Gravitational waves in teleparallel theories of gravity

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Tartu cosmology workshop - 6. June 2018
Outline

1. Introduction
2. Waves in torsion gravity
3. Waves in non-metricity gravity
4. Conclusion
1 Introduction

2 Waves in torsion gravity

3 Waves in non-metricity gravity

4 Conclusion
Open questions in cosmology and gravity:
- Accelerating phases in the history of the Universe?
- Relation between gravity and gauge theories?
- How to quantize gravity?
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Teleparallel gravity \cite{Moller61}:
- Based on tetrad and flat spin connection.
- Describes gravity as gauge theory of the translation group.
- First order action, second order field equations.
- Spin connection as Lorentz gauge degree of freedom.
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Symmetric teleparallel gravity [Nester, Yo '99]
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Gravity formulated as gauge theories.
Overview of geometries

- Riemann-Cartan: $Q_{\rho\mu\nu}=0$
- Minkowski: symmetric teleparallel
  - Weitzenböck: $w_{\rho\mu\nu}=0$, $w_{\rho\mu\nu}=0$
  - teleparallel: $R^\sigma{}_{\rho\mu\nu}=0$
- Riemann: $T^\lambda{}_{\mu\nu}=0$, $Q_{\rho\mu\nu}=0$
- Torsion free: $T^\lambda{}_{\mu\nu}=0$

Waves in teleparallel gravity

Tartu - 6. June 2018
Newman-Penrose formalism

- Complex double null basis of the tangent bundle:

\[
I = \partial_t + \partial_z, \quad n = \frac{\partial_t - \partial_z}{2}, \quad m = \frac{\partial_x + i\partial_y}{\sqrt{2}}, \quad \bar{m} = \frac{\partial_x - i\partial_y}{\sqrt{2}}.
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- Consider plane null wave with \( k_\mu = -\omega l_\mu \) and \( u = t - z \):

\[ h_{\mu\nu} = H_{\mu\nu} e^{ik_\mu x_\mu} = H_{\mu\nu} e^{i\omega u}. \]
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- Riemann tensor determined by “electric” components:

\[ \Psi_2 = -\frac{1}{6} R_{nlnl} = \frac{1}{12} \ddot{h}_{ll} , \quad \psi_3 = -\frac{1}{2} R_{nln\bar{m}} = \frac{1}{4} \ddot{h}_{l\bar{m}} , \quad \psi_4 = -R_{n\bar{m}n\bar{m}} = \frac{1}{2} \ddot{h}_{\bar{m}\bar{m}} , \quad \Phi_{22} = -R_{nmn\bar{m}} = \frac{1}{2} \ddot{h}_{m\bar{m}} . \]
Polarisations of gravitational waves

\[ \Psi_4, \bar{\Psi}_4, \Phi_{22}, \Psi_3, \bar{\Psi}_3, \Psi_2 \]
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Field content and geometry

- Fundamental fields in the gravity sector:
  - Coframe field \( \theta^a = \theta^a_\mu dx^\mu \).
  - Flat spin connection \( \omega^a_b = \omega^a_{b\mu} dx^\mu \).
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Derived quantities:
- Frame field $e_a = e_a^\mu \partial_\mu$ with $\iota_{e_a} \theta^b = \delta^b_a$.
- Metric $g_{\mu\nu} = \eta_{ab} \theta^a_{\mu} \theta^b_{\nu}$.
- Volume form $\theta d^4 x = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$.
- Levi-Civita connection
  \[
  \omega_{ab} = -\frac{1}{2} (\iota_{e_b} \iota_{e_c} d\theta_a + \iota_{e_c} \iota_{e_a} d\theta_b - \iota_{e_a} \iota_{e_b} d\theta_c) \theta^c.
  \]

- Torsion $T^a = d\theta^a + \omega^a_b \wedge \theta^b$. 
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- **Torsion** \( T^a = d\theta^a + \omega^a_{b} \wedge \theta^b \).

- **Gauge fixing**
  - Perform local Lorentz transformation:
    \[
    \theta'^a = \Lambda^a_b \theta^b, \quad \omega'^a_{b} = \Lambda^a_c \omega^c_d \Lambda_b^d + \Lambda^a_c d\Lambda_b^c.
    \]
  - \( \Rightarrow \) Weitzenböck gauge: set \( \omega^a_{b} \equiv 0 \).
Most general action and corresponding field equations

- Most general action:

\[
S = \frac{1}{2\kappa^2} \int d^4x \ e \left( c_1 T^\mu_\nu^\rho \ T^\nu_\mu^\rho + c_2 T^\mu_\nu^\rho \ T^\rho_\nu^\mu + c_3 T^\mu_\mu^\rho \ T^\nu_\nu^\rho \right).
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\partial_\sigma (F^{\mu\rho\sigma} + B^{\mu\rho\sigma}) = 0 .
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- Field tensors:
  - Symmetric perturbation part:

\[ F^\mu{}_{\rho\sigma} = (2c_1 + c_2) \left( \partial^\sigma \phi^{\mu\rho} - \partial^\rho \phi^{\mu\sigma} \right) + c_3 \left[ (\partial^\sigma \phi^{\alpha\alpha} - \partial_\alpha \phi^{\alpha\sigma}) \eta^{\mu\rho} - (\partial^\rho \phi^{\alpha\alpha} - \partial_\alpha \phi^{\alpha\rho}) \eta^{\mu\sigma} \right]. \]
  - Antisymmetric perturbation part:

\[ B^\mu{}_{\rho\sigma} = (2c_1 - c_2) \left( \partial^\sigma a^{\mu\rho} - \partial^\rho a^{\mu\sigma} \right) + (2c_2 + c_3) \partial^\mu a^{\sigma\rho}. \]
Newman-Penrose decomposition

Field equations expressed in Newman-Penrose basis

\begin{align*}
0 &= E_{nn} = (2c_1 + c_2 + c_3) \partial_n^2 \phi_{nl} + 2c_3 \phi_{\bar{m}n} + (2c_1 + c_2 + c_3) \partial_n^2 a_{nl}, \\
0 &= E_{mn} = (2c_1 + c_2) \partial_n^2 \phi_{ml} + (2c_1 - c_2) \partial_n^2 a_{ml}, \\
0 &= E_{\bar{m}n} = (2c_1 + c_2) \partial_n^2 \phi_{\bar{m}l} + (2c_1 - c_2) \partial_n^2 a_{\bar{m}l}, \\
0 &= E_{nm} = -c_3 \partial_n^2 \phi_{lm} - (2c_2 + c_3) \partial_n^2 a_{lm}, \\
0 &= E_{n\bar{m}} = -c_3 \partial_n^2 \phi_{l\bar{m}} - (2c_2 + c_3) \partial_n^2 a_{l\bar{m}}, \\
0 &= E_{m\bar{m}} = -c_3 \partial_n^2 \phi_{ll}, \\
0 &= E_{ln} = (2c_1 + c_2) \partial_n^2 \phi_{ll}, .
\end{align*}
Gravitational wave polarisations

\[ c_1 = \sin \theta \cos \phi \]
\[ c_2 = \sin \theta \sin \phi \]
\[ c_3 = \cos \theta \]
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  - Volume form $\sqrt{-\det g} d^4 x$.
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    \[
    \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) .
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  - Non-metricity $Q_{\rho\mu\nu} = \nabla_\rho g_{\mu\nu}$.
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- Non-metricity $Q_{\rho\mu\nu} = \nabla_\rho g_{\mu\nu}$.
- Gauge fixing
  - Perform local coordinate transformation:
    $$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}, \quad \Gamma'_{\rho\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x'^\rho}{\partial x^\gamma} \Gamma^\gamma_{\alpha\beta} + \frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu} \frac{\partial x'^\rho}{\partial x^\alpha}.$$ 
  - Coincident gauge: set $\Gamma_{\rho\mu\nu} \equiv 0 \Rightarrow Q_{\rho\mu\nu} = \partial_\rho g_{\mu\nu}$. 

Most general action and corresponding field equations

- **Most general action:**

\[
S = - \int d^4 x \sqrt{-g} \left[ c_1 Q^\alpha_{\mu\nu} + c_2 Q_{(\mu}^\alpha \nu) + c_3 Q^\alpha g_{\mu\nu} + c_4 \delta_{(\mu}^\alpha \tilde{Q}_{\nu)} + \frac{c_5}{2} \left( \tilde{Q}^\alpha g_{\mu\nu} + \delta_{(\mu}^\alpha Q_{\nu)} \right) \right]
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Linearized field equations:

\[ 0 = 2c_1 \eta^{\alpha\sigma} \partial_{\alpha} \partial_{\sigma} h_{\mu\nu} + c_2 \eta^{\alpha\sigma} \left( \partial_{\alpha} \partial_{\mu} h_{\sigma\nu} + \partial_{\alpha} \partial_{\nu} h_{\sigma\mu} \right) + 2c_3 \eta_{\mu\nu} \eta^{\tau\omega} \eta^{\alpha\sigma} \partial_{\alpha} \partial_{\sigma} h_{\tau\omega} + c_4 \eta^{\omega\sigma} \left( \partial_{\mu} \partial_{\omega} h_{\nu\sigma} + \partial_{\nu} \partial_{\omega} h_{\mu\sigma} \right) + c_5 \eta_{\mu\nu} \eta^{\omega\gamma} \eta^{\alpha\sigma} \partial_{\alpha} \partial_{\omega} h_{\sigma\gamma} + c_5 \eta^{\omega\sigma} \partial_{\mu} \partial_{\nu} h_{\omega\sigma} \]
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+ c_4 \eta^{\omega \sigma} \left( \partial_\mu \partial_\omega h_{\nu \sigma} + \partial_\nu \partial_\omega h_{\mu \sigma} \right) + c_5 \eta_{\mu \nu} \eta^{\omega \gamma} \eta^{\alpha \sigma} \partial_\alpha \partial_\omega h_{\sigma \gamma} + c_5 \eta^{\omega \sigma} \partial_\mu \partial_\nu h_{\omega \sigma}
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- Fields are tetrad and flat spin connection.
- Only torsion, no curvature or non-metricity.
- Most general theory needs 3 parameters at linearized level.

Symmetric teleparallel gravity:
- Fields are metric and flat, symmetric affine connection.
- Only non-metricity, no curvature or torsion.
- Most general theory needs 5 parameters at linearized level.

Results:
- Gravitational waves propagate at the speed of light (not shown in this talk).
- Polarisation classes $N_2$, $N_3$, III_5, II_6: tensor modes always exist, maybe more.
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Teleparallel gravity workshop

June 25-29, 2018 - Tartu, Estonia
http://hexagon.fi.tartu.ee/~telegrav2018/
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