Scalar-torsion theories of gravity

Manuel Hohmann

Laboratory of Theoretical Physics - Institute of Physics - University of Tartu
Center of Excellence “The Dark Side of the Universe”

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2 General scalar-torsion gravity
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6 Conclusion
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4 “Scalar-curvature”-like class

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6 Conclusion
Motivation

Open questions in cosmology and gravity:
- Accelerating phases in the history of the Universe?
- Relation between gravity and gauge theories?
- How to quantize gravity?
Motivation

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  - Relation between gravity and gauge theories?
  - How to quantize gravity?

- Teleparallel gravity \cite{Moller61}:
  - Based on tetrad and flat spin connection.
  - Describes gravity as gauge theory of the translation group.
  - Gravitational field strength is torsion.
  - First order action, second order field equations.
  - Spin connection as Lorentz gauge degree of freedom.

Scalar field non-minimally coupled to torsion \cite{Geng11}:
- Possibly arises from more fundamental theory.
- Differs from non-minimal coupling to curvature.
- Possible model for so far unexplained observations.

Arising questions:
- Most general class of scalar-torsion gravity theories?
- Behavior under conformal transformations?
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Ingredients of scalar-torsion gravity

Fundamental fields:
- Coframe field $\theta^a = \theta^a_\mu dx^\mu$.
- Flat spin connection $\omega^a_b = \omega^a_{b\mu} dx^\mu$.
- $N$ scalar fields $\phi = (\phi^A; A = 1, \ldots, N)$.
- Arbitrary matter fields $\chi^I$. 
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Derived quantities:
- Frame field $e^a = e^a_\mu \frac{\partial}{\partial \mu}$ with $\iota_{e^a} \theta^b = \delta^b_a$.
- Metric $g_{\mu\nu} = \eta_{ab} \theta^a_\mu \theta^b_\nu$.
- Volume form $\theta d^4x = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3$.
- Levi-Civita connection

$$\omega_{ab} = -\frac{1}{2} \left( \iota_{e^b} \iota_{e^c} d\theta^a + \iota_{e^c} \iota_{e^a} d\theta^b - \iota_{e^a} \iota_{e^b} d\theta^c \right) \theta^c.$$  

- Torsion $T^a = d\theta^a + \omega^a_b \wedge \theta^b$.  

\[ S_g \left[ \theta^a, \omega^a_{\cdot b}, \phi^A \right] + S_m \left[ \theta^a, \phi^A, \chi^I \right] \]

\[ L(T, X, Y, \phi) \]

\[ \mathcal{A}(\phi) T + B(\phi) \partial_\mu \phi \partial^\mu \phi + C(\phi) T^\mu \partial_\mu \phi + \mathcal{V}(\phi) \]

\[ f(T, \phi) + Z(\phi) \partial_\mu \phi \partial^\mu \phi \]
Structure of the action [MH '18]:

\[
S \left[ \theta^a, \omega^a_{\ b}, \phi^A, \chi^I \right] = S_g \left[ \theta^a, \omega^a_{\ b}, \phi^A \right] + S_m \left[ \theta^a, \phi^A, \chi^I \right].
\]
General scalar-torsion gravity - action

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\]

- **Variation of the action:**
  - Gravitational part:

\[
\delta S_g = \int_M \left( \Delta_a \wedge \delta \theta^a + \frac{1}{2} \Xi^b_a \wedge \delta \omega^a_b + \Phi_A \wedge \delta \phi^A \right)
\]

\[
= \int_M \left( \gamma_a \wedge \delta \theta^a + \Pi_a \wedge \delta T^a + \Phi_A \wedge \delta \phi^A \right).
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General scalar-torsion gravity - action

Structure of the action $[\text{MH} \ '18]$:

$$ S \left[ \theta^a, \omega^a_{\ b}, \phi^A, \chi^I \right] = S_g \left[ \theta^a, \omega^a_{\ b}, \phi^A \right] + S_m \left[ \theta^a, \phi^A, \chi^I \right]. $$

Variation of the action:

- Gravitational part:

$$ \delta S_g = \int_M \left( \Delta_a \wedge \delta \theta^a + \frac{1}{2} \Xi^a_{\ b} \wedge \delta \omega^a_{\ b} + \Phi_A \wedge \delta \phi^A \right), $$

$$ = \int_M \left( \Upsilon_a \wedge \delta \theta^a + \Pi_a \wedge \delta T^a + \Phi_A \wedge \delta \phi^A \right). $$

- Matter part:

$$ \delta S_m = \int_M \left( \Sigma_a \wedge \delta \theta^a + \Psi_A \wedge \delta \phi^A + \Omega_I \wedge \delta \chi^I \right). $$
General scalar-torsion gravity - field equations

- Relation between different terms used to write field equations:

\[ \Delta_a = \gamma_a - \mathcal{D}\Pi_a, \quad \Xi^{ab} = -2\Pi^{[a} \theta^{b]}, \]
\[ \Pi^a = \frac{1}{4} \epsilon_c \epsilon_b \Xi^{bc} \wedge \theta^a - \epsilon_b \Xi^{ab}, \quad \gamma^a = \Delta^a + \mathcal{D} \left( \frac{1}{4} \epsilon_c \epsilon_b \Xi^{bc} \wedge \theta^a - \epsilon_b \Xi^{ab} \right). \]
General scalar-torsion gravity - field equations

- Relation between different terms used to write field equations:

\[ \Delta_a = \gamma_a - \dot{\Delta} \Pi_a, \quad \Xi^{ab} = -2\Pi^{[a} \wedge \theta^{b]}, \]

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- Field equations:
  - Tetrad field equations:

\[ \Delta_a + \Sigma_a = 0 \Leftrightarrow \gamma_a - \dot{\Delta} \Pi_a + \Sigma_a = 0. \]
General scalar-torsion gravity - field equations

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Field equations:

- Tetrad field equations:
  \[
  \Delta_a + \Sigma_a = 0 \iff \gamma_a - \mathbf{D} \Pi_a + \Sigma_a = 0.
  \]

- Antisymmetric part \(=\) connection field equations:
  \[
  \mathbf{D}\Xi^{ab} = 0 \iff \mathbf{D}\Pi^{[a} \wedge \theta^{b]} + \Pi^{[a} \wedge T^{b]} = 0.
  \]
General scalar-torsion gravity - field equations

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- Field equations:
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  - Antisymmetric part \equiv connection field equations:

\[
\dot{\Pi}^{[a \wedge \theta^b]} + \Pi^{[a \wedge T^b]} = 0.
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- Scalar field equations: \(\Phi_A + \Psi_A = 0\).
Relation between different terms used to write field equations:

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Field equations:

- Tetrad field equations:
  $$\Delta_a + \Sigma_a = 0 \quad \Leftrightarrow \quad \gamma_a - \dot{D} \Pi_a + \Sigma_a = 0.$$

- Antisymmetric part $\equiv$ connection field equations:
  $$\dot{D} \Xi^{ab} = 0 \quad \Leftrightarrow \quad \dot{D} \Pi^{[a \wedge \theta^b]} + \Pi^{[a \wedge T^b]} = 0.$$

- Scalar field equations: $\Phi_A + \Psi_A = 0$.
- Matter field equations: $\Omega_I = 0$. 
Local Lorentz transformation of the fundamental fields:

\[ \delta \lambda \theta^a = \lambda^a_b \theta^b , \quad \delta \lambda \omega^a_b = \lambda^a_c \omega^c_b - \omega^a_c \lambda^c_b - d\lambda^a_b = -D\lambda^a_b . \]
Local Lorentz transformation of the fundamental fields:

\[ \delta_{\lambda} \theta^a = \lambda^a_b \theta^b, \quad \delta_{\lambda} \omega^a_b = \lambda^a_c \omega^c_b - \omega^a_c \lambda^c_b - d \lambda^a_b = -\mathbf{D} \lambda^a_b. \]

Local Lorentz transformation of the action:

\[ \delta_{\lambda} S_m = \int_M \sum_a \wedge (\lambda^a_b \theta^b) = \int_M [\wedge^{a \wedge \theta^b}] \lambda_{ab}, \]

\[ \delta_{\lambda} S_g = \int_M \left[ \gamma_a \wedge (\lambda^a_b \theta^b) + \Pi_a \wedge (\lambda^a_b T^b) \right] = \int_M \left( \gamma^{[a \wedge \theta^b]} + \Pi^{[a \wedge T^b]} \right) \lambda_{ab} \]

\[ = \int_M \left[ \Delta_a \wedge (\lambda^a_b \theta^b) - \frac{1}{2} \Xi_{a b} \wedge \mathbf{D} \lambda^a_b \right] = \int_M \left( \Delta^{[a \wedge \theta^b]} - \frac{1}{2} \mathbf{D} \Xi^{a b} \right) \lambda_{ab}. \]
Local Lorentz transformation of the fundamental fields:

\[ \delta_\lambda \theta^a = \lambda^a_b \theta^b, \quad \delta_\lambda \omega^a_b = \lambda^a_c \omega^c_b - \omega^a_c \lambda^c_b - d \lambda^a_b = -\hat{D} \lambda^a_b. \]

Local Lorentz transformation of the action:

\[ \delta_\lambda S_m = \int_M \Sigma_a \wedge (\lambda^a_b \theta^b) = \int_M [\Sigma^{[a} \wedge \theta^{b]}] \lambda^{ab}, \]

\[ \delta_\lambda S_g = \int_M \left[ \gamma_a \wedge (\lambda^a_b \theta^b) + \Pi_a \wedge (\lambda^a_b T^b) \right] = \int_M \left( \gamma^{[a} \wedge \theta^{b]} + \Pi^{[a} \wedge T^{b]} \right) \lambda^{ab} \]

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Consequences:

Symmetry of the energy-momentum tensor:

\[ \Sigma^{[a} \wedge \theta^{b]} = 0. \]
Local Lorentz transformation of the fundamental fields:
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\[ = \int_M \left[ \Delta_a \wedge (\lambda^a_b \theta^b) - \frac{1}{2} \Xi_a^b \wedge \mathbf{D}\lambda^a_b \right] = \int_M \left( \Delta^{[a \wedge \theta^b]} - \frac{1}{2} \mathbf{D}\Xi^{ab} \right) \lambda_{ab}. \]

Consequences:
- Symmetry of the energy-momentum tensor:
\[ \Sigma^{[a \wedge \theta^b]} = 0. \]
- Connection equations \( \equiv \) antisymmetric part of tetrad equations:
\[ \gamma^{[a \wedge \theta^b]} + \Pi^{[a \wedge T^b]} = 0, \quad \Delta^{[a \wedge \theta^b]} - \frac{1}{2} \mathbf{D}\Xi^{ab} = 0. \]
Variation of the matter action under infinitesimal diffeomorphisms $\xi$:

$$\delta_\xi S_m = \int_M \left( \Sigma_a \wedge \mathcal{L}_\xi \theta^a + \Psi_A \wedge \mathcal{L}_\xi \phi^A + \Omega_I \wedge \mathcal{L}_\xi \chi^I \right).$$
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Euler-Lagrange equations $\Omega_I = 0$ hold on-shell - what follows is valid on-shell only.
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On-shell variation of the matter action:

$$\delta_\xi S_m = \int_M \left( \Sigma_a \wedge \mathcal{L}_\xi \theta^a + \Psi_A \wedge \mathcal{L}_\xi \phi^A \right)$$

$$= \int_M \left( d\Sigma_a + \Sigma_b \wedge \dot{\omega}^b_a + \Psi_A \wedge \iota_{e_a} d\phi^A \right) \xi^a$$

$$= \int_M \left( D\Sigma_a + \Psi_A \wedge \iota_{e_a} d\phi^A \right) \xi^a.$$
Variation of the matter action under infinitesimal diffeomorphisms $\xi$:

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$$= \int_M \left( D\Sigma_a + \Psi_A \wedge \iota_{e_a} d\phi^A \right) \xi^a.$$

Energy-momentum conservation:

$$D\Sigma_a + \Psi_A \wedge \iota_{e_a} d\phi^A = 0.$$
Scalar field redefinition:

- Change of fundamental fields: \( \bar{\phi} = f(\phi) \)
Scalar field redefinition:

Change of fundamental fields: \( \bar{\phi} = f(\phi) \Rightarrow \delta \bar{\phi} = f' \delta \phi. \)
Scalar field redefinition:

- Change of fundamental fields: $\bar{\phi} = f(\phi) \Rightarrow \delta \bar{\phi} = f' \delta \phi$.
- Change of equations: $\bar{\Phi} = (f')^{-1} \Phi$, $\bar{\Psi} = (f')^{-1} \Psi$. 
General scalar-torsion gravity - transformations

- Scalar field redefinition:
  - Change of fundamental fields: \( \bar{\phi} = f(\phi) \Rightarrow \delta \bar{\phi} = f' \delta \phi. \)
  - Change of equations: \( \bar{\Phi} = (f')^{-1} \Phi, \bar{\Psi} = (f')^{-1} \Psi. \)

- Conformal transformation:
  - Change of fundamental fields: \( \bar{\theta}^a = e^{\gamma(\phi)} \theta^a \)

\[\]
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Conformal transformation:
- Change of fundamental fields: $\bar{\theta}^a = e^{\gamma(\phi)} \theta^a \Rightarrow \delta \bar{\theta}^a = e^\gamma (\delta \theta^a + \gamma' \theta^a \delta \phi)$.
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- **Conformal transformation:**
  - Change of fundamental fields: $\bar{\theta}^a = e^{\gamma(\phi)} \theta^a \Rightarrow \delta \bar{\theta}^a = e^\gamma (\delta \theta^a + \gamma' \theta^a \delta \phi)$.
  - Change of equations:
    \[
    \bar{\Sigma}_a = e^{-\gamma} \Sigma_a, \quad \bar{\Psi} = \Psi - \gamma' \Sigma_a \wedge \theta^a, \\
    \bar{\Delta}_a = e^{-\gamma} \Delta_a, \quad \bar{\Phi} = \Phi - \gamma' \Delta_a \wedge \theta^a.
    \]
Scalar field redefinition:
- Change of fundamental fields: $\bar{\phi} = f(\phi) \Rightarrow \delta \bar{\phi} = f' \delta \phi$.
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Conformal transformation:
- Change of fundamental fields: $\bar{\theta}^a = e^{\gamma(\phi)} \theta^a \Rightarrow \delta \bar{\theta}^a = e^\gamma (\delta \theta^a + \gamma' \theta^a \delta \phi)$.
- Change of equations:

\[
\begin{align*}
\bar{\Sigma}_a &= e^{-\gamma} \Sigma_a , \\
\bar{\Delta}_a &= e^{-\gamma} \Delta_a , \\
\bar{\Psi} &= \Psi - \gamma' \Sigma_a \wedge \theta^a , \\
\bar{\Phi} &= \Phi - \gamma' \Delta_a \wedge \theta^a .
\end{align*}
\]

Disformal transformation:
- Change of fundamental fields:

\[
\bar{\theta}^a = C(\phi, X) \theta^a + D(\phi, X) \eta^{ab} (\iota_{e_b} d\phi) d\phi , \quad X = -\frac{1}{2} \eta^{ab} (\iota_{e_a} d\phi) (\iota_{e_b} d\phi)
\]

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Manuel Hohmann (University of Tartu)
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Scalar field redefinition:
- Change of fundamental fields: $\bar{\phi} = f(\phi) \Rightarrow \delta \bar{\phi} = f'(\phi) \delta \phi$.
- Change of equations: $\bar{\Phi} = (f')^{-1}\Phi$, $\bar{\Psi} = (f')^{-1}\Psi$.

Conformal transformation:
- Change of fundamental fields: $\bar{\theta}^a = e^{\gamma(\phi)}\theta^a \Rightarrow \delta \bar{\theta}^a = e^{\gamma} (\delta \theta^a + \gamma' \theta^a \delta \phi)$.
- Change of equations:
  - $\bar{\Sigma}_a = e^{-\gamma} \Sigma_a$,
  - $\bar{\Delta}_a = e^{-\gamma} \Delta_a$,
  - $\bar{\Psi} = \Psi - \gamma' \Sigma_a \wedge \theta^a$,
  - $\bar{\phi} = \Phi - \gamma' \Delta_a \wedge \theta^a$.

Disformal transformation:
- Change of fundamental fields:
  $$\bar{\theta}^a = C(\phi, X)\theta^a + D(\phi, X)\eta^{ab}(\iota_{eb} d\phi) d\phi,$$
  $$X = -\frac{1}{2} \eta^{ab}(\iota_{ea} d\phi)(\iota_{eb} d\phi)$$
- Leads to more lengthy relation between original and transformed variables.
Condition for symmetry under $\xi$ of fundamental fields [MH '15]:

$$\mathcal{L}_\xi \theta^a = -\lambda^a_b \theta^b, \quad \mathcal{L}_\xi \omega^{ab} = \dot{D}\lambda^a_b, \quad \mathcal{L}_\xi \phi = 0.$$
Condition for symmetry under $\xi$ of fundamental fields $[MH\ '15]$:

$$\mathcal{L}_\xi \theta^a = -\lambda^a_{\ b} \theta^b, \quad \mathcal{L}_\xi \omega^a_{\ b} = \tilde{D} \lambda^a_{\ b}, \quad \mathcal{L}_\xi \phi = 0.$$  

Six generating vector fields $\xi_1, \ldots, \xi_6$ of cosmological symmetry.

Translation generators:

$$\xi_1 = \chi \sin \vartheta \cos \varphi \partial_r + \frac{\chi}{r} \cos \vartheta \cos \varphi \partial_\vartheta - \frac{\chi \sin \varphi}{r \sin \vartheta},$$

$$\xi_2 = \chi \sin \vartheta \sin \varphi \partial_r + \frac{\chi}{r} \cos \vartheta \sin \varphi \partial_\vartheta + \frac{\chi \cos \varphi}{r \sin \vartheta},$$

$$\xi_3 = \chi \cos \vartheta \partial_r - \frac{\chi}{r} \sin \vartheta \partial_\vartheta.$$  

Rotation generators:

$$\xi_4 = \sin \varphi \partial_\vartheta + \frac{\cos \varphi}{\tan \vartheta} \partial_\varphi,$$

$$\xi_5 = -\cos \varphi \partial_\vartheta + \frac{\sin \varphi}{\tan \vartheta} \partial_\varphi,$$

$$\xi_6 = \partial_\varphi.$$
General scalar-torsion gravity - cosmology

- **Condition for symmetry under** $\xi$ **of fundamental fields** [MH '15]:

\[
\mathcal{L}_\xi \theta^a = -\lambda^a_{\ b} \theta^b, \quad \mathcal{L}_\xi \omega^a_{\ b} = \delta^a_{\ b}, \quad \mathcal{L}_\xi \phi = 0.
\]

- **Six generating vector fields** $\xi_1, \ldots, \xi_6$ **of cosmological symmetry**.
  - **Translation generators**:
    \[
    \begin{align*}
    \xi_1 &= \chi \sin \vartheta \cos \varphi \partial_r + \frac{\chi}{r} \cos \vartheta \cos \varphi \partial_\vartheta - \frac{\chi \sin \varphi}{r \sin \vartheta}, \\
    \xi_2 &= \chi \sin \vartheta \sin \varphi \partial_r + \frac{\chi}{r} \cos \vartheta \sin \varphi \partial_\vartheta + \frac{\chi \cos \varphi}{r \sin \vartheta}, \\
    \xi_3 &= \chi \cos \vartheta \partial_r - \frac{\chi}{r} \sin \vartheta \partial_\vartheta.
    \end{align*}
    \]
  - **Rotation generators**:
    \[
    \begin{align*}
    \xi_4 &= \sin \varphi \partial_\varphi + \frac{\cos \varphi}{\tan \vartheta} \partial_\vartheta, \\
    \xi_5 &= -\cos \varphi \partial_\varphi + \frac{\sin \varphi}{\tan \vartheta} \partial_\vartheta, \\
    \xi_6 &= \partial_\varphi.
    \end{align*}
    \]

- **Any 2-form constructed from** $\theta, \omega, \phi$ **with cosmological symmetry vanishes**.
Condition for symmetry under $\xi$ of fundamental fields [MH '15]:

$$\mathcal{L}_\xi \theta^a = -\lambda^a{}_b \theta^b, \quad \mathcal{L}_\xi \omega^a{}_b = \bar{D} \lambda^a{}_b, \quad \mathcal{L}_\xi \phi = 0.$$ 

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Translation generators:

$$\begin{align*}
\xi_1 &= \chi \sin \vartheta \cos \varphi \partial_r + \frac{\chi}{r} \cos \vartheta \cos \varphi \partial_\vartheta - \frac{\chi \sin \varphi}{r \sin \vartheta}, \\
\xi_2 &= \chi \sin \vartheta \sin \varphi \partial_r + \frac{\chi}{r} \cos \vartheta \sin \varphi \partial_\vartheta + \frac{\chi \cos \varphi}{r \sin \vartheta}, \\
\xi_3 &= \chi \cos \vartheta \partial_r - \frac{\chi}{r} \sin \vartheta \partial_\vartheta.
\end{align*}$$

Rotation generators:

$$\begin{align*}
\xi_4 &= \sin \varphi \partial_\vartheta + \frac{\cos \varphi}{\tan \vartheta} \partial_\varphi, \\
\xi_5 &= -\cos \varphi \partial_\vartheta + \frac{\sin \varphi}{\tan \vartheta} \partial_\varphi, \\
\xi_6 &= \partial_\varphi.
\end{align*}$$

Any 2-form constructed from $\theta, \omega, \phi$ with cosmological symmetry vanishes.

Connection field equation solved identically [MH, L. Järv, M. Krššák, C. Pfeifer ’18 to appear].
Diagonal tetrad in spherical coordinates:

\[ \theta^a_{\mu} = \text{diag} \left( n(t), \frac{a(t)}{\sqrt{1-kr^2}}, a(t)r, a(t)r \sin \vartheta \right) \, , \]
General scalar-torsion gravity - cosmological spin connections

- Diagonal tetrad in spherical coordinates:

\[ \theta^a_{\mu} = \text{diag} \left( n(t), \frac{a(t)}{\sqrt{1-kr^2}}, a(t)r, a(t)r \sin \vartheta \right), \]

- Cosmological spin connections [MH, L. Järv, U. Ualikhanova '18]:
  - Spatially flat spacetime \( k = 0 \):
    \[
    \omega^1_{2\vartheta} = -\omega^2_{1\vartheta} = -1, \quad \omega^1_{3\varphi} = -\omega^3_{1\varphi} = -\sin \vartheta, \quad \omega^2_{3\varphi} = -\omega^3_{2\varphi} = -\cos \vartheta.
    \]
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  \]

- Spatially closed spacetime \( k = 1 \):
  \[
  \omega^1_{2\vartheta} = -\omega^2_{1\vartheta} = -\sqrt{1-r^2}, \quad \omega^1_{2\varphi} = -\omega^2_{1\varphi} = -r \sin \vartheta, \quad \omega^1_{3\vartheta} = -\omega^3_{1\vartheta} = r,
  
  \omega^1_{3\varphi} = -\omega^3_{1\varphi} = -\sqrt{1-r^2} \sin \vartheta, \quad \omega^2_{3r} = -\omega^3_{2r} = -\frac{1}{\sqrt{1-r^2}}, \quad \omega^2_{3\varphi} = -\omega^3_{2\varphi} = -\cos \vartheta.
  \]
Diagonal tetrad in spherical coordinates:

\[ \theta^a_\mu = \text{diag} \left( n(t), \frac{a(t)}{\sqrt{1-kr^2}}, a(t)r, a(t)r \sin \vartheta \right) , \]

Cosmological spin connections [MH, L. Järv, U. Ualikhanova ’18]:

Spatially flat spacetime \( k = 0 \):

\[ \omega^{12}_\vartheta = -\omega^{21}_\vartheta = -1 , \quad \omega^{13}_\varphi = -\omega^{31}_\varphi = -\sin \vartheta , \quad \omega^{23}_\varphi = -\omega^{32}_\varphi = -\cos \vartheta . \]

Spatially closed spacetime \( k = 1 \):

\[ \omega^{12}_\vartheta = -\omega^{21}_\vartheta = -\sqrt{1-r^2} , \quad \omega^{12}_\varphi = -\omega^{21}_\varphi = -r \sin \vartheta , \quad \omega^{13}_\varphi = -\omega^{31}_\varphi = r , \]
\[ \omega^{13}_\vartheta = -\omega^{31}_\vartheta = -\sqrt{1-r^2} \sin \vartheta , \quad \omega^{23}_r = -\omega^{32}_r = -\frac{1}{\sqrt{1-r^2}} , \quad \omega^{23}_\varphi = -\omega^{32}_\varphi = -\cos \vartheta . \]

Spatially open spacetime \( k = -1 \):

\[ \omega^{01}_r = \omega^{10}_r = \frac{1}{\sqrt{1+r^2}} , \quad \omega^{02}_\vartheta = \omega^{20}_\vartheta = r , \quad \omega^{03}_\varphi = \omega^{30}_\varphi = r \sin \vartheta , \]
\[ \omega^{12}_\vartheta = -\omega^{21}_\vartheta = -\sqrt{1+r^2} , \quad \omega^{13}_\varphi = -\omega^{31}_\varphi = -\sqrt{1+r^2} \sin \vartheta , \quad \omega^{23}_\varphi = -\omega^{32}_\varphi = -\cos \vartheta . \]
Outline

1. Introduction
2. General scalar-torsion gravity
3. $L(T, X, Y, \phi)$ theory
4. “Scalar-curvature”-like class
5. Scalar-torsion gravity without derivative coupling
6. Conclusion
Gravitational part of the action [MH, C. Pfeifer '18]:

\[ S_g \left[ \theta^a, \omega^a_{\cdot b}, \phi^A \right] = \int_M L \left( T, X^{AB}, Y^A, \phi^A \right) \theta d^4x. \]

- Torsion scalar: \( T = \frac{1}{2} T_{\rho\mu\nu} S^\rho_{\mu\nu}. \)
- Superpotential:

\[ S_{\rho\mu\nu} = \frac{1}{2} (T_{\nu\mu\rho} + T_{\rho\mu\nu} - T_{\mu\nu\rho}) - g_{\rho\mu} T^\sigma_{\sigma\nu} + g_{\rho\nu} T^\sigma_{\sigma\mu}. \]

- Scalar field kinetic term: \( X^{AB} = -\frac{1}{2} g^{\mu\nu} \phi^A_{,\mu} \phi^B_{,\nu}. \)
- Kinetic coupling term: \( Y^A = T^\mu_{\mu\nu} \phi^A_{,\nu}. \)
Gravitational part of the action [MH, C. Pfeifer '18]:

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Torsion scalar: \( T = \frac{1}{2} T^\rho_{\mu\nu} S^\rho_{\mu\nu}. \)

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\[ S_{\rho\mu\nu} = \frac{1}{2} \left( T_{\nu\mu\rho} + T^\rho_{\mu\nu} - T_{\mu\nu\rho} \right) - g^\rho_{\mu\nu} T^{\sigma}_{\sigma\nu} + g^\rho_{\nu\sigma} T^{\sigma}_{\sigma\mu}. \]

Scalar field kinetic term: \( X^{AB} = -\frac{1}{2} g^\mu\nu \phi^A_{,\mu} \phi^B_{,\nu}. \)

Kinetic coupling term: \( Y^A = T^\mu_{\mu\nu} \phi^A_{,\nu}. \)

Matter action variation expressed in components:

\[ \delta S_m[\theta^a, \phi^A, \chi^I] = \int_M \left( \Theta^\mu a \delta \theta^a_{\mu} + \vartheta_A \delta \phi^A + \varpi_I \delta \chi^I \right) \theta d^4 x. \]
Symmetric part of tetrad equations:

\[
\begin{align*}
\nabla^\sigma \left( L_{YA} \phi_A^{\mu} \right) - \nabla^\rho \left( L_{YA} \phi_A^{\mu} \right) g^\rho_\sigma g_{\mu\nu} + L_{YA} \left( T_{\mu\nu}^\rho \phi_A^\rho + T^\rho_{\mu\nu} \phi_A^\rho \right) \\
- Lg_{\mu\nu} - 2\nabla^\rho \left( L_T S_{\mu\nu}^\rho \right) + L_T S_{\mu^\rho_\sigma T_{\nu}}^\rho_\sigma - L_{X_A} \phi_A^{\mu} \phi_B^\nu = \Theta_{\mu\nu}.
\end{align*}
\]
\[ \nabla_{(\mu} \left( L_{YA} \phi^A_{,\nu)} \right) - \nabla_{\sigma} \left( L_{YA} \phi^A_{,\rho} \right) g^{\rho\sigma} g_{\mu\nu} + L_{YA} \left( T_{(\mu\nu)}^\rho \phi^A_{,\rho} + T^\rho_{\rho(\mu\phi^A_{,\nu})} \right) \\
- Lg_{\mu\nu} - 2\nabla_{\rho} \left( L_{T} S_{(\mu\nu)}^\rho \right) + L_{T} S_{(\rho\sigma} T_{\nu)\rho\sigma} - L_{XAB} \phi^A_{,\mu} \phi^B_{,\nu} = \Theta_{\mu\nu} . \]

Antisymmetric part of tetrad equations \equiv connection equations:

\[ 3 \partial_{[\rho} L_{T} T^\rho_{\mu\nu]} + \partial_{[\mu} L_{YA} \phi^A_{,\nu]} - \frac{3}{2} L_{YA} T^\rho_{[\mu\nu} \phi^A_{,\rho]} = 0 . \]
\( L(T, X, Y, \phi) \) theory - field equations

- **Symmetric part of tetrad equations:**

\[
\nabla^\mu \left( L_{Y^A \phi^A_{\mu \nu}} \right) - \nabla^\nu \left( L_{Y^A \phi^A_{\mu \rho}} \right) g^{\rho \sigma} g_{\mu \nu} + L_{Y^A} \left( T_{(\mu \nu)}^{\rho \phi^A_{\rho \nu}} + T_{\rho (\mu \phi^A_{\nu \rho})} \right) \nabla^\phi - L g_{\mu \nu} - 2 \nabla^\rho \left( L_{T S_{(\mu \nu)}^{\rho}} \right) + L_{T S_{(\mu \rho \sigma \tau \nu)}^{\rho \sigma}} - L_{X^{AB} \phi^A_{\mu \nu}} = \Theta_{\mu \nu}.
\]

- **Antisymmetric part of tetrad equations \equiv connection equations:**

\[
3 \partial_{[ \rho} L_{T^{\rho \mu \nu]} \phi^A_{\nu \rho]} - \partial_{[ \mu} L_{Y^A \phi^A_{\nu \rho]} - \frac{3}{2} L_{Y^A} T_{\rho (\mu \phi^A_{\nu \rho})} = 0.
\]

- **Scalar field equations:**

\[
g^{\mu \nu} \nabla^\mu \left( L_{Y^A} T^{\rho \phi^B_{\rho \nu}} - L_{X^{AB} \phi^B_{\mu \nu}} \right) - L_{\phi^A} = \vartheta^A.
\]
\( L(T, X, Y, \phi) \) theory - field equations

- **Symmetric part of tetrad equations:**

\[
\nabla_{(\mu} \left( L_{YA} \phi^A_{,\nu)} \right) - \nabla_{\sigma} \left( L_{YA} \phi^A_{,\rho} \right) g^{\rho\sigma} g_{\mu\nu} + L_{YA} \left( T_{(\mu\nu)}^\rho \phi^A_{,\rho} + T^\rho_{\mu\nu} \phi^A_{,\rho} \right) \\
- Lg_{\mu\nu} - 2\nabla_{\rho} \left( L_{T} S_{(\mu\nu)}^\rho \right) + L_{T} S_{(\mu\rho\sigma} T_{\nu)\rho\sigma} - L_{XAB} \phi^A_{,\mu} \phi^B_{,\nu} = \Theta_{\mu\nu}.
\]

- **Antisymmetric part of tetrad equations \( \equiv \) connection equations:**

\[
3\partial_{[\rho} L_{T} T^\rho_{\mu\nu]} + \partial_{[\mu} L_{YA} \phi^A_{,\nu]} - \frac{3}{2} L_{YA} T^\rho_{[\mu\nu} \phi^A_{,\rho]} = 0.
\]

- **Scalar field equations:**

\[
g^{\mu\nu} \nabla_{\mu} \left( L_{YA} T^\rho_{\rho\nu} - L_{XAB} \phi^B_{,\nu} \right) - L_{\phi^A} = \vartheta_A.
\]

- **Matter field equations:** \( \varpi_I = 0 \).
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Action $[MH\ '18]$: 
- Gravitational part:
  \[ S_g[\theta^a, \omega^a{}_b, \phi^A] = \frac{1}{2\kappa^2} \int_M \left[ -A(\phi) T + 2B_{AB}(\phi)X^{AB} + 2C_A(\phi)Y^A - 2\kappa^2 V(\phi) \right] \theta^4 x. \]
- Matter part:
  \[ S_m[\theta^a, \phi^A, \chi^I] = S_m^3 \left[ e^{\alpha(\phi)} \theta^a, \chi^I \right]. \]
Action \[^{[MH '18]}\]:

- **Gravitational part:**
  \[
  S_g \left[ \theta^a, \omega^a b, \phi^A \right] = \frac{1}{2\kappa^2} \int_M \left[ -A(\phi) T + 2B_{AB}(\phi) X^{AB} + 2C_A(\phi) Y^A - 2\kappa^2 V(\phi) \right] \theta^4 x.
  \]

- **Matter part:**
  \[
  S_m[\theta^a, \phi^A, \chi^I] = S^3_m \left[ e^{\alpha(\phi)} \theta^a, \chi^I \right].
  \]

- **Free functions** \( A, B_{AB}, C_A, V, \alpha \) of scalar fields.
“Scalar-curvature”-like class - action

- **Action [MH ’18]**:
  - Gravitational part:
    \[
    S_g[\theta^a, \omega^a{}_b, \phi^A] = \frac{1}{2\kappa^2} \int_M \left[ -\mathcal{A}(\phi) T + 2\mathcal{B}_{AB}(\phi) X^{AB} + 2\mathcal{C}_A(\phi) Y^A - 2\kappa^2 \mathcal{V}(\phi) \right] \theta d^4x.
    \]
  - Matter part:
    \[
    S_m[\theta^a, \phi^A, \chi^I] = S^3_m[\epsilon^\alpha(\phi) \theta^a, \chi^I].
    \]

- Free functions \(\mathcal{A}, \mathcal{B}_{AB}, \mathcal{C}_A, \mathcal{V}, \alpha\) of scalar fields.
- \(\mathcal{C}_A \equiv -\mathcal{A},_A \Leftrightarrow\) theory reduces to scalar-curvature gravity.
“Scalar-curvature”-like class - action

**Action [MH ’18]:**

- **Gravitational part:**
  \[
  S_g \left[ \theta^a, \omega^a{}_b, \phi^A \right] = \frac{1}{2\kappa^2} \int_M \left[ -A(\phi) T + 2B_{AB}(\phi)X^{AB} + 2C_A(\phi)Y^A - 2\kappa^2V(\phi) \right] \theta d^4x.
  \]

- **Matter part:**
  \[
  S_m[\theta^a, \phi^A, \chi^I] = S^3_m \left[ e^{\alpha(\phi)} \theta^a, \chi^I \right].
  \]

**Free functions** $A, B_{AB}, C_A, V, \alpha$ of scalar fields.

**$C_A \equiv -A, A \iff$ theory reduces to scalar-curvature gravity.**

**Special subclass of $L(T, X, Y, \phi)$ class of theories.**
“Scalar-curvature”-like class - field equation

Symmetric part of the tetrad equations:

\[
(A_A + C_A) S_{(\mu\nu)}^\rho \phi_A^{,\rho} + A \left( \partial_{\mu} R_{\rho\nu} - \frac{1}{2} \partial_{\rho} R_{\mu\nu} \right) + \left( \frac{1}{2} B_{AB} - C_{(A,B)} \right) \phi_A^{,\rho} \phi_B^{,\sigma} g^{\rho\sigma} g_{\mu\nu}
\]

\[
- \left( B_{AB} - C_{(A,B)} \right) \phi_A^{,\mu} \phi_B^{,\nu} + C_A \left( \nabla_{\mu} \nabla_{\nu} \phi_A - \Box \phi_A g_{\mu\nu} \right) + \kappa^2 \gamma g_{\mu\nu} = \kappa^2 \Theta_{\mu\nu},
\]
"Scalar-curvature"-like class - field equation

- Symmetric part of the tetrad equations:

\[
(A, A + C_A) S_{(\mu\nu)}^\rho \phi_A^\rho + A \left( \overset{\circ}{R}_{\mu\nu} - \frac{1}{2} \overset{\circ}{R} g_{\mu\nu} \right) + \left( \frac{1}{2} B_{AB} - C_{(A,B)} \right) \phi_A^A \phi_B^B g^{\rho\sigma} g_{\mu\nu} \\
- (B_{AB} - C_{(A,B)}) \phi_A^A \phi_B^B + C_A \left( \overset{\circ}{\nabla}_\mu \overset{\circ}{\nabla}_\nu \phi_A^B - \square \phi_A^B g_{\mu\nu} \right) + \kappa^2 \nu g_{\mu\nu} = \kappa^2 \Theta_{\mu\nu},
\]

- Antisymmetric part of the tetrad field equations:

\[
3(A, A + C_A) T^\rho_{[\mu\nu] \phi_A^\rho} + 2C_{[A,B]} \phi_A^A \phi_B^B = 0.
\]
Scalar-curvature"-like class - field equation

- Symmetric part of the tetrad equations:

\[
\left(A, A + C_A\right) S_{(\mu \nu)}^\rho \phi^A + A \left(\overset{\circ}{R}^{\rho}_{\mu \nu} - \frac{1}{2} R g_{\mu \nu}\right) + \left(\frac{1}{2} B_{AB} - C_{(A, B)}\right) \phi^A_{, \rho} \phi^B_{, \sigma} g^{\rho \sigma} g_{\mu \nu} \\
- \left(B_{AB} - C_{(A, B)}\right) \phi^A_{, \mu} \phi^B_{, \nu} + C_A \left(\overset{\circ}{\nabla}_\mu \overset{\circ}{\nabla}_\nu \phi^A - \Box \phi^A g_{\mu \nu}\right) + \kappa^2 \mathcal{V} g_{\mu \nu} = \kappa^2 \Theta_{\mu \nu},
\]

- Antisymmetric part of the tetrad field equations:

\[
3\left(A, A + C_A\right) T^\rho_{[\mu \nu} \phi^A_{, \rho]} + 2C_{[A, B]} \phi^A_{, \mu} \phi^B_{, \nu} = 0.
\]

- Scalar field equation:

\[
\frac{1}{2} A, A T - B_{AB} \Box \phi^B - \left(B_{AB, C} - \frac{1}{2} B_{BC, A}\right) g^{\mu \nu} \phi^B_{, \mu} \phi^C_{, \nu} \\
+ C_A \overset{\circ}{\nabla}_\mu T^{\nu \mu}_\nu + 2C_{[A, B]} T^{\mu \nu}_{\mu} \phi^B_{, \nu} + \kappa^2 \mathcal{V}, A = \kappa^2 \alpha, A \Theta.
\]
Conformal transformation and scalar field redefinition:

\[
\tilde{\theta}^a = e^{\gamma(\phi)} \theta^a, \quad \tilde{e}_a = e^{-\gamma(\phi)} e_a, \quad \tilde{\phi}^A = f^A(\phi).
\]
Conformal transformation and scalar field redefinition:

\[ \bar{\theta}^a = e^{\gamma(\phi)} \theta^a, \quad \bar{e}_a = e^{-\gamma(\phi)} e_a, \quad \bar{\phi}^A = f^A(\phi). \]

Transformation of geometry:

\[ \bar{T} = e^{-2\gamma} \left( T + 4\gamma_{,A} Y^A + 12\gamma_{,A,\gamma,B} X^{AB} \right), \quad \bar{\phi}^A = f^A, \]
\[ \bar{X}^{AB} = e^{-2\gamma} \frac{\partial \bar{\phi}^A}{\partial \phi^C} \frac{\partial \bar{\phi}^B}{\partial \phi^D} X^{CD}, \quad \bar{Y}^A = e^{-2\gamma} \frac{\partial \bar{\phi}^A}{\partial \phi^B} \left( Y^B + 6\gamma_{,C} X^{BC} \right), \]
Conformal transformation and scalar field redefinition:

\[
\bar{\theta}^a = e^{\gamma(\phi)} \theta^a, \quad \bar{e}_a = e^{-\gamma(\phi)} e_a, \quad \bar{\phi}^A = f^A(\phi).
\]

Transformation of geometry:

\[
\bar{T} = e^{-2\gamma} \left( T + 4\gamma_A Y^A + 12\gamma_A \gamma_B X^{AB} \right), \quad \bar{\phi}^A = f^A,
\]

\[
\bar{X}^{AB} = e^{-2\gamma} \frac{\partial \bar{\phi}^A}{\partial \phi^C} \frac{\partial \bar{\phi}^B}{\partial \phi^D} X^{CD}, \quad \bar{Y}^A = e^{-2\gamma} \frac{\partial \bar{\phi}^A}{\partial \phi^B} \left( Y^B + 6\gamma_C X^{BC} \right),
\]

Transformation of parameter functions to preserve action:

\[
\mathcal{A} = e^{2\gamma} \bar{A},
\]

\[
\mathcal{B} = e^{2\gamma} \left( \bar{B}f'^2 - 6\bar{A} \gamma'^2 + 6\bar{C}f' \gamma' \right),
\]

\[
\mathcal{C} = e^{2\gamma} \left( \bar{C}f' - 2\bar{A} \gamma' \right),
\]

\[
\mathcal{V} = e^{4\gamma} \bar{V},
\]

\[
\alpha = \bar{\alpha} + \gamma.
\]
Quantities invariant under conformal transformations $\gamma$:

- "Scalar" quantities:
  \[ I_1 = \frac{e^{2\alpha}}{A}, \quad I_2 = \frac{\nu}{A^2}. \]
Quantities invariant under conformal transformations $\gamma$:

- "Scalar" quantities:

$$I_1 = \frac{e^{2\alpha}}{A}, \quad I_2 = \frac{\nu}{A^2}.$$ 

- "Covector" quantities:

$$H_A = \frac{C_A + A_{,A}}{2A}, \quad K_A = \frac{C_A + 2\alpha_{,A}A}{2e^{2\alpha}}.$$
“Scalar-curvature”-like class - invariants

- Quantities invariant under conformal transformations $\gamma$:
  - “Scalar” quantities:
    \[ I_1 = \frac{e^{2\alpha}}{A}, \quad I_2 = \frac{\mathcal{V}}{A^2}. \]
  - “Covector” quantities:
    \[ H_A = \frac{C_A + A_A}{2A}, \quad K_A = \frac{C_A + 2\alpha A_A}{2e^{2\alpha}}. \]
  - “Metric” quantities:
    \[ F_{AB} = \frac{2A\mathcal{B}_{AB} - 6A_{(A}C_{B)} - 3A_A A_B}{4A^2}, \]
    \[ G_{AB} = \frac{\mathcal{B}_{AB} - 6\alpha_{(A}C_{B)} - 6\alpha A_A A_B}{2e^{2\alpha}}. \]
Quantities invariant under conformal transformations $\gamma$:

- "Scalar" quantities:
  \[ I_1 = \frac{e^{2\alpha}}{A}, \quad I_2 = \frac{\nu}{A^2}. \]

- "Covector" quantities:
  \[ H_A = \frac{C_A + A_A}{2A}, \quad K_A = \frac{C_A + 2\alpha_A A}{2e^{2\alpha}}. \]

- "Metric" quantities:
  \[ F_{AB} = \frac{2A B_{AB} - 6A_{A(A} C_{B)} - 3A_A A_B}{4A^2}, \]
  \[ G_{AB} = \frac{B_{AB} - 6\alpha_A A_{A(A} C_{B)} - 6\alpha_A A_{A(B} A}{2e^{2\alpha}}. \]

Covariance under scalar field redefinitions:

\[ \bar{I}_{1,2} = I_{1,2}, \quad (\bar{H}, \bar{K})_A = \frac{\partial \phi^B}{\partial \phi^A} (H, K)_B, \quad (\bar{F}, \bar{G})_{AB} = \frac{\partial \phi^C}{\partial \phi^A} \frac{\partial \phi^D}{\partial \phi^B} (F, G)_{CD}. \]
Jordan frame: minimal coupling to matter.

\[ A^J = \frac{1}{I_1}, \quad B^J_{AB} = 2G_{AB}, \quad C^J_A = 2K_A, \quad \nu^J = \frac{I_2}{I_1}, \quad \alpha^J = 0. \]
“Scalar-curvature”-like class - special frames

- Jordan frame: minimal coupling to matter.
  \[
  A^3 = \frac{1}{I_1}, \quad B^3_{AB} = 2G_{AB}, \quad C^3_A = 2\mathcal{K}_A, \quad \nu^3 = \frac{I_2}{I_1^2}, \quad \alpha^3 = 0.
  \]

- Einstein frame: no coupling to torsion scalar.
  \[
  A^e = 1, \quad B^e_{AB} = 2\mathcal{F}_{AB}, \quad C^e_A = 2\mathcal{H}_A, \quad \nu^e = I_2, \quad \alpha^e = \frac{1}{2}\ln I_1.
  \]
“Scalar-curvature”-like class - special frames

- Jordan frame: minimal coupling to matter.

\[ A^\jmath = \frac{1}{I_1}, \quad B^\jmath_{\alpha\beta} = 2G_{\alpha\beta}, \quad C^\jmath_\alpha = 2K_\alpha, \quad \nu^\jmath = \frac{I_2}{I_1^2}, \quad \alpha^\jmath = 0. \]

- Einstein frame: no coupling to torsion scalar.

\[ A^\epsilon = 1, \quad B^\epsilon_{\alpha\beta} = 2F_{\alpha\beta}, \quad C^\epsilon_\alpha = 2H_\alpha, \quad \nu^\epsilon = I_2, \quad \alpha^\epsilon = \frac{1}{2} \ln I_1. \]

- “Debraiding frame” (for \( H_{[\alpha,\beta]} \equiv 0 \)): minimal coupling to torsion.

\[ \left( \ln A^\omega \right)_\alpha = 2H_\alpha, \quad \left( \ln B^\omega \right)^\alpha_{\beta\gamma} = \left[ \ln (F + 3H \otimes H) \right]^\alpha_{\beta\gamma} + 2 \delta^\alpha_{\beta} H_\gamma, \quad C^\omega_\alpha = 0, \quad \left( \ln \nu^\omega \right)_\alpha = (\ln I_2)_\alpha + 4H_\alpha, \quad \alpha^\omega_\alpha = I_1 K_\alpha. \]
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Gravitational action [MH, L. Järv, U. Ualikhanova ’18]:

\[ S = \frac{1}{2\kappa^2} \int_M \left[ f(T, \phi) + Z_{AB}(\phi)g^{\mu\nu} \phi^A_{,\mu} \phi^B_{,\nu} \right] \theta^4 + S_m[\theta^a, \chi^I]. \]
Scalar-torsion gravity without derivative coupling

- Gravitational action \([MH, L. Järv, U. Ualikhanova '18]\):
  \[
  S = \frac{1}{2\kappa^2} \int_M \left[ f(T, \phi) + Z_{AB}(\phi)g^{\mu\nu}\phi_A^{\mu}\phi_B^{\nu} \right] \theta d^4x + S_m[\theta^a, \chi^I].
  \]

- Field equations:
  - Symmetric part of the tetrad field equations:
    \[
    \frac{1}{2} fg_{\mu\nu} + \overset{\diamond}{\nabla}_\rho \left( f_T S_{(\mu\nu)^\rho} \right) - \frac{1}{2} f_T S_{(\mu^\rho T_\nu)^\rho\sigma} - Z_{AB}\phi^{A}_{,\mu}\phi^{B}_{,\nu} + \frac{1}{2} Z_{AB}\phi^{A}_{,\rho}\phi^{B}_{,\sigma} g^{\rho\sigma} g_{\mu\nu} = \kappa^2 \Theta_{\mu\nu},
    \]
  - Antisymmetric part of the tetrad field equations:
    \[
    \partial_{[\rho} f_T T^\rho_{\mu\nu]} = 0 \iff \partial_{\mu} f_T \left[ \partial_\nu \left( \theta e_{[a}^{\mu} e_{b]}^{\nu} \right) + 2\theta e_c^{[\mu} e_{[a}^{\nu]} \omega_{b]}^{c} \right] = 0.
    \]
  - Scalar field equation:
    \[
    f_{\phi^A} - (2Z_{AB,\phi^C} - Z_{BC,\phi^A}) g^{\mu\nu}\phi^{B}_{,\mu}\phi^{C}_{,\nu} - 2Z_{AB}\overset{\diamond}{\Box}\phi^B = 0.
    \]
Scalar-torsion gravity without derivative coupling

- **Gravitational action** [MH, L. Järv, U. Ualikhanova '18]:
  \[ S = \frac{1}{2\kappa^2} \int_M \left[ f(T, \phi) + Z_{AB}(\phi) g^{\mu\nu} \phi^A_{\mu} \phi^B_{\nu} \right] \theta^4 x + S_m[\theta^a, \chi^l]. \]

- **Field equations:**
  - **Symmetric part of the tetrad field equations:**
    \[ \frac{1}{2} f g_{\mu\nu} + \check{\nabla}_{\rho} \left( f_T S_{(\mu\nu)^\rho} \right) - \frac{1}{2} f_T S_{(\mu}{}^{\rho\sigma} T_{\nu)}{}^{\rho\sigma} - Z_{AB} \phi^A_{\mu} \phi^B_{\nu} + \frac{1}{2} Z_{AB} \phi^A_{\rho} \phi^B_{\sigma} g^{\rho\sigma} g_{\mu\nu} = \kappa^2 \Theta_{\mu\nu}, \]
  - **Antisymmetric part of the tetrad field equations:**
    \[ \partial_{[\rho} f_T T^{\rho \mu\nu]} = 0 \iff \partial_{\mu} f_T \left[ \partial_{\nu} \left( \theta e[a]{}^\mu e[b]{}^\nu \right) + 2 \theta e[c]{}^\mu e[a]{}^\nu \omega[c]{}_{b\nu} \right] = 0. \]
  - **Scalar field equation:**
    \[ f_{\phi^A} - (2 Z_{AB,\phi^C} - Z_{BC,\phi^A}) g^{\mu\nu} \phi^B_{\mu} \phi^C_{\nu} - 2 Z_{AB} \Box \phi^B = 0. \]

- **Contains various interesting examples:** \( f(T) \) equivalent, teleparallel dark energy, ...
Outline

1 Introduction
2 General scalar-torsion gravity
3 $L(T, X, Y, \phi)$ theory
4 “Scalar-curvature”-like class
5 Scalar-torsion gravity without derivative coupling
6 Conclusion
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Most general class of theories based on tetrad, flat spin connection, scalar field(s).
No direct coupling between matter and spin connection.
Local Lorentz invariance: symmetric energy-momentum, dependence of equations.
Energy-momentum conservation up to exchange between matter and scalar field(s).
Theories related by transformations of scalar field(s) and tetrad.
Generic cosmological solutions to antisymmetric field equations.

Theory without derivative couplings:
Simple, yet interesting class of scalar-torsion theories.
Includes $f(T)$, teleparallel dark energy, other studied models.
Good test case for application and development of formalisms.
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- Constructed from torsion scalar, kinetic term, vector torsion coupling.
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  • Dynamical systems analysis and stable fixed points
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  • Inflation and related parameters
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  - Potential appearance of ghosts?
  - Work towards numerical simulations.


MH, L. Järv, C. Pfeifer, M. Krššák; Modified teleparallel theories of gravity in symmetric spacetimes; to appear.