Tetrads and spacetime symmetries in $f(T)$ gravity

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Motivation

- Open questions in cosmology and gravity:
  - Accelerating phases in the history of the Universe?
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  - Based on tetrad and flat spin connection.
  - Describes gravity as gauge theory of the translation group.
  - First order action, second order field equations.
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  - Simple class of teleparallel theories beyond general relativity.
  - Cosmology typically features de Sitter attractors [MH, Järv, Ualikhanova '17].
  - ↯ Cumbersome equation relating tetrad and spin connection.
  - Use notion of symmetry to find particular solutions?

Teleparallel geometry

- Fields required to define the geometry:
  - tetrad $e^a = e^a{}_\mu dx^\mu$,
  - spin connection $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu$.
- Spin connection chosen to be flat:

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = 0\,.$$

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- Derived quantities:
  - Metric: $g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu$.
  - Spacetime connection: $\Gamma^\rho{}_{\mu\nu} = e_a{}^\rho \left(\partial_\nu e^a{}_\mu + \omega^a{}_{b\nu} e^b{}_\mu\right)$.
  - Torsion: $T^\rho{}_{\mu\nu} = \Gamma^\rho{}_{\nu\mu} - \Gamma^\rho{}_{\mu\nu}$.
  - Gauge covariant derivative:
    $D_\mu e^a{}_\nu = \partial_\mu e^a{}_\nu - \Gamma^\rho{}_{\nu\mu} e^a{}_\rho + \omega^a{}_{b\mu} e^b{}_\nu = 0$.

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- Local Lorentz (gauge) invariance with transformation $\Lambda^a{}_b$:

$$e^a{}_\mu \mapsto \Lambda^a{}_b e^b{}_\mu\,, \quad \omega^a{}_{b\mu} \mapsto \Lambda^a{}_c \Lambda_b{}^d \omega^c{}_{d\mu} + \Lambda^a{}_c \partial_\mu \Lambda_b{}^c\,.$$

Symmetry of the geometry

- Diffeomorphisms generated by vector field ξ.
- Invariance of spacetime geometry:
  - Metric:

$$0 = (\mathcal{L}_\xi g)_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho} \,.$$

  - Connection:

$$0 = (\mathcal{L}_\xi \Gamma)^\mu{}_{\nu\rho} = \xi^\sigma \partial_\sigma \Gamma^\mu{}_{\nu\rho} - \partial_\sigma \xi^\mu \Gamma^\sigma{}_{\nu\rho} + \partial_\nu \xi^\sigma \Gamma^\mu{}_{\sigma\rho} + \partial_\rho \xi^\sigma \Gamma^\mu{}_{\nu\sigma} + \partial_\nu \partial_\rho \xi^\mu \,.$$

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- Satisfied if and only if $\exists \lambda : M \to \mathfrak{so}(1,3)$ such that [MH '15]

$$(\mathcal{L}_\xi e)^a{}_\mu = -\lambda^a{}_b e^b{}_\mu \,, \quad (\mathcal{L}_\xi \omega)^a{}_{b\mu} = D_\mu \lambda^a{}_b \,.$$

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- Several symmetry generators ξ form Lie algebra $\mathfrak{g} \subset \mathrm{Vect}(M)$.
- Local Lie algebra homomorphism $\boldsymbol{\lambda} : \mathfrak{g} \times M \to \mathfrak{so}(1,3)$.

Teleparallel $f(T)$ action and field equations

- Action:

$$S_g[e, \omega] = \int_M f(T) \det e \, d^4x \,.$$

- Torsion scalar:

$$T = \frac{1}{4} T^{\mu\nu\rho} T_{\mu\nu\rho} + \frac{1}{2} T^{\mu\nu\rho} T_{\rho\nu\mu} - T^{\mu}{}_{\rho\mu} T^{\nu\rho}{}_{\nu} \,.$$

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- Field equations obtained from variations δ_e and $\delta_\omega = D\lambda$.
- Antisymmetric part of δ_e equation $\equiv \delta_\omega$ equation:

$$f_{TT}\left(T^{\rho}{}_{\mu\nu}\partial_\rho T + T^{\rho}{}_{\rho\mu}\partial_\nu T - T^{\rho}{}_{\rho\nu}\partial_\mu T\right) = 0\,.$$

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- ϟ Cumbersome 2-nd order differential equation unless $f_{TT} \equiv 0$.
- Find solutions through symmetry?

Weitzenböck gauge and “good tetrads”

- Use local Lorentz invariance to choose simple spin connection.
- Weitzenböck gauge: ${\omega^a}_{b\mu} \equiv 0$.

- First order differential equation for ${e^a}_\mu$.

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“Good tetrad” [Tamanini & Böhmer ’12]

A tetrad is called *good tetrad* if it satisfies the antisymmetric part

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of the $f(T)$ field equations in the Weitzenböck gauge.

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- Symmetry condition in Weitzenböck gauge:

$$(\mathcal{L}_\xi e)^a{}_\mu = -\lambda^a{}_b e^b{}_\mu\,, \quad 0 = (\mathcal{L}_\xi\omega)^a{}_{b\mu} = D_\mu\lambda^a{}_b = \partial_\mu\lambda^a{}_b\,.$$

- First order differential equation for $e^a{}_\mu$.

$\Rightarrow$ Lie algebra homomorphism $\boldsymbol{\lambda} : \mathfrak{g} \to \mathfrak{so}(1,3)$ (independent of M).

Example: spatially flat FLRW

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- Representation: translations $\mapsto 0$, rotations $\to \mathfrak{so}(3) \subset \mathfrak{so}(1,3)$.
- Symmetry condition fixes tetrad up to $n(t), a(t)$.

$$e^a{}_\mu = \begin{pmatrix} n(t) & 0 & 0 & 0 \\ 0 & a(t)\sin\theta\cos\phi & a(t)r\cos\theta\cos\phi & -a(t)r\sin\theta\sin\phi \\ 0 & a(t)\sin\theta\sin\phi & a(t)r\cos\theta\sin\phi & a(t)r\sin\theta\cos\phi \\ 0 & a(t)\cos\theta & -a(t)r\sin\theta & 0 \end{pmatrix}.$$

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- Minkowski spacetime: $n(t) = a(t) = 1$.

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$$e^a{}_\mu = \begin{pmatrix} n(t) & 0 & 0 & 0 \\ 0 & \frac{a(t)\sin\theta\cos\phi}{\sqrt{1-r^2}} & a(t)r\left(\sqrt{1-r^2}\cos\theta\cos\phi - r\sin\phi\right) & -a(t)r\sin\theta\left(\sqrt{1-r^2}\sin\phi + r\cos\theta\cos\phi\right) \\ 0 & \frac{a(t)\sin\theta\sin\phi}{\sqrt{1-r^2}} & a(t)r\left(\sqrt{1-r^2}\cos\theta\sin\phi + r\cos\phi\right) & a(t)r\sin\theta\left(\sqrt{1-r^2}\cos\phi - r\cos\theta\sin\phi\right) \\ 0 & \frac{a(t)\cos\theta}{\sqrt{1-r^2}} & -a(t)r\sqrt{1-r^2}\sin\theta & a(t)r^2\sin^2\theta \end{pmatrix}.$$

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- de Sitter spacetime: $n(t) = 1, a(t) = \cosh t$.

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$$e^a{}_\mu = \begin{pmatrix} n(t)\sqrt{1+r^2} & \frac{a(t)r}{\sqrt{1+r^2}} & 0 & 0 \\ n(t)r\sin\theta\cos\phi & a(t)\sin\theta\cos\phi & a(t)r\cos\theta\cos\phi & -a(t)r\sin\theta\sin\phi \\ n(t)r\sin\theta\sin\phi & a(t)\sin\theta\sin\phi & a(t)r\cos\theta\sin\phi & a(t)r\sin\theta\cos\phi \\ n(t)r\cos\theta & a(t)\cos\theta & -a(t)r\sin\theta & 0 \end{pmatrix}.$$

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- Anti de Sitter spacetime: $n(t) = 1, a(t) = \cos t$.

Maximally symmetric spacetimes

- Minkowski spacetime:
  - Symmetry algebra $\mathfrak{g} \cong \mathfrak{iso}(1,3)$.
  - Repr.: translations $\mapsto 0$, Lorentz transformations $\to \mathfrak{so}(1,3)$.

$$e^a{}_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & \cos\theta & -r\sin\theta & 0 \end{pmatrix}.$$

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- de Sitter and anti-de Sitter spacetimes:
  - Symmetry algebra $\mathfrak{g} \cong \mathfrak{so}(2,3), \mathfrak{so}(1,4)$.
  - ↯ No non-trivial homomorphism $\boldsymbol{\lambda} : \mathfrak{g} \to \mathfrak{so}(1,3)$.
  - ↯ No solutions to symmetry condition!
  - ↯ Solutions to antisymmetric field equations exist (previous slides).

Conclusion

- Summary:
  - Try to find solutions of $f(T)$ gravity theories.
  - Consider symmetry of metric and connection.
  - Work in Weitzenböck gauge $\omega^a{}_{b\mu} = 0$.
  - *Some* symmetric tetrads solve antisymmetric field equations.
  - ↯ Relation does not hold for dS and AdS spacetimes.

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  - ↯ Relation does not hold for dS and AdS spacetimes.
- Outlook:
  - Relation between symmetry condition and field equations?
  - Further solutions with other symmetries?