gauge $S_\epsilon : M_0 \to M_\epsilon$:
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- Define gauge-invariant metric $g_\epsilon = S g_\epsilon = S_\epsilon^* g_\epsilon$. 
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- Metric in arbitrary gauge \( \mathcal{X} \):
  \[
  \mathcal{X} g_\epsilon = \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} \frac{\epsilon^{l_1+\cdots+k l_k+\cdots}}{(1!)^{l_1} (k!) l_k \cdots l_1 \cdots l_k} \mathcal{E}_{l_1}^{X(1)} \cdots \mathcal{E}_{l_k}^{X(k)} \cdots g_\epsilon.
  \]
Gauge invariant perturbations

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  - E.g., impose gauge conditions on the metric.
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  \mathcal{X} g_\epsilon = \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} \frac{\epsilon^{l_1+\cdots+k l_k+\cdots}}{(1!)^{l_1} \cdots (k!)^{l_k} l_1! \cdots l_k! \cdots} \mathcal{E}^{l_1} \cdots \mathcal{E}^{l_k} X_{(1)}(1) \cdots X_{(k)}(k) g_\epsilon .
  \]
- Metric components split into two parts:
  - Gauge-invariant part \( g_\epsilon \): physical content.
  - Gauge defining vector fields \( X_{(k)} \): coordinate choice.
Choose a fixed “distinguished” gauge $S_\epsilon : M_0 \to M_\epsilon$:
- E.g., impose gauge conditions on the metric.
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Define gauge-invariant metric $g_\epsilon = S g_\epsilon = S_\epsilon^* g_\epsilon$.

Metric in arbitrary gauge $\mathcal{X}$:

$$\mathcal{X} g_\epsilon = \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} \frac{\epsilon^{l_1+\cdots+k l_k+\cdots}}{(1!)^{l_1} \cdots (k!)^{l_k} \cdots l_1! \cdots l_k! \cdots} \mathcal{L}^{l_1}_{X(1)} \cdots \mathcal{L}^{l_k}_{X(k)} \cdots g_\epsilon.$$ 

Metric components split into two parts:
- Gauge-invariant part $g_\epsilon :$ physical content.
- Gauge defining vector fields $X_{(k)} :$ coordinate choice.

Number # of independent components:

$$\#(\mathcal{X} g_\epsilon) = \#(g_\epsilon) + \#(X_{(k)}).$$
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Post-Newtonian matter and velocity orders

- Energy-momentum tensor of a perfect fluid:

\[ T^{\mu\nu} = (\rho + \rho \Pi + p) u^{\mu} u^{\nu} + p g^{\mu\nu}. \]

- Rest mass density \( \rho \).
- Specific internal energy \( \Pi \).
- Pressure \( p \).
- Four-velocity \( u^{\mu} \).
Post-Newtonian matter and velocity orders

- Energy-momentum tensor of a perfect fluid:

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- Rest mass density \( \rho \).
- Specific internal energy \( \Pi \).
- Pressure \( p \).
- Four-velocity \( u^\mu \).

- Universe rest frame and slow-moving source matter:
  - Consider some gauge \( \mathcal{X} : M_0 \rightarrow M \) ("universe rest frame").
  - Pullback of metric and matter variables along \( \mathcal{X} \).
  - Velocity of the source matter: \( \mathcal{X} \tilde{v}^i = \mathcal{X} u^i / \mathcal{X} u^0 \).
  - Assume that source matter is slow-moving: \( |\mathcal{X} \tilde{v}| \ll 1 \).
Energy-momentum tensor of a perfect fluid:

\[ T^{\mu \nu} = \left( \rho + \rho \Pi + p \right) u^\mu u^\nu + pg^{\mu \nu}. \]

- Rest mass density $\rho$.
- Specific internal energy $\Pi$.
- Pressure $p$.
- Four-velocity $u^\mu$.

Universe rest frame and slow-moving source matter:

- Consider some gauge $\mathcal{X} : M_0 \to M$ (“universe rest frame”).
- Pullback of metric and matter variables along $\mathcal{X}$.
- Velocity of the source matter: $\mathcal{X} v^i = \mathcal{X} u^i / \mathcal{X} u^0$.
- Assume that source matter is slow-moving: $|\mathcal{X} \tilde{v}| \ll 1$.

Use $\epsilon = |\mathcal{X} \tilde{v}|$ as perturbation parameter.
Post-Newtonian matter and velocity orders

- Energy-momentum tensor of a perfect fluid:

\[ T^{\mu\nu} = (\rho + \rho\Pi + p) u^\mu u^\nu + pg^{\mu\nu}. \]

  - Rest mass density \( \rho \sim \mathcal{O}(2). \)
  - Specific internal energy \( \Pi \sim \mathcal{O}(2). \)
  - Pressure \( p \sim \mathcal{O}(4). \)
  - Four-velocity \( u^\mu. \)

- Universe rest frame and slow-moving source matter:

  - Consider some gauge \( \chi : M_0 \to M \) ("universe rest frame").
  - Pullback of metric and matter variables along \( \chi. \)
  - Velocity of the source matter: \( \chi v^i = \chi u^i / \chi u^0. \)
  - Assume that source matter is slow-moving: \( |\chi \tilde{v}| \ll 1. \)

- Use \( \epsilon = |\chi \tilde{v}| \) as perturbation parameter.

- Assign velocity orders \( \mathcal{O}(n) \sim \epsilon^n \) to all quantities.
Post-Newtonian matter and velocity orders

- Energy-momentum tensor of a perfect fluid:

\[ T^{\mu\nu} = (\rho + \rho \Pi + p) u^\mu u^\nu + pg^{\mu\nu}. \]

- Rest mass density \( \rho \sim O(2) \).
- Specific internal energy \( \Pi \sim O(2) \).
- Pressure \( p \sim O(4) \).
- Four-velocity \( u^\mu \).

- Universe rest frame and slow-moving source matter:
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- Use \( \epsilon = |\mathcal{X} v| \) as perturbation parameter.

- Assign velocity orders \( O(n) \sim \epsilon^n \) to all quantities.

- Quasi-static: assign additional \( O(1) \) to time derivatives \( \partial_0 \).
Standard post-Newtonian metric

- PPN formalism assumes fixed standard gauge $\mathcal{P}$.

\begin{align*}
  P_2 \ g_{00} &= 2 P \ U, \\
  P_2 \ g_{ij} &= 2 \gamma P \ U \delta_{ij}, \\
  P_3 \ g_{0i} &= -\frac{1}{2} \left( 3 + 4 \gamma + \alpha_1 - \alpha_2 + \zeta_1 - 2 \xi \right) P V^i - \frac{1}{2} \left( 1 + \alpha_2 - \zeta_1 + 2 \xi \right) P W^i, \\
  P_4 \ g_{00} &= -2 \beta P U^2 + \left( 2 + 2 \gamma + \alpha_3 + \zeta_1 - 2 \xi \right) P \Phi_1 + 2 \left( 1 + 3 \gamma - 2 \beta + \zeta_2 + \xi \right) P \Phi_2 + 2 \left( 3 \gamma + 3 \zeta_4 - 2 \xi \right) P \Phi_3 - 2 \xi P \Phi_4 - \left( \zeta_1 - 2 \xi \right) P A,
\end{align*}

- Metric contains PPN parameters and PPN potentials.
  - PPN potentials describe source matter distribution.
  - PPN parameters characterize gravity theory.
Standard post-Newtonian metric

- PPN formalism assumes fixed standard gauge $\mathcal{P}$.
- Metric in standard PPN gauge:

\[
\begin{align*}
\mathcal{P}^2 g_{00} &= 2^\mathcal{P} U, \\
\mathcal{P}^2 g_{ij} &= 2^\mathcal{P} U \delta_{ij}, \\
\mathcal{P}^3 g_{0i} &= -\frac{1}{2} (3 + 4\gamma + \alpha_1 - \alpha_2 + \zeta_1 - 2\xi)^^\mathcal{P} V_i - \frac{1}{2} (1 + \alpha_2 - \zeta_1 + 2\xi)^^\mathcal{P} W_i, \\
\mathcal{P}^4 g_{00} &= -2^\mathcal{P} U^2 + (2 + 2\gamma + \alpha_3 + \zeta_1 - 2\xi)^^\mathcal{P} \Phi_1 + 2(1 + 3\gamma - 2\beta + \zeta_2 + \xi)^^\mathcal{P} \Phi_2 \\
&\quad + 2(1 + \zeta_3)^^\mathcal{P} \Phi_3 + 2(3\gamma + 3\zeta_4 - 2\xi)^^\mathcal{P} \Phi_4 - 2^\mathcal{P} \Phi_W - (\zeta_1 - 2\xi)^^\mathcal{P} \mathcal{A},
\end{align*}
\]

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- PPN formalism assumes fixed standard gauge $\mathcal{P}$.
- Metric in standard PPN gauge:

\[
\begin{align*}
\mathcal{P}^2 g_{00} &= 2^\mathcal{P} U , \\
\mathcal{P}^2 g_{ij} &= 2\gamma^\mathcal{P} U \delta_{ij} , \\
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\mathcal{P}^4 g_{00} &= -2\beta^\mathcal{P} U^2 + (2 + 2\gamma + \alpha_3 + \zeta_1 - 2\xi)^\mathcal{P} \Phi_1 + 2(1 + 3\gamma - 2\beta + \zeta_2 + \xi)^\mathcal{P} \Phi_2 \\
&\quad + 2(1 + \zeta_3)^\mathcal{P} \Phi_3 + 2(3\gamma + 3\zeta_4 - 2\xi)^\mathcal{P} \Phi_4 - 2\xi^\mathcal{P} \Phi_W - (\zeta_1 - 2\xi)^\mathcal{P} \mathcal{A} ,
\end{align*}
\]

- Metric contains **PPN parameters** and **PPN potentials**.
  - **PPN potentials** describe source matter distribution.
  - **PPN parameters** characterize gravity theory.
Standard post-Newtonian metric

- PPN formalism assumes fixed standard gauge $P$.
- Metric in standard PPN gauge:

\[ P^2 g_{00} = 2^P U, \]
\[ P^2 g_{ij} = 2^P \gamma^P U \delta_{ij}, \]
\[ P^3 g_{0i} = -\frac{1}{2} (3 + 4 \gamma + \alpha_1 - \alpha_2 + \zeta_1 - 2 \xi)^P V_i - \frac{1}{2} (1 + \alpha_2 - \zeta_1 + 2 \xi)^P W_i, \]
\[ P^4 g_{00} = -2^P \beta^P U^2 + (2 + 2 \gamma + \alpha_3 + \zeta_1 - 2 \xi)^P \Phi_1 + 2(1 + 3 \gamma - 2 \beta + \zeta_2 + \xi)^P \Phi_2 \]
\[ + 2(1 + \zeta_3)^P \Phi_3 + 2(3 \gamma + 3 \zeta_4 - 2 \xi)^P \Phi_4 - 2^P \Phi W - (\zeta_1 - 2 \xi)^P \mathcal{A}, \]

- Metric contains PPN parameters and PPN potentials.
  - PPN potentials describe source matter distribution.
  - PPN parameters characterize gravity theory.

\sim Decompose metric into gauge-invariant and pure gauge parts.
Gauge-invariant metric

- Definition of gauge-invariant metric components:

\[ g_{00} = g^*, \quad g_{0i} = g_i^\circ, \quad g_{ij} = g^\bullet \delta_{ij} + g_{ij}^+ . \]
Gauge-invariant metric

- Definition of gauge-invariant metric components:
  \[ g_{00} = g^*, \quad g_{0i} = g_i^\odot, \quad g_{ij} = g^\bullet \delta_{ij} + g_{ij}^\dagger. \]

- Conditions imposed on components:
  \[ \partial^i g_i^\odot = 0, \quad \partial^i g_{ij}^\dagger = 0, \quad g_{[ij]}^\dagger = 0, \quad g_{ii}^\dagger = 0. \]
Gauge-invariant metric

- Definition of gauge-invariant metric components:
  \[ g_{00} = g^*, \quad g_{0i} = g_i^\circ, \quad g_{ij} = g^* \delta_{ij} + g_{ij}^\dagger. \]

- Conditions imposed on components:
  \[ \partial_i g_i^\circ = 0, \quad \partial_i g_{ij}^\dagger = 0, \quad g_{[ij]} = 0, \quad g_{ii}^\dagger = 0. \]

- Relation to arbitrary gauge \( \chi \):
  \[ \chi^2 g_{00} = 2^* g^*, \]
  \[ \chi^2 g_{ij} = 2^* \delta_{ij} + 2^\dagger g_{ij} + 2 \partial_i \partial_j 2^\chi + 2 \partial (i 2^\chi_j), \]
  \[ \chi^3 g_{0i} = 3^\circ g_i + \partial_i 3^* + \partial_0 \partial_i 2^\chi + \partial_0 2^\chi_i, \]
  \[ \chi^4 g_{00} = 4^* + 2 \partial_0 3^* + (\partial_i 2^\chi + 2^\chi_i) \partial_i 2^* g^*, \]
  \[ \chi^4 g_{ij} = 4^* \delta_{ij} + 4^\dagger g_{ij} + 2 \partial_i \partial_j 4^\chi + \mathcal{O}(2) \cdot \mathcal{O}(2). \]
Gauge-invariant metric

- Definition of gauge-invariant metric components:
  \[ g_{00} = g^*, \quad g_{0i} = g^i, \quad g_{ij} = g^\delta_{ij} + g^\dagger_{ij}. \]

- Conditions imposed on components:
  \[ \partial^i g^\diamond_i = 0, \quad \partial^i g^\dagger_{ij} = 0, \quad g^\dagger_{[ij]} = 0, \quad g^\dagger_{ii} = 0. \]

- Relation to arbitrary gauge \( \mathcal{X} \):
  \[
  \mathcal{X}^2 g_{00} = g^2*, \\
  \mathcal{X}^2 g_{ij} = g^\delta_{ij} + g^\dagger_{ij} + 2 \partial_i \partial_j \mathcal{X}^\dagger + 2 \partial_i \partial_j \mathcal{X}^\diamond, \\
  \mathcal{X}^3 g_{0i} = g^i + \partial_i \mathcal{X}^* + \partial_0 \partial_i \mathcal{X}^\dagger + \partial_0 \mathcal{X}^\diamond, \\
  \mathcal{X}^4 g_{00} = g^* + 2 \partial_0 \mathcal{X}^* + (\partial_i \mathcal{X}^\dagger + \mathcal{X}^\diamond) \partial_i g^*, \\
  \mathcal{X}^4 g_{ij} = g^\delta_{ij} + g^\dagger_{ij} + 2 \partial_i \partial_j \mathcal{X}^\dagger + \mathcal{O}(2) \cdot \mathcal{O}(2). \]

- Gauge defining vector fields:
  \[ X_i = \partial_i \mathcal{X}^\dagger + \mathcal{X}^\diamond, \quad X_0 = \mathcal{X}^*, \quad \partial^i \mathcal{X}^\dagger = 0. \]
Count number of independent components at each order:

<table>
<thead>
<tr>
<th>$\chi \frac{2}{g_{00}}$</th>
<th>$\chi \frac{2}{g_{ij}}$</th>
<th>$\chi \frac{3}{g_{0i}}$</th>
<th>$\chi \frac{4}{g_{00}}$</th>
<th>$\chi \frac{4}{g_{ij}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>3</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>$\mathbf{2}^{*}$</td>
<td>$\mathbf{2}^{*}$</td>
<td>$\mathbf{3}^{\diamond}$</td>
<td>$\mathbf{4}^{*}$</td>
<td>$\mathbf{4}^{*}$</td>
</tr>
<tr>
<td>1</td>
<td>$\mathbf{2}^{*}$</td>
<td>2</td>
<td>1</td>
<td>$\mathbf{2}^{\star}$</td>
</tr>
<tr>
<td>pure gauge</td>
<td>invariant</td>
<td>total</td>
<td>total</td>
<td>total</td>
</tr>
<tr>
<td>-</td>
<td>$\mathbf{2}^{\star}$</td>
<td>$\mathbf{2}^{\star}$</td>
<td>$\mathbf{3}^{\star}$</td>
<td>$\mathbf{4}^{\star}$</td>
</tr>
<tr>
<td>0</td>
<td>$\mathbf{2}^{\star}$</td>
<td>1</td>
<td>1</td>
<td>$\mathbf{4}^{\star}$</td>
</tr>
</tbody>
</table>

Components split into invariant and gauge parts.

Possible to separate physical information from coordinate choice.
Decomposition of metric components

- Count number of independent components at each order:

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<th>Invariant</th>
<th>Pure Gauge</th>
</tr>
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<tr>
<td>$\chi_2^0 g_{00}$</td>
<td>1</td>
<td>$g^*$</td>
<td>-</td>
</tr>
<tr>
<td>$\chi_2 g_{ij}$</td>
<td>6</td>
<td>$g^*, g^\dagger_{ij}$</td>
<td>$X^\diamond, X_i^\circ$</td>
</tr>
<tr>
<td>$\chi_3^0 g_{0i}$</td>
<td>3</td>
<td>$g_i^\circ$</td>
<td>$X^*$</td>
</tr>
<tr>
<td>$\chi_4^0 g_{00}$</td>
<td>1</td>
<td>$g^*$</td>
<td>-</td>
</tr>
<tr>
<td>$\chi_4 g_{ij}$</td>
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⇒ Components split into invariant and gauge parts.
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<td>$1 + 2$</td>
</tr>
<tr>
<td>$\chi^3 g_{0i}$</td>
<td>$3^\circ$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\chi^4 g_{00}$</td>
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<td>$1$</td>
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⇒ Components split into invariant and gauge parts.
⇒ Possible to separate physical information from coordinate choice.
Relation to standard PPN gauge

- Use relation between expansion coefficients:

\[ P^k g = \sum_{0 \leq l_1 + 2l_2 + \ldots + k} \frac{1}{l_1! l_2! \ldots} \mathcal{P}^{l_1} \ldots \mathcal{P}^{l_k} g = \mathcal{P}^{k-l_1-2l_2-\ldots} g. \]
Relation to standard PPN gauge

- Use relation between expansion coefficients:

\[
P^k g = \sum_{0 \leq l_1 + 2l_2 + \ldots \leq k} \frac{1}{l_1! l_2! \ldots} \mathcal{E}_{l_1} \mathcal{E}_{l_2} \cdots \mathcal{E}_{l_k} \mathcal{P}_{k-l_1-2l_2-\ldots} g.
\]

- Split components of \( P g_{\mu\nu} \) into \( g_{\mu\nu} \) and \( P^\mu \).
Relation to standard PPN gauge

- Use relation between expansion coefficients:

\[ \mathcal{P}^k g = \sum_{0 \leq l_1 + 2l_2 + \ldots \leq k} \frac{1}{l_1! l_2! \ldots} \mathcal{P}^l_1 \ldots \mathcal{P}^l_k \mathcal{P}^{k-l_1-2l_2-\ldots} g. \]

- Split components of \( \mathcal{P} g_{\mu\nu} \) into \( g_{\mu\nu} \) and \( P^\mu \).

⇒ Gauge defining vector fields:

\[
\begin{align*}
P^\diamond &= 0, \\
P^\circ_i &= 0, \\
P^* &= -\frac{1}{4} (2 + 4\gamma + \alpha_1 - 2\alpha_2 + 2\zeta_1 - 4\xi) \chi_0.
\end{align*}
\]
Relation to standard PPN gauge

- Use relation between expansion coefficients:
  \[ P^k g = \sum_{0 \leq l_1 + 2l_2 + \ldots \leq k} \frac{1}{l_1! l_2! \ldots} \xi_1^{l_1} \xi_2^{l_2} \ldots g. \]

- Split components of \( P g_{\mu\nu} \) into \( g_{\mu\nu} \) and \( P^\mu \).

  \[ P^\mu = 0, \quad P_i^\nu = 0, \quad P^* = -\frac{1}{4} (2 + 4\gamma + \alpha_1 - 2\alpha_2 + 2\zeta_1 - 4\xi) \chi,0. \]

  \[ g^* = 2U, \quad g^\nu = 2\gamma U, \quad g^\dagger_{ij} = 0, \quad g_i^\nu = - \left( 1 + \gamma + \frac{\alpha_1}{4} \right) (V_i + W_i), \]

  \[ g^* = \frac{1}{2} (2 - \alpha_1 + 2\alpha_2 + 2\alpha_3) \Phi_1 + 2 (1 + 3\gamma - 2\beta + \zeta_2 + \xi) \Phi_2 + 2 (1 + \zeta_3) \Phi_3 + 2 (3\gamma + 3\zeta_4 - 2\xi) \Phi_4 - 2\xi \Phi_W - 2\beta U^2 + \frac{1}{2} (2 + 4\gamma + \alpha_1 - 2\alpha_2) \mathfrak{A} + \frac{1}{2} (2 + 4\gamma + \alpha_1 - 2\alpha_2 + 2\zeta_1 - 4\xi) \mathfrak{B}. \]
Gauge-invariant field equations

- Perform similar decomposition of energy-momentum tensor:

\[
T^* = T_{00} = \rho \left( 1 - g_{00} + v^2 + \Pi \right) + \mathcal{O}(6),
\]

\[
T^\circ_i + \partial_i T^\dagger = T_{0i} = -\rho v_i + \mathcal{O}(5),
\]

\[
T^* \delta_{ij} + \triangle_{ij} T^\wedge + 2 \partial(i T^\Delta_{j}) + T^\dagger_{ij} = T_{ij} = \rho v_i v_j + p \delta_{ij} + \mathcal{O}(6).
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• Perform similar decomposition of energy-momentum tensor:

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T_i^\diamond + \partial_i T^\dagger = T_{0i} = -\rho \mathbf{v}_i + \mathcal{O}(5), \\
T^\dagger \delta_{ij} + \nabla_{ij} T^\bigtriangledown + 2 \partial_i (T_j^\bigtriangledown) + T^\dagger_{ij} = T_{ij} = \rho \mathbf{v}_i \mathbf{v}_j + p \delta_{ij} + \mathcal{O}(6).
\]

• Express components in terms of PPN potentials:

\[
\begin{align*}
\frac{2}{4\pi} T^* &= \rho = -\frac{1}{4\pi} \Delta U, \\
\frac{3}{4\pi} T^\dagger &= -\frac{1}{4\pi} \partial U, \\
\frac{3}{8\pi} T_i^\diamond &= \frac{1}{8\pi} \Delta (V_i + W_i), \\
\frac{4}{4\pi} T^* &= \rho \left( \Pi + \mathbf{v}^2 - \frac{2}{\Pi} \right) = -\frac{1}{4\pi} \Delta (\Phi_3 + \Phi_1 - 2\Phi_2), \\
\frac{4}{3} T^\dagger &= \frac{1}{3} \rho \mathbf{v}^2 + p = -\frac{1}{12\pi} \Delta (\Phi_1 + 3\Phi_4), \\
T^\bigtriangledown &= \frac{1}{16\pi} (3\Phi - \Phi_1).
\end{align*}
\]
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- Perform similar decomposition of energy-momentum tensor:

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T^* = T_{00} = \rho \left( 1 - \frac{2}{\Pi} + \frac{v^2}{\Pi} \right) + O(6),
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T^* \delta_{ij} + \triangle_{ij} T^\dagger + 2\partial_j T^\triangle_i + T^\dagger_{ij} = T_{ij} = \rho v_i v_j + p \delta_{ij} + O(6).
\]

- Express components in terms of PPN potentials:

\[
\begin{align*}
2T^* &= \rho = -\frac{1}{4\pi} \triangle U, \\
3T^\circ &= -\frac{1}{4\pi} \partial_0 U, \\
3T^\dagger &= \frac{1}{8\pi} \triangle (V_i + W_i), \\
4T^* &= \rho \left( \Pi + v^2 - \frac{2}{\Pi} \right) = -\frac{1}{4\pi} \triangle (\Phi_3 + \Phi_1 - 2\Phi_2), \\
4T^\circ &= \frac{1}{3} \rho v^2 + p = -\frac{1}{12\pi} \triangle (\Phi_1 + 3\Phi_4), \\
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\end{align*}
\]

- Decompose also gravity side \( \simeq \triangle g \).
Gauge-invariant field equations

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\[ T^\star \delta_{ij} + \Delta_{ij} T^{\Delta} + 2\partial (i T^{\Delta}_j) + T^{\dagger}_{ij} = T_{ij} = \rho v_i v_j + p \delta_{ij} + \mathcal{O}(6). \]

2. Express components in terms of PPN potentials:

\[ \frac{2}{\Delta} T^* = \rho = -\frac{1}{4\pi} \Delta U, \quad \frac{3}{\Delta} T^\star = -\frac{1}{4\pi} \partial_0 U, \quad \frac{3}{\Delta} T^\circ_i = \frac{1}{8\pi} \Delta (V_i + W_i), \]

\[ \frac{4}{\Delta} T^* = \rho \left( \Pi + v^2 - \frac{2}{\Delta} g^* \right) = -\frac{1}{4\pi} \Delta \left( \Phi_3 + \Phi_1 - 2\Phi_2 \right), \]

\[ \frac{4}{\Delta} T^\star = \frac{1}{3} \rho v^2 + p = -\frac{1}{12\pi} \Delta \left( \Phi_1 + 3\Phi_4 \right), \quad \frac{4}{\Delta} T^{\Delta} = \frac{1}{16\pi} (3\Delta - \Phi_1). \]

3. Decompose also gravity side \( \simeq \Delta g \).

\[ \Rightarrow \quad \text{Find PPN parameters by comparing coefficients on both sides.} \]
Action and field equations

- Action of scalar-tensor gravity with massless scalar field: [Nordtvedt ’70]

\[ S = \frac{1}{2\kappa^2} \int_M d^4x \sqrt{-g} \left( \psi R - \frac{\omega(\psi)}{\psi} \partial_{\rho} \psi \partial^\rho \psi \right) + S_m[g_{\mu\nu}, \chi]. \]
Action and field equations

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S = \frac{1}{2\kappa^2} \int_M d^4x \sqrt{-g} \left( \psi R - \frac{\omega(\psi)}{\psi} \partial_\rho \psi \partial^\rho \psi \right) + S_m[g_{\mu\nu}, \chi].
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- Free function \(\omega(\psi)\) of the scalar field \(\psi\).
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- Work in Jordan conformal frame: no direct coupling between matter and scalar field.
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  \]

- Free function \( \omega(\psi) \) of the scalar field \( \psi \).
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⇒ Field equations:
  \[
  \psi R_{\mu\nu} - \nabla_\mu \partial_\nu \psi - \frac{\omega}{\psi} \partial_\mu \psi \partial_\nu \psi + \frac{g_{\mu\nu}}{4\omega + 6} \frac{d\omega}{d\psi} \partial_\rho \psi \partial^\rho \psi = \kappa^2 \left( T_{\mu\nu} - \frac{\omega + 1}{2\omega + 3} g_{\mu\nu} T \right)
  \]
  \[
  (2\omega + 3) \Box \psi + \frac{d\omega}{d\psi} \partial_\rho \psi \partial^\rho \psi = \kappa^2 T.
  \]
Action and field equations

- Action of scalar-tensor gravity with massless scalar field: \[ \text{[Nordtvedt '70]} \]

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\[ \Rightarrow \text{Field equations:} \]

\[
\psi R_{\mu\nu} - \nabla_\mu \partial_\nu \psi - \frac{\omega}{\psi} \partial_\mu \psi \partial_\nu \psi + \frac{g_{\mu\nu}}{4\omega + 6} \frac{d\omega}{d\psi} \partial_\rho \psi \partial^\rho \psi = \kappa^2 \left( T_{\mu\nu} - \frac{\omega + 1}{2\omega + 3} g_{\mu\nu} T \right),
\]

\[
(2\omega + 3) \Box \psi + \frac{d\omega}{d\psi} \partial_\rho \psi \partial^\rho \psi = \kappa^2 T.
\]

\[ \Rightarrow \text{Decompose into gauge-invariant field equations.} \]
Perturbative solution ansatz

- Consider arbitrary gauge $\mathcal{X}$ to pull equations and fields to reference spacetime $M_0$. 

- Relevant components of scalar field:
  - $X^0 \psi = \Psi$
  - $X^2 \psi$
  - $X^4 \psi$

- Cosmological background value $\Psi$ assumed to be constant.

- Relation to gauge-invariant scalar field perturbations:
  - $X^2 \psi = 2 \psi$
  - $X^4 \psi = 4 \psi + (\partial_i X^2 \psi + 2 \Box_i X^2 \psi)^2$

- Taylor expansion of free function $\omega$ around cosmological background value:
  - $\omega_0 = \omega(\Psi)$
  - $\omega_1 = \omega'(\Psi)$

$\Rightarrow$ Zeroth order $X^0 \psi = \Psi$, $X^0 g_{\mu\nu} = \eta_{\mu\nu}$ solves (vacuum) field equations.

$\Rightarrow$ Remaining equations determine gauge-invariant metric components.
Perturbative solution ansatz

- Consider arbitrary gauge $\mathcal{X}$ to pull equations and fields to reference spacetime $M_0$.
- Relevant components of scalar field: $\mathcal{X}_0 \psi = \Psi$, $\mathcal{X}_2 \psi$, $\mathcal{X}_4 \psi$.

Cosmological background value $\Psi$ assumed to be constant.

Relation to gauge-invariant scalar field perturbations $\mathcal{X}_2 \psi$ and $\mathcal{X}_4 \psi$:

$\mathcal{X}_2 \psi = 2 \psi$, $\mathcal{X}_4 \psi = 4 \psi + \left( \partial_i^2 \mathcal{X}_2 \psi \right)$.

Taylor expansion of free function $\omega$ around cosmological background value:

$\omega_0 = \omega(\Psi)$, $\omega_1 = \omega'(\Psi)$.

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- Relevant components of scalar field: $\mathcal{X}_{}^0 \psi = \Psi, \mathcal{X}_{}^2 \psi, \mathcal{X}_{}^4 \psi$.
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Consider arbitrary gauge $\mathcal{X}$ to pull equations and fields to reference spacetime $M_0$.

- Relevant components of scalar field: $\mathcal{X}^0 \psi = \psi$, $\mathcal{X}^2 \psi$, $\mathcal{X}^4 \psi$.

- Cosmological background value $\Psi$ assumed to be constant.

- Relation to gauge-invariant scalar field perturbations $\psi$ and $\bar{\psi}$:

$$\mathcal{X}^2 \psi = \bar{\psi}, \quad \mathcal{X}^4 \psi = \psi + (\partial_i \mathcal{X}^\iota + \mathcal{X}_i^\iota) \bar{\psi}_{,i}.$$

Taylor expansion of free function $\omega$ around cosmological background value:

$$\omega_0 = \omega(\Psi), \quad \omega_1 = \omega'(\Psi).$$
Perturbative solution ansatz

- Consider arbitrary gauge $\mathcal{X}$ to pull equations and fields to reference spacetime $M_0$.
- Relevant components of scalar field: $\mathcal{X}^0\psi = \psi$, $\mathcal{X}^2\psi$, $\mathcal{X}^4\psi$.
- Cosmological background value $\Psi$ assumed to be constant.
- Relation to gauge-invariant scalar field perturbations $\psi$ and $\tilde{\psi}$:
  \[ \mathcal{X}^2\psi = \tilde{\psi} , \quad \mathcal{X}^4\psi = \psi + (\partial_i \mathcal{X}^\gamma + \mathcal{X}^\gamma_i) \psi, \]
- Taylor expansion of free function $\omega$ around cosmological background value:
  \[ \omega_0 = \omega(\Psi) , \quad \omega_1 = \omega'(\Psi) . \]
Perturbative solution ansatz

- Consider arbitrary gauge $\mathcal{X}$ to pull equations and fields to reference spacetime $M_0$.
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- Cosmological background value $\Psi$ assumed to be constant.
- Relation to gauge-invariant scalar field perturbations $\psi^2$ and $\psi^4$:
  $$\mathcal{X}^2_\psi = \psi^2, \quad \mathcal{X}^4_\psi = \psi^4 + (\partial_i \mathcal{X}^\psi + \mathcal{X}^\psi_i) \psi_{,i}^2.$$
- Taylor expansion of free function $\omega$ around cosmological background value:
  $$\omega_0 = \omega(\Psi), \quad \omega_1 = \omega'(\Psi).$$

$\Rightarrow$ Zeroth order $\mathcal{X}^0_\psi = \psi, \mathcal{X}^0 g_{\mu\nu} = \eta_{\mu\nu}$ solves (vacuum) field equations.
Perturbative solution ansatz

- Consider arbitrary gauge $\mathcal{X}$ to pull equations and fields to reference spacetime $M_0$.
- Relevant components of scalar field: $\mathcal{X}_0 \psi = \psi$, $\mathcal{X}_2 \psi$, $\mathcal{X}_4 \psi$.
- Cosmological background value $\Psi$ assumed to be constant.
- Relation to gauge-invariant scalar field perturbations $\tilde{\psi}$ and $\tilde{\psi}$:

$$\mathcal{X}^0 \psi = \tilde{\psi}, \quad \mathcal{X}^2 \psi = \psi + (\partial_i \mathcal{X}^0 + \mathcal{X}^i_2) \psi,i.$$

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$\Rightarrow$ Remaining equations determine gauge-invariant metric components.
Post-Newtonian metric and PPN parameters

- Normalization of the gravitational constant:

\[ \kappa^2 = 4\pi \psi \frac{2\omega_0 + 3}{\omega_0 + 2}. \]
Post-Newtonian metric and PPN parameters

- Normalization of the gravitational constant:

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⇒ Metric in terms of PPN potentials:

\[
\begin{align*}
\mathbf{g}^* &= 2U, \\
\mathbf{g}^* &= 2\frac{\omega_0 + 1}{\omega_0 + 2}U, \\
\mathbf{g}^t &= 0, \\
\mathbf{g}^\phi &= -\frac{2\omega_0 + 3}{\omega_0 + 2}(V_i + W_i), \\
\mathbf{g}^4 &= \frac{3\omega_0 + 4}{\omega_0 + 2}(\mathbf{A} + \mathbf{B}) + \Phi_1 + \left(4\frac{\omega_0 + 2}{\omega_0 + 2} - \frac{\omega_1 \Psi}{(2\omega_0 + 3)(\omega_0 + 2)^2}\right)\Phi_2 \\
&\quad + 3\Phi_3 + 6\frac{\omega_0 + 1}{\omega_0 + 2}\Phi_4 - 2\left(1 + \frac{\omega_1 \Psi}{4(2\omega_0 + 3)(\omega_0 + 2)^2}\right)U^2.
\end{align*}
\]
Post-Newtonian metric and PPN parameters

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⇒ Metric in terms of PPN potentials:

\[ g^* = 2U, \quad g^\omega = 2\omega_0 + 1 \frac{U}{\omega_0 + 2}, \quad g^{\uparrow} = 0, \quad g^\phi = -\frac{2\omega_0 + 3}{\omega_0 + 2} (V_i + W_i), \]

\[ g^4 = \frac{3\omega_0 + 4}{\omega_0 + 2} (\mathbf{A} + \mathbf{B}) + \Phi_1 + \left( \frac{4\omega_0 + 2}{\omega_0 + 2} - \frac{\omega_1 \psi}{(2\omega_0 + 3)(\omega_0 + 2)^2} \right) \Phi_2 \]

\[ + 3 \Phi_3 + 6 \frac{\omega_0 + 1}{\omega_0 + 2} \Phi_4 - 2 \left( 1 + \frac{\omega_1 \psi}{4(2\omega_0 + 3)(\omega_0 + 2)^2} \right) U^2. \]

⇒ PPN parameters reproduce well-known result: [Nordtvedt '70]

\[ \gamma = \frac{\omega_0 + 1}{\omega_0 + 2}, \quad \beta = 1 + \frac{\omega_1 \psi}{4(2\omega_0 + 3)(\omega_0 + 2)^2}, \quad \alpha_1 = \alpha_2 = \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \xi = 0. \]
Outline

1 Introduction
2 Gauge-invariant higher order perturbations
3 Parametrized post-Newtonian formalism
4 Gauge-invariant PPN formalism
5 Example: PPN limit of scalar-tensor gravity
6 Conclusion
Summary

- **Gauge-invariant perturbation theory:**
  - Distinguish between physical and background spacetime.
  - Gauge pulls physical metric to background spacetime.
  - Gauge dependent comparison between both metrics.
  - Decompose perturbations into physical data and gauge data.

- **Parametrized post-Newtonian formalism:**
  - Weak-field approximation of metric gravity theories.
  - Characterizes gravity theories by 10 (constant) parameters.
  - Parameters closely related to solar system observations.

- **Gauge-invariant PPN formalism:**
  - Apply gauge-invariant perturbation theory to PPN formalism.
  - Decomposition of metric and field equations.
  - Avoids issues arising from necessity to choose a gauge.
  - Simpler set of equations to determine PPN parameters.
  - Can also be formulated for tetrad instead of metric.
Summary

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  - Decomposition of metric and field equations.
  - Avoids issues arising from necessity to choose a gauge.
  - Simpler set of equations to determine PPN parameters.
  - Can also be formulated for tetrad instead of metric.
Summary

- **Gauge-invariant perturbation theory:**
  - Distinguish between physical and background spacetime.
  - Gauge pulls physical metric to background spacetime.
  - Gauge dependent comparison between both metrics.
  - Decompose perturbations into physical data and gauge data.

- **Parametrized post-Newtonian formalism:**
  - Weak-field approximation of metric gravity theories.
  - Characterizes gravity theories by 10 (constant) parameters.
  - Parameters closely related to solar system observations.

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- **Post-Newtonian limit of scalar-tensor gravity:**
  - Perturbative field equations simplify in gauge-invariant formulation.
  - Consistency check: obtain well-known PPN parameters.
  - Also possible to use tetrad formulation to calculate solution.
Outlook

- Extend formalism by including higher perturbation orders:
  - General covariant expansion instead of space-time split.
  - Allow also for fast-moving source masses.
  - Consider inspiral phase of black hole merger event.
  - Devise method for calculating gravitational waves.

- Further possible extensions and modifications:
  - Apply to formalism based on modified density $\rho^* = \rho \sqrt{-g_{00}}$.
    - [Will '18]
  - Allow for multiple dynamical metrics / tetrads.
  - Take into account Vainshtein or other screening mechanisms.
  - Consider cosmological background and time variability of PPN parameters.
  - Include further PPN potentials appearing in higher derivative theories.

- Apply formalism to complicated gravity theories:
  - Bimetric and multimetric gravity theories.
  - Multi-scalar Horndeski generalizations.
  - Theories involving generalized Proca fields.
  - Extensions based on metric-affine geometry.
  - Extensions of teleparallel and symmetric teleparallel gravity.
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Further reading

MH,
“Gauge invariant approach to the parametrized post-Newtonian formalism”,