Cosmological backgrounds and their perturbations in teleparallel gravity

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Metric-affine geometry and spacetime symmetries

- Fundamental fields in metric-affine geometry:
  - Metric tensor $g_{\mu\nu}$.
  - Connection with coefficients $\Gamma^{\mu}{}_{\nu\rho}$.

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  - Curvature:

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}{}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\tau\rho}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\mu}{}_{\tau\sigma}\Gamma^{\tau}{}_{\nu\rho} \,. \tag{1}$$

  - Torsion:

$$T^{\mu}{}_{\nu\rho} = \Gamma^{\mu}{}_{\rho\nu} - \Gamma^{\mu}{}_{\nu\rho} \,. \tag{2}$$

  - Nonmetricity:

$$Q_{\mu\nu\rho} = \nabla_{\mu}g_{\nu\rho} = \partial_{\mu}g_{\nu\rho} - \Gamma^{\sigma}{}_{\nu\mu}g_{\sigma\rho} - \Gamma^{\sigma}{}_{\rho\mu}g_{\nu\sigma} \,. \tag{3}$$

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- Some special classes of connections used in gravity theory:
  - Levi-Civita connection: $T = Q = 0$.
  - Metric teleparallelism: $R = Q = 0$.
  - Symmetric teleparallelism: $R = T = 0$.
  - General teleparallelism: $R = 0$.

Generators of cosmological symmetry

- Symmetry under action of a vector field X^μ:
  - Metric:

$$0 = (\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + \partial_\mu X^\rho g_{\rho\nu} + \partial_\nu X^\rho g_{\mu\rho} \,. \tag{4}$$

  - Connection coefficients:

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- Generating vector fields:
  - Rotations:

$$R_1 = \sin\varphi \partial_\vartheta + \frac{\cos\varphi}{\tan\vartheta} \partial_\varphi \,, \tag{6a}$$

$$R_2 = -\cos\varphi \partial_\vartheta + \frac{\sin\varphi}{\tan\vartheta} \partial_\varphi \,, \tag{6b}$$

$$R_3 = -\partial_\varphi \,, \tag{6c}$$

  - Translations:

$$T_1 = \chi \sin\vartheta \cos\varphi \partial_r + \frac{\chi}{r} \cos\vartheta \cos\varphi \partial_\vartheta - \frac{\chi \sin\varphi}{r \sin\vartheta} \partial_\varphi \,, \tag{7a}$$

$$T_2 = \chi \sin\vartheta \sin\varphi \partial_r + \frac{\chi}{r} \cos\vartheta \sin\varphi \partial_\vartheta + \frac{\chi \cos\varphi}{r \sin\vartheta} \partial_\varphi \,, \tag{7b}$$

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Cosmologically symmetric metric-affine geometry

1. Most general metric with cosmological symmetry:
   - Metric in space-time split:
   $$g_{\mu\nu} = -n_\mu n_\nu + h_{\mu\nu}\,. \tag{8}$$
   - Unit normal covector field:
   $$n_\mu \mathrm{d}x^\mu = -N\,\mathrm{d}t\,. \tag{9}$$
   - Spatial metric (gives projection onto spatial slices):
   $$h_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = A^2\left[\frac{\mathrm{d}r \otimes \mathrm{d}r}{\chi^2} + r^2(\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta \mathrm{d}\varphi \otimes \mathrm{d}\varphi)\right]\,. \tag{10}$$

⇒ Metric depends on lapse $N(t)$ and scale factor $A(t)$.

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   $\Rightarrow$ Metric depends on lapse $N(t)$ and scale factor $A(t)$.
2. Most general affine connection with cosmological symmetry:
   - Connection characterized by cosmologically symmetric torsion and nonmetricity:
   $$T^\mu{}_{\nu\rho} = \frac{2}{A}(\mathcal{T}_1 h^\mu_{[\nu} n_{\rho]} + \mathcal{T}_2 n_\sigma \varepsilon^{\sigma\mu}{}_{\nu\rho}) \,, \quad Q_{\rho\mu\nu} = \frac{2}{A}(\mathcal{Q}_1 n_\rho n_\mu n_\nu + 2\mathcal{Q}_2 n_\rho h_{\mu\nu} + 2\mathcal{Q}_3 h_{\rho(\mu} n_{\nu)}) \,. \tag{11}$$

   $\Rightarrow$ Connection depends on five free functions $\mathcal{T}_1(t), \mathcal{T}_2(t), \mathcal{Q}_1(t), \mathcal{Q}_2(t), \mathcal{Q}_3(t)$.

Cosmologically symmetric teleparallel geometry

- Define time derivatives and Hubble parameters:

$$F' = \frac{A}{N}\frac{\mathrm{d}F}{\mathrm{d}t}\,, \quad \mathcal{H} = \frac{A'}{A}\,, \qquad \dot{F} = \frac{1}{N}\frac{\mathrm{d}F}{\mathrm{d}t}\,, \quad H = \frac{\dot{A}}{A}\,. \tag{12}$$

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- $R^{\mu}{}_{\nu\rho\sigma} = 0$ if and only if:

$$(\mathcal{Q}_1 + \mathcal{Q}_2)(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2) + (\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2)' = 0\,, \tag{13a}$$

$$\mathcal{T}_2(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2) = 0\,, \tag{13b}$$

$$(\mathcal{Q}_1 + \mathcal{Q}_2)(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3) - (\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3)' = 0\,, \tag{13c}$$

$$\mathcal{T}_2(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3) = 0\,, \tag{13d}$$

$$(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2)(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3) - \mathcal{T}_2^2 + u^2 = 0\,, \tag{13e}$$

$$\mathcal{T}_2' = 0\,. \tag{13f}$$

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$\Rightarrow$ Different branches of solutions for $u = 0$ and $u \neq 0$.

Solution branches

$$\mathcal{T}_2 = u \neq 0$$
$$\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 = 0$$
$$\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3 = 0$$

$$u = 0$$
$$\mathcal{T}_2 = 0$$
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$$\mathcal{Q}_1 + \mathcal{Q}_2 = -\frac{(\mathcal{H}-\mathcal{T}_1+\mathcal{Q}_2)'}{\mathcal{H}-\mathcal{T}_1+\mathcal{Q}_2}$$

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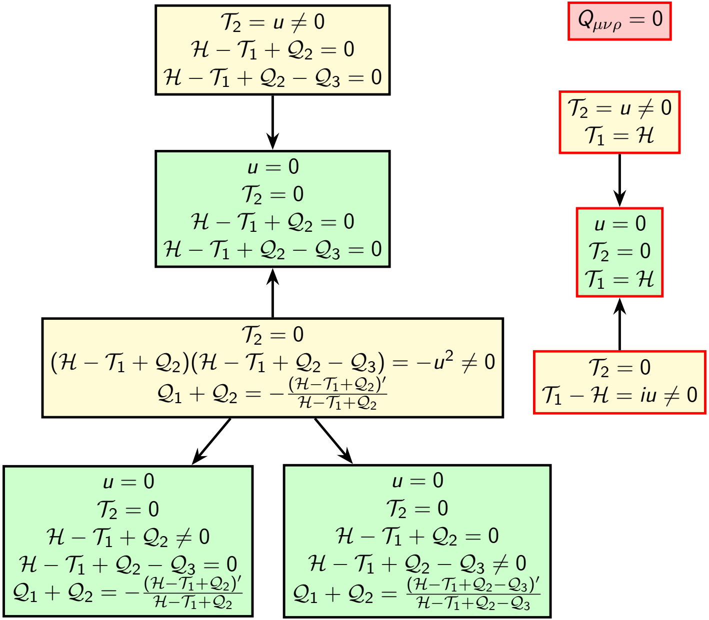
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Solution branches

$$T^{\mu}{}_{\nu\rho} = 0$$

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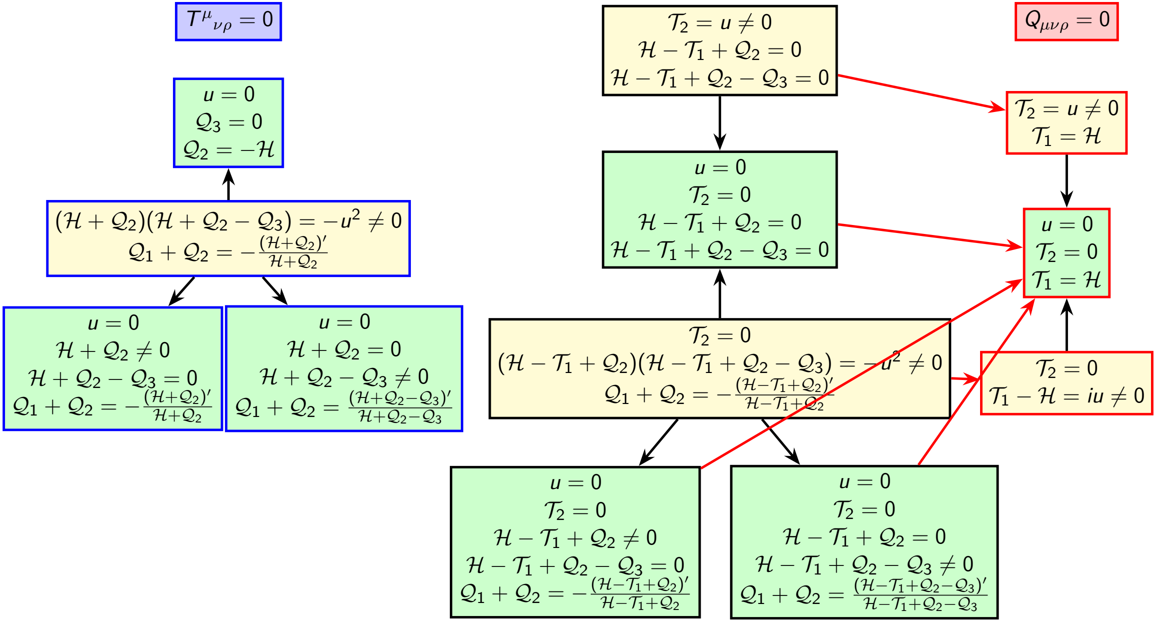

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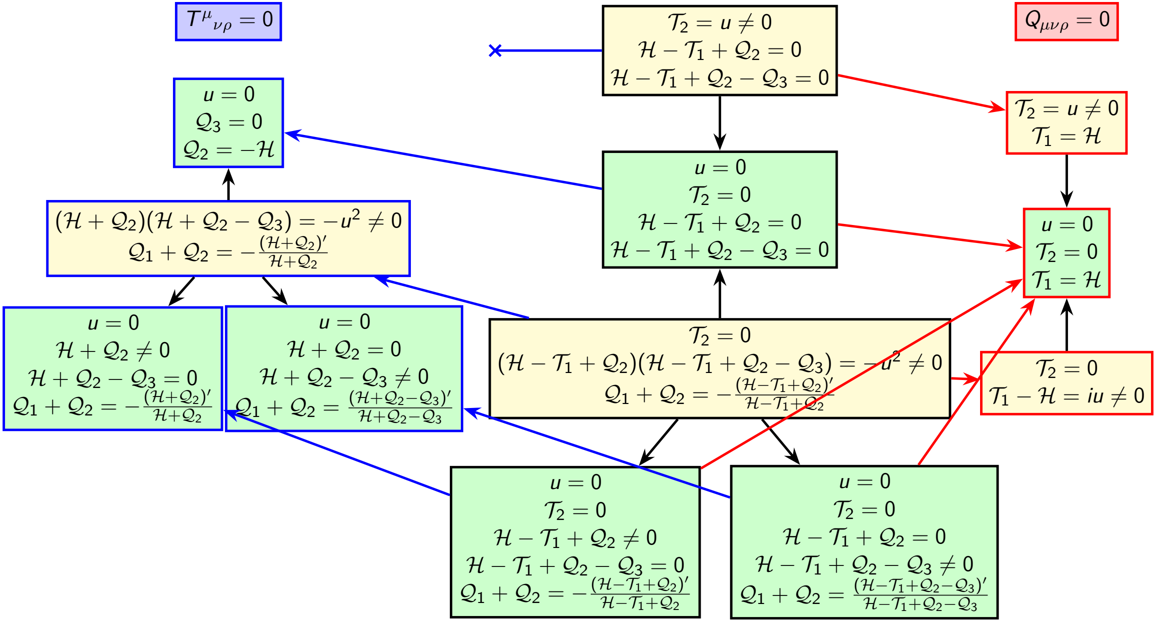

- Consider linear perturbation $\delta g_{\mu\nu}, \delta\Gamma^{\mu}{}_{\nu\rho}$ around background $\bar{g}_{\mu\nu}, \bar{\Gamma}^{\mu}{}_{\nu\rho}$:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,, \quad \Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\rho}\,. \tag{14}$$

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- Preserve flatness of affine connection ($\delta R^{\mu}{}_{\nu\rho\sigma} = 0$):

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- Further conditions in metric and symmetric teleparallel gravity:
  - Metric case $\delta Q_{\mu\nu\rho} = 0$:

$$\lambda_{\mu\nu} + \lambda_{\nu\mu} = \delta g_{\mu\nu} \quad \Rightarrow \quad \lambda_{\mu\nu} = \frac{1}{2}(\delta g_{\mu\nu} + a_{\mu\nu})\,. \tag{16}$$

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  - Symmetric case $\delta T^{\mu}{}_{\nu\rho} = 0$:

$$\nabla_{[\rho}\lambda^{\mu}{}_{\nu]} = 0 \quad \Rightarrow \quad \lambda^{\mu}{}_{\nu} = \nabla_{\nu}\zeta^{\mu}\,. \tag{17}$$

Irreducible decomposition of perturbations

- Algebraic $3+1$ split of perturbation tensor fields:
  - Metric: $\widehat{\delta g}_{00}, \widehat{\delta g}_{0a}, \widehat{\delta g}_{ab}$.
  - General teleparallel: $\hat{\lambda}_{00}, \hat{\lambda}_{0a}, \hat{\lambda}_{a0}, \hat{\lambda}_{ab}$.
  - Metric teleparallel: $\hat{a}_{0a}, \hat{a}_{ab}$.
  - Symmetric teleparallel $\hat{\zeta}_0, \hat{\zeta}_a$.

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- Differential decomposition of spatial algebraic components:
  - Vector $U_a = \mathrm{d}_a\tilde{U} + \hat{U}_a$, $\mathrm{d}_a\hat{U}^a = 0 \rightsquigarrow$ scalar + divergence-free vector.
  - Symmetric tensor $U_{ab} = \bar{U}\gamma_{ab} + (\mathrm{d}_a\mathrm{d}_b - \gamma_{ab}\triangle/3)\tilde{U} + \mathrm{d}_{(a}\hat{U}_{b)} + \check{U}_{ab}$.
  - Antisymmetric tensor $U^{ab} = \upsilon^{abc}(\mathrm{d}_c\tilde{U} + \hat{U}_c)$.

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$\Rightarrow$ Number of irreducible components:

	scalar	pseudoscalar	vector	pseudovector	tensor
$\delta g_{\mu\nu}$	4	0	2	0	1
$\lambda_{\mu\nu}$	5	1	3	1	1
$a_{\mu\nu}$	1	1	1	1	0
ζ_μ	2	0	1	0	0

Infinitesimal coordinate transformations

- Transformation of perturbations under coordinate changes:
  - Fields transform under infinitesimal coordinate change $x'^{\mu} = x^{\mu} + X^{\mu}(x)$:

$$g_{\mu\nu} - g'_{\mu\nu} = (\mathcal{L}_X g)_{\mu\nu}\,, \quad \Gamma^{\mu}{}_{\nu\rho} - \Gamma'^{\mu}{}_{\nu\rho} = (\mathcal{L}_X \Gamma)^{\mu}{}_{\nu\rho}\,. \tag{18}$$

  - Linear perturbation expansion of fields around common background:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}\,, \qquad \Gamma^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\rho}\,, \tag{19a}$$

$$g'_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g'_{\mu\nu}\,, \qquad \Gamma'^{\mu}{}_{\nu\rho} = \bar{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma'^{\mu}{}_{\nu\rho}\,. \tag{19b}$$

  - Consider X^{μ} to be of same order as linear perturbations:

$$\delta_X \delta g_{\mu\nu} = \delta g_{\mu\nu} - \delta g'_{\mu\nu} = (\mathcal{L}_X \bar{g})_{\mu\nu}\,, \quad \delta_X \delta\Gamma^{\mu}{}_{\nu\rho} = \delta\Gamma^{\mu}{}_{\nu\rho} - \delta\Gamma'^{\mu}{}_{\nu\rho} = (\mathcal{L}_X \bar{\Gamma})^{\mu}{}_{\nu\rho}\,. \tag{20}$$

Infinitesimal coordinate transformations

- Transformation of perturbations under coordinate changes:
  - Fields transform under infinitesimal coordinate change $x'^{\mu} = x^{\mu} + X^{\mu}(x)$:

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- Transformation of connection perturbations:
  - Use Lie derivative of flat connection:

$$(\mathcal{L}_X \bar{\Gamma})^{\mu}{}_{\nu\rho} = \bar{\nabla}_{\rho}\bar{\nabla}_{\nu}X^{\mu} - \bar{\nabla}_{\rho}(X^{\sigma}\,\bar{T}^{\mu}{}_{\nu\sigma})\,. \tag{21}$$

  ⇒ Transformation of perturbation tensor fields:

$$\delta_X \lambda^{\mu}{}_{\nu} = \lambda^{\mu}{}_{\nu} - \lambda'^{\mu}{}_{\nu} = \bar{\nabla}_{\nu}X^{\mu} - X^{\sigma}\,\bar{T}^{\mu}{}_{\nu\sigma}\,. \tag{22}$$

- Perform 3 + 1 decomposition of coordinate transformation:
  - Metric transformation:

$$\begin{aligned} A\delta_X\widehat{\delta g}_{00} &= 2\hat{X}'_{\perp}\,, && \text{(23a)}\\ A\delta_X\widehat{\delta g}_{a0} &= \mathrm{d}_a\hat{X}_{\perp} + \mathrm{d}_a\hat{X}'_{\parallel} + \hat{Z}'_a - \mathcal{H}(\mathrm{d}_a\hat{X}_{\parallel} + \hat{Z}_a)\,, && \text{(23b)}\\ A\delta_X\widehat{\delta g}_{ab} &= 2\mathrm{d}_a\mathrm{d}_b\hat{X}_{\parallel} + 2\mathrm{d}_{(a}\hat{Z}_{b)} - 2\mathcal{H}\hat{X}_{\perp}\gamma_{ab}\,. && \text{(23c)} \end{aligned}$$

  - Connection transformation:

$$\begin{aligned} A\delta_X\hat{\lambda}_{00} &= \hat{X}'_{\perp} - \mathcal{Q}_1\hat{X}_{\perp}\,, && \text{(24a)}\\ A\delta_X\hat{\lambda}_{0b} &= \mathrm{d}_b\hat{X}_{\perp} - (\mathcal{H} + \mathcal{Q}_2 - \mathcal{Q}_3 - \mathcal{T}_1)(\mathrm{d}_b\hat{X}_{\parallel} + \hat{Z}_b)\,, && \text{(24b)}\\ A\delta_X\hat{\lambda}_{a0} &= \mathrm{d}_a\hat{X}'_{\parallel} + \hat{Z}'_a + (\mathcal{Q}_2 - \mathcal{T}_1)(\mathrm{d}_a\hat{X}_{\parallel} + \hat{Z}_a)\,, && \text{(24c)}\\ A\delta_X\hat{\lambda}_{ab} &= \mathrm{d}_b(\mathrm{d}_a\hat{X}_{\parallel} + \hat{Z}_a) - (\mathcal{H} + \mathcal{Q}_2)\hat{X}_{\perp}\gamma_{ab} - \mathcal{T}_2\upsilon_{abc}(\mathrm{d}^c\hat{X}_{\parallel} + \hat{Z}^c)\,. && \text{(24d)} \end{aligned}$$

$3+1$ split and gauge transformations

- Perform $3+1$ decomposition of coordinate transformation:
  - Metric transformation:

$$A\delta_X\widehat{\delta g}_{00} = 2\hat{X}'_{\perp}\,, \tag{23a}$$

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$$A\delta_X\widehat{\delta g}_{ab} = 2\mathrm{d}_a\mathrm{d}_b\hat{X}_{\parallel} + 2\mathrm{d}_{(a}\hat{Z}_{b)} - 2\mathcal{H}\hat{X}_{\perp}\gamma_{ab}\,. \tag{23c}$$

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$\rightsquigarrow$ Further decompose into transformation of irreducible components.

Gauge fixing and gauge-invariant variables

- Construction of gauge-invariant quantities for gauge G:
  - Decompose irreducible components into gauge-invariant and gauge-dependent part:

$$\hat{Y} = \underset{\mathrm{G}}{\hat{\boldsymbol{Y}}} + \delta_{\underset{\mathrm{G}}{\hat{X}}} \hat{Y} \,. \tag{25}$$

  - Gauge condition fixing $\underset{\mathrm{G}}{\hat{\boldsymbol{Y}}}$ $\Leftrightarrow$ gauge transformation $\underset{\mathrm{G}}{\hat{X}}$ from arbitrary gauge:

$$0 = \underset{\mathrm{G}}{\hat{C}}(\underset{\mathrm{G}}{\hat{\boldsymbol{Y}}}) = \underset{\mathrm{G}}{\hat{C}}(\hat{Y} - \delta_{\underset{\mathrm{G}}{\hat{X}}} \hat{Y}) \quad \Leftrightarrow \quad \underset{\mathrm{G}}{\hat{X}} = \underset{\mathrm{G}}{\hat{f}}(\hat{Y}) \,. \tag{26}$$

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- Number of independent components:
  - n perturbation components $\hat{Y}$ before gauge fixing.
  - 4 components of gauge-defining vector field $\underset{\mathrm{G}}{\hat{X}}$.
  - 4 gauge conditions $\underset{\mathrm{G}}{\hat{C}}$.

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- Example: coincident (perturbation) gauge:

$$(\underset{\mathrm{C}}{\hat{\boldsymbol{\zeta}}}_0, \underset{\mathrm{C}}{\hat{\boldsymbol{\zeta}}}_a) \equiv 0 \quad \Leftrightarrow \quad (\underset{\mathrm{C}}{\hat{X}}_0, \underset{\mathrm{C}}{\hat{X}}_a) = (\hat{\zeta}_0, \hat{\zeta}_a) \,. \tag{27}$$

Example: equivalent branches of $f(X)$ theories

- Consider similarly constructed gravity theories:

$$\int_M \frac{f(Q)}{2\kappa^2}\sqrt{-g}\mathrm{d}^4x \longleftarrow \int_M \frac{f(G)}{2\kappa^2}\sqrt{-g}\mathrm{d}^4x \longrightarrow \int_M \frac{f(T)}{2\kappa^2}\sqrt{-g}\mathrm{d}^4x$$

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- Consider flat branch of cosmological teleparallel geometries:

$$\begin{array}{c} u = 0 \\ \mathcal{Q}_3 = 0 \\ \mathcal{Q}_2 = -\mathcal{H} \end{array} \longleftarrow \begin{array}{c} u = 0 \\ \mathcal{T}_2 = 0 \\ \mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 = 0 \\ \mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3 = 0 \end{array} \longrightarrow \begin{array}{c} u = 0 \\ \mathcal{T}_2 = 0 \\ \mathcal{T}_1 = \mathcal{H} \end{array}$$

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⇒ Gravity scalars agree: $G = T = Q = -6H^2$.

⇒ Identical dynamics for cosmological background evolution:

$$\kappa^2\rho = -\frac{1}{2}f + 6f'H^2\,, \tag{28a}$$

$$\kappa^2 p = \frac{1}{2}f - 2f'(\dot{H} + 3H^2) - 24f''H^2\dot{H}\,. \tag{28b}$$

Example: inequivalent branches of $f(X)$ theories

- Gravity scalars:

$$G = \frac{3}{A^2}[2\mathcal{T}_2^2 - 2(\mathcal{Q}_2 - \mathcal{T}_1)^2 - \mathcal{Q}_3(\mathcal{Q}_1 - \mathcal{Q}_2 + 2\mathcal{T}_1)]\,, \tag{29a}$$

$$Q = -\frac{3}{A^2}[2\mathcal{Q}_2^2 + \mathcal{Q}_3(\mathcal{Q}_1 - \mathcal{Q}_2)]\,, \tag{29b}$$

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$\Rightarrow$ G, Q, T depend on u and further scalar functions of time:

- metric teleparallel: different dynamics for axial and vector torsion branches.
- symmetric teleparallel: 1 additional scalar function of time.
- general teleparallel: 2 additional scalar functions of time.

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↯ Some background scalars decouple, but enter in perturbations.

Example: tensor perturbations in $f(X)$ theories

- General form of tensor propagation equation:

$$2\kappa^2 A^2 \hat{\mathcal{T}}_{ab} = f' \left(\triangle \hat{q}_{ab} - 2u^2 \hat{q}_{ab} - 2\mathcal{H}\hat{q}'_{ab} - \hat{q}''_{ab}\right) + \frac{f''}{A^2}\Delta \,. \tag{30}$$

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- Modification depends on teleparallel class and cosmological background branch:

$$\Delta = 3[4\mathcal{H}^2(\mathcal{H} - \mathcal{K}) - 2(2\mathcal{H} - \mathcal{K})\mathcal{K}' + \mathcal{K}'']\hat{q}'_{ab}$$

$$\Delta = 3\{4\mathcal{H}\mathcal{K}(\mathcal{H} - \mathcal{K})(\mathcal{H}\mathcal{K} + u^2) + 2\mathcal{K}[(\mathcal{K}^2 - 2\mathcal{H}\mathcal{K} - u^2)\mathcal{H}' + (u^2 + \mathcal{K}^2)\mathcal{K}'']\}(\mathcal{K}\hat{q}'_{ab} - 2u^2\hat{q}_{ab})$$

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⇒ Modification of Hubble friction $\hat{q}'_{ab}$ and curvature $\hat{q}_{ab}$ terms.

Summary

- Cosmologically symmetric teleparallel background geometry:
  - Metric takes familiar Robertson-Walker form.
  - Different branches for flat connection:

	general	symmetric	metric
spatially flat	3	3	1
spatially curved	2	1	2
scalar functions	2	1	0

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  - $3 + 1$-split and irreducible decomposition.
  - Gauge transformations: universal prescription for gauge-invariant variables.
- Application to teleparallel gravity (example):
  - $f(G)$, $f(T)$, $f(Q)$ yield same cosmological dynamics on one branch.
  - $f(G)$, $f(T)$, $f(Q)$ cosmological dynamics differ for other branches.
  - Strong coupling problem in $f(T)$ gravity.
  - Even stronger coupling problem in $f(Q)$ and $f(G)$?
  - Modified Hubble friction terms in tensor perturbation equations.

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