Gravitational waves in teleparallel gravity

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1. Introduction
2. Principal symbol: speed of gravitational waves
3. Newman-Penrose formalism: polarization of gravitational waves
4. Waves in non-metricity teleparallel gravity
5. Waves in torsion teleparallel gravity
6. Conclusion
Outline

1 Introduction

2 Principal symbol: speed of gravitational waves

3 Newman-Penrose formalism: polarization of gravitational waves

4 Waves in non-metricity teleparallel gravity

5 Waves in torsion teleparallel gravity

6 Conclusion
Open questions in cosmology and gravity:

- Accelerating phases in the history of the Universe?
- Relation between gravity and gauge theories?
- How to quantize gravity?

Teleparallel gravity

- Based on tetrad and flat spin connection.
- Describes gravity as gauge theory of the translation group.
- First order action, second order field equations.
- Spin connection as Lorentz gauge degree of freedom.

Symmetric teleparallel gravity

- Based on metric and flat, symmetric connection.
- Describes gravity as non-metricity of the connection.
- First order action, second order field equations.
- Contains diffeomorphisms as gauge group.
Motivation

- Open questions in cosmology and gravity:
  - Accelerating phases in the history of the Universe?
  - Relation between gravity and gauge theories?
  - How to quantize gravity?
- Teleparallel gravity [Møller '61]:
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Gravity formulated as gauge theories.
Overview of geometries

- **Riemann-Cartan**: $Q_{\rho \mu \nu} = 0$
- **Riemann**: $T^\lambda_{\mu \nu} = 0$, $Q_{\rho \mu \nu} = 0$
- **Minkowski**: $Q_{\rho \mu \nu} = 0$
- **Weitzenböck**: $Q_{\rho \mu \nu} = 0$, $R^\sigma_{\rho \mu \nu} = 0$
- **Teleparallel**: $R^\sigma_{\rho \mu \nu} = 0$
- **Torsion free**: $T^\lambda_{\mu \nu} = 0$
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$$D^A_B \psi^B(x) = 0.$$ 

- $N$ equations labeled by $A = 1, \ldots, N$.
- $N$ fields $\psi^B$ labeled by $B = 1, \ldots, N$. 

**Principal symbol of a linear PDE**
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Partial derivatives \( \partial_\mu \) with respect to spacetime coordinates.
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**Structure of the linear partial differential operator:**

\[ D^A_B = M^A_B(x) + M^A_B{}^{\mu_1}(x) \partial_{\mu_1} + \ldots + M^A_B{}^{\mu_1\cdots\mu_m}(x) \partial_{\mu_1} \cdots \partial_{\mu_m}. \]
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Consider plane wave ansatz \( \Psi^A(x) = \hat{\Psi}^A e^{ik_\mu x^\mu} \) for the field:
\[
D^A_B \Psi^B(x) = \left( M^A_B(x) + \ldots + i^p M^A_B{}^{\mu_1 \cdots \mu_m}(x) k_{\mu_1} \cdots k_{\mu_m} \right) \hat{\Psi}^A e^{ik_\mu x^\mu} .
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Principal symbol is the highest order term in wave covector \( k_\mu \):

\[ P^A_B(x, k) = M^A_B{}^{\mu_1\ldots\mu_m}(x) k_{\mu_1} \ldots k_{\mu_m}. \]
Principal polynomial and propagation speed

- Principal polynomial:
  \[ p(x, k) = \det P^A_B(x, k). \]
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PDE of order \( p \) is called strictly hyperbolic if there exists a covector \( \tilde{k}_\mu \) such that for all non-zero covectors \( k_\mu \) the polynomial \( p(x, k + tk) \) in \( t \) has \( m \) distinct real roots.
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- Hyperbolic PDE has well-defined initial value problem:
  - Foliation of spacetime by spacelike hypersurfaces with covector \( \tilde{k}_\mu \).
  - Initial data on chosen hypersurface \( t = 0 \).
  - Non-vanishing initial data only on compact subset.
  - PDE determines propagation of initial data over time.
  - Wave front: outer shell of non-vanishing propagating field.
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\[ \Rightarrow \] Propagation speed determined by zeros of principal polynomial.
Gauge theories lead to $p(x, k) \equiv 0$ for all $k_\mu$!
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$\iff$ There exist directions $\hat{\psi}^A$ with $P^A_B(x, k) \hat{\psi}^B \equiv 0$ for all $k$. 

Gauge degrees of freedom.

No propagation - not physical modes.

$\iff$ Endomorphism $P^A_B(x, k)$ has non-trivial kernel.

Block decomposition of principal symbol:

\[
P^A_B(x, k) = \begin{pmatrix}
\bar{P}^A_B(x, k) & 0 \\
0 & 0
\end{pmatrix}
\]

Gauge degrees of freedom.

Physical degrees of freedom.

Non-trivial restricted principal polynomial:

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\bar{p}(x, k) = \det \bar{P}^A_B(x, k).
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Covectors $k_\mu$ of physical wave front satisfy $\bar{p}(x, k) = 0$. 

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Newman-Penrose formalism

- Complex double null basis of the tangent bundle:

\[ l = \partial_t + \partial_z, \quad n = \frac{\partial_t - \partial_z}{2}, \quad m = \frac{\partial_x + i\partial_y}{\sqrt{2}}, \quad \bar{m} = \frac{\partial_x - i\partial_y}{\sqrt{2}}. \]
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- Consider plane null wave with \( k_\mu = -\omega l_\mu \) and \( u = t - z \):
  \[ h_{\mu\nu} = \hat{h}_{\mu\nu} e^{ik_\mu x^\mu} = \hat{h}_{\mu\nu} e^{i\omega u}. \]
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- Effect of the wave on test particles - geodesic deviation:
  \[ v^\rho \nabla_\rho (v^\sigma \nabla_\sigma s^\mu) = -R^{\mu}_{\rho\nu\sigma} v^\rho v^\sigma s^\nu. \]
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- Riemann tensor determined by “electric” components:
  \[ \psi_2 = -\frac{1}{6} R_{nnlnl} = \frac{1}{12} \hat{h}_{ll} , \quad \psi_3 = -\frac{1}{2} R_{nnln\bar{m}} = \frac{1}{4} \hat{h}_{l\bar{m}} , \]
  \[ \psi_4 = -R_{n\bar{m}n\bar{m}} = \frac{1}{2} \hat{h}_{\bar{m}\bar{m}} , \quad \phi_{22} = -R_{nmn\bar{m}} = \frac{1}{2} \hat{h}_{m\bar{m}} . \]
Polarisations of gravitational waves

Effect of the different polarizations on spherical shell of test masses:

\[ \psi_4, \bar{\psi}_4 \]
- tensors

\[ \phi_{22} \]
- breathing

\[ \psi_3, \bar{\psi}_3 \]
- vectors

\[ \psi_2 \]
- longitudinal
E(2) classification of gravitational waves

- Consider Lorentz transformation which fixes wave covector $k_\mu$:
  - Rotations around wave covector & null rotations (= boost + rotation).
  - Set of transformations isomorphic to Euclidean group E(2).
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- Transformation of NP components under basis transformation:

\[
\begin{align*}
\Psi_2 &\rightarrow \Phi_{22} \\
\Psi_3 &\rightarrow \Psi_4 \\
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```
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```

- Possible sets of non-vanishing NP components:
  - $\Pi_6$: 6 polarizations, all modes are allowed.
  - $\Pi_5$: 5 polarizations, $\Psi_2 = 0$, no longitudinal mode.
  - $\Pi_3$: 3 polarizations, $\Psi_2 = \Psi_3 = 0$, only tensor and breathing modes.
  - $\Pi_2$: 2 polarizations, $\Psi_2 = \Psi_3 = \Phi_{22} = 0$, only tensor modes.
  - $\Pi_1$: 1 polarization, $\Psi_2 = \Psi_3 = \Psi_4 = 0$, only breathing mode.
  - $\Pi_0$: no gravitational waves.
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Possible sets of non-vanishing NP components:
- II$_6$: 6 polarizations, all modes are allowed.
- III$_5$: 5 polarizations, $\Psi_2 = 0$, no longitudinal mode.
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  $\Psi_2 \downarrow \downarrow \downarrow \rightarrow \Psi_3 \rightarrow \Phi_{22}$

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Field content and geometry

- Fundamental fields in the gravity sector:
  - Metric $g_{\mu\nu}$.
  - Flat, symmetric affine connection $\Gamma^\mu_{\nu\rho}$.

- Derived quantities:
  - Volume form $\sqrt{-\det g} d^4x$.
  - Levi-Civita connection $\nabla^\rho \Gamma_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right)$.
  - Non-metricity $Q_{\rho\mu\nu} = \nabla_\rho g_{\mu\nu}$.

- Gauge fixing
  - Perform local coordinate transformation:
    $g'_{\mu\nu} = \partial_\alpha x^\mu \partial_\beta x^\nu g_{\alpha\beta}$,
    $\Gamma'_{\rho\mu\nu} = \partial_\alpha x^\mu \partial_\beta x^\nu \partial_\gamma x^\rho \Gamma_{\gamma\alpha\beta} + \partial^2_\alpha x^\mu \partial_\beta x^\nu \partial_\gamma x^\rho$.
  - Coincident gauge: set $\times \Gamma_{\rho\mu\nu} \equiv 0 \Rightarrow Q_{\rho\mu\nu} = \partial_\rho g_{\mu\nu}$.
Field content and geometry

- **Fundamental fields in the gravity sector:**
  - Metric $g_{\mu\nu}$.
  - Flat, symmetric affine connection $\Gamma^\mu_{\nu\rho}$.

- **Derived quantities:**
  - Volume form $\sqrt{-\det g} d^4x$.
  - Levi-Civita connection
    \[
    \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right).
    \]
  - Non-metricity $Q_{\rho\mu\nu} = \nabla_\rho g_{\mu\nu}$.
Field content and geometry

- Fundamental fields in the gravity sector:
  - Metric $g_{\mu\nu}$.
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  - Volume form $\sqrt{-\det g} d^4x$.
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    \[
    \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left( \partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu} \right).
    \]
  - Non-metricity $Q_{\rho\mu\nu} = \nabla_\rho g_{\mu\nu}$.

- Gauge fixing
  - Perform local coordinate transformation:
    \[
    g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'\mu} \frac{\partial x^\beta}{\partial x'\nu} g_{\alpha\beta}, \quad \Gamma'_{\rho\mu\nu} = \frac{\partial x^\alpha}{\partial x'\mu} \frac{\partial x^\beta}{\partial x'\nu} \frac{\partial x'^\rho}{\partial x^\gamma} \Gamma_{\alpha\beta}^\gamma + \frac{\partial^2 x^\alpha}{\partial x'\mu \partial x'\nu} \frac{\partial x'^\rho}{\partial x^\alpha}.
    \]

  \Rightarrow \text{Coincident gauge: set } \Gamma_{\rho\mu\nu} \equiv 0 \Rightarrow Q_{\rho\mu\nu} = \partial_\rho g_{\mu\nu}.$
Most general action and corresponding field equations

- Most general action:

\[
S = - \int d^4 x \frac{\sqrt{-g}}{2} \left[ c_1 Q^\alpha_{\mu \nu} + c_2 Q_{(\mu}^\alpha_{\nu)} 
+ c_3 Q^\alpha g_{\mu \nu} + c_4 \delta_{(\mu}^\alpha Q_{\nu)} + \frac{c_5}{2} \left( \tilde{Q}^\alpha g_{\mu \nu} + \delta_{(\mu}^\alpha Q_{\nu)} \right) \right] Q^{\alpha \mu \nu}.
\]
Most general action and corresponding field equations

- Most general action:

\[
S = - \int d^4x \sqrt{-g} \left[ c_1 Q^\alpha_{\mu\nu} + c_2 Q_{(\mu}^{\alpha \nu)} 
+ c_3 Q^\alpha g_{\mu\nu} + c_4 \delta_{(\mu}^\alpha Q_{\nu)} + \frac{c_5}{2} \left( \tilde{Q}^\alpha g_{\mu\nu} + \delta_{(\mu}^\alpha Q_{\nu)} \right) \right] Q^{\mu\nu}_\alpha.
\]

- Consider linear perturbation of the metric:

\[
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.
\]
Most general action:

$$S = - \int d^4x \sqrt{-g} \left[ c_1 Q^\alpha_{\mu\nu} + c_2 Q_{(\mu}^{\alpha \nu)} + c_3 Q^\alpha g_{\mu\nu} + c_4 \delta^\alpha_{(\mu} \tilde{Q}^{\nu)} + \frac{c_5}{2} \left( \tilde{Q}^\alpha g_{\mu\nu} + \delta^\alpha_{(\mu} Q^{\nu)} \right) \right] Q_{\alpha \mu\nu}.$$ 

Consider linear perturbation of the metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$ 

Linearized vacuum field equations:

$$0 = 2c_1 \eta^{\alpha\sigma} \partial_\alpha \partial_\sigma h_{\mu\nu} + c_2 \eta^{\alpha\sigma} \left( \partial_\alpha \partial_\mu h_{\sigma\nu} + \partial_\alpha \partial_\nu h_{\sigma\mu} \right)$$

$$+ 2c_3 \eta_{\mu\nu} \eta^{\tau\omega} \eta^{\alpha\sigma} \partial_\alpha \partial_\sigma h_{\tau\omega} + c_4 \eta^{\omega\sigma} \left( \partial_\mu \partial_\omega h_{\nu\sigma} + \partial_\nu \partial_\omega h_{\mu\sigma} \right)$$

$$+ c_5 \eta_{\mu\nu} \eta^{\omega\gamma} \eta^{\alpha\sigma} \partial_\alpha \partial_\omega h_{\sigma\gamma} + c_5 \eta^{\omega\sigma} \partial_\mu \partial_\nu h_{\omega\sigma}.$$
Principal polynomial and speed of propagation

- Decomposition of amplitude $\hat{h}_{\lambda\rho}$ in irreducible components:

$$\hat{h}_{\lambda\rho} = S_{\lambda\rho} + 2 k_\lambda V_\rho + \frac{1}{3} \left( \eta_{\lambda\rho} - \frac{k_\lambda k_\rho}{\eta_{\mu\nu} k_{\mu\nu}} \right) T + \left( k_\lambda k_\rho - \frac{1}{4} \eta_{\lambda\rho} \eta^{\alpha\beta} k_\alpha k_\beta \right) U.$$
Decomposition of amplitude $\hat{h}_{\lambda\rho}$ in irreducible components:

$$\hat{h}_{\lambda\rho} = S_{\lambda\rho} + 2k(\lambda V_{\rho}) + \frac{1}{3} \left( \eta_{\lambda\rho} - \frac{k_\lambda k_\rho}{\eta_{\mu\nu} k_{\mu\nu}} \right) T + \left( k_\lambda k_\rho - \frac{1}{4} \eta_{\lambda\rho} \eta^{\alpha\beta} k_\alpha k_\beta \right) U. $$

Conditions imposed on irreducible components:

$$\eta^{\lambda\rho} S_{\lambda\rho} = 0, \quad k^\lambda S_{\lambda\rho} = 0, \quad k^\rho V_{\rho} = 0.$$
Principals polynomial and speed of propagation

- Decomposition of amplitude $\hat{h}_{\lambda \rho}$ in irreducible components:

\[
\hat{h}_{\lambda \rho} = S_{\lambda \rho} + 2k_\lambda V_\rho + \frac{1}{3} \left( \eta_{\lambda \rho} - \frac{k_\lambda k_\rho}{\eta_{\mu \nu} k_{\mu \nu}} \right) T + \left( k_\lambda k_\rho - \frac{1}{4} \eta_{\lambda \rho} \eta^{\alpha \beta} k_\alpha k_\beta \right) U.
\]

- Conditions imposed on irreducible components:

\[
\eta^{\lambda \rho} S_{\lambda \rho} = 0, \quad k^{\lambda} S_{\lambda \rho} = 0, \quad k^\rho V_\rho = 0.
\]

- Decomposed field equations:

\[
0 = (2c_3 + c_5)(\eta^{\alpha \beta} k_\alpha k_\beta)^2 T + \frac{3}{4} [c_5 + 2(c_1 + c_2 + c_4)](\eta^{\alpha \beta} k_\alpha k_\beta)^3 U,
\]

\[
0 = (2c_1 + 8c_3 + c_5)(\eta^{\alpha \beta} k_\alpha k_\beta) T + \frac{3}{2} (2c_5 + c_2 + c_4)(\eta^{\alpha \beta} k_\alpha k_\beta)^2 U,
\]

\[
0 = (2c_1 + c_2 + c_4)(\eta^{\alpha \beta} k_\alpha k_\beta)^2 V_\nu,
\]

\[
0 = 2c_1 \eta^{\alpha \beta} k_\alpha k_\beta S_{\mu \nu}.
\]
Principal polynomial and speed of propagation

- Decomposition of amplitude \( \hat{h}_{\lambda \rho} \) in irreducible components:

\[
\hat{h}_{\lambda \rho} = S_{\lambda \rho} + 2k_{(\lambda} V_{\rho)} + \frac{1}{3} \left( \eta_{\lambda \rho} - \frac{k_{\lambda} k_{\rho}}{\eta_{\mu \nu} k_{\mu \nu}} \right) T + \left( k_{\lambda} k_{\rho} - \frac{1}{4} \eta_{\lambda \rho} \eta^{\alpha \beta} k_{\alpha} k_{\beta} \right) U.
\]

- Conditions imposed on irreducible components:

\[
\eta^{\lambda \rho} S_{\lambda \rho} = 0, \quad k^{\lambda} S_{\lambda \rho} = 0, \quad k^\rho V_\rho = 0.
\]

- Decomposed field equations:

\[
0 = (2c_3 + c_5) (\eta^{\alpha \beta} k_\alpha k_\beta)^2 T + \frac{3}{4} [c_5 + 2(c_1 + c_2 + c_4)] (\eta^{\alpha \beta} k_\alpha k_\beta)^3 U,
\]
\[
0 = (2c_1 + 8c_3 + c_5) (\eta^{\alpha \beta} k_\alpha k_\beta) T + \frac{3}{2} (2c_5 + c_2 + c_4) (\eta^{\alpha \beta} k_\alpha k_\beta)^2 U,
\]
\[
0 = (2c_1 + c_2 + c_4) (\eta^{\alpha \beta} k_\alpha k_\beta)^2 V_\nu,
\]
\[
0 = 2c_1 \eta^{\alpha \beta} k_\alpha k_\beta S_{\mu \nu}.
\]

- Principal polynomial \( p(x, k) = \text{const.} \cdot (\eta^{\alpha \beta} k_\alpha k_\beta)^{15} \).
Principal polynomial and speed of propagation

- Decomposition of amplitude $\hat{h}_{\lambda\rho}$ in irreducible components:

$$\hat{h}_{\lambda\rho} = S_{\lambda\rho} + 2k_{(\lambda} V_{\rho)} + \frac{1}{3} \left( \eta_{\lambda\rho} - \frac{k_{\lambda} k_{\rho}}{\eta_{\mu\nu} k_{\mu\nu}} \right) T + \left( k_{\lambda} k_{\rho} - \frac{1}{4} \eta_{\lambda\rho} \eta^{\alpha\beta} k_{\alpha} k_{\beta} \right) U .$$

- Conditions imposed on irreducible components:

$$\eta^{\lambda\rho} S_{\lambda\rho} = 0 , \quad k^{\lambda} S_{\lambda\rho} = 0 , \quad k^{\rho} V_{\rho} = 0 .$$

- Decomposed field equations:

$$0 = (2c_3 + c_5)(\eta^{\alpha\beta} k_{\alpha} k_{\beta})^2 T + \frac{3}{4} [c_5 + 2(c_1 + c_2 + c_4)](\eta^{\alpha\beta} k_{\alpha} k_{\beta})^3 U ,$$

$$0 = (2c_1 + 8c_3 + c_5)(\eta^{\alpha\beta} k_{\alpha} k_{\beta}) T + \frac{3}{2} (2c_5 + c_2 + c_4)(\eta^{\alpha\beta} k_{\alpha} k_{\beta})^2 U ,$$

$$0 = (2c_1 + c_2 + c_4)(\eta^{\alpha\beta} k_{\alpha} k_{\beta})^2 V_{\nu} ,$$

$$0 = 2c_1 \eta^{\alpha\beta} k_{\alpha} k_{\beta} S_{\mu\nu} .$$

- Principal polynomial $p(x, k) = \text{const.} \cdot (\eta^{\alpha\beta} k_{\alpha} k_{\beta})^{15} .$

$$\eta^{\alpha\beta} k_{\alpha} k_{\beta} = 0 \iff \text{propagation at the speed of light.}$$
Newman-Penrose formalism

- Assume plane null wave $h_{\mu\nu} = \hat{h}_{\mu\nu}e^{ik_\mu x^\mu}$ with $\eta^{\alpha\beta}k_\alpha k_\beta = 0$. 

Possible $E^2$ classes:

- $c_2 + c_4 = c_5 = 0$: all six modes are allowed $\Rightarrow$ II$_6$.
- $c_2 + c_4 = 0$, $c_5 \neq 0$: only scalar $\Psi^2 \sim \ddot{h}_{ll}$ prohibited $\Rightarrow$ III$_5$.
- $c_2 + c_4 \neq 0$, $c_2 + c_4 + c_5 \neq 0$: also vector $\Psi^3 \sim \ddot{h}_{lm}$ prohibited $\Rightarrow$ N$_3$.
- $c_2 + c_4 + c_5 = 0$, $c_5 \neq 0$: also scalar $\Phi^22 \sim \ddot{h}_{\bar{m}m}$ prohibited $\Rightarrow$ N$_2$. 

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Newman-Penrose formalism

- Assume plane null wave $h_{\mu\nu} = \hat{h}_{\mu\nu} e^{ik_{\mu}x^{\mu}}$ with $\eta^{\alpha\beta}k_{\alpha}k_{\beta} = 0$.

$\Rightarrow$ Terms involving $c_1$ and $c_3$ do not contribute for $\Box h_{\mu\nu} = 0$. 

Possible $E(2)$ classes:

- $c_2 + c_4 = c_5 = 0$: all six modes are allowed $\Rightarrow II_6$.
- $c_2 + c_4 = 0$, $c_5 \neq 0$: only scalar $\Psi_2 \sim \ddot{h}_{ll}$ prohibited $\Rightarrow III_5$.
- $c_2 + c_4 \neq 0$, $c_2 + c_4 + c_5 \neq 0$: also vector $\Psi_3 \sim \ddot{h}_{lm}$ prohibited $\Rightarrow N_3$.
- $c_2 + c_4 + c_5 = 0$, $c_5 \neq 0$: also scalar $\Phi_{22} \sim \ddot{h}_{m\bar{m}}$ prohibited $\Rightarrow N_2$. 

Manuel Hohmann (University of Tartu)
Newman-Penrose formalism

- Assume plane null wave \( h_{\mu\nu} = \hat{h}_{\mu\nu} e^{ik_{\mu} x^\mu} \) with \( \eta^{\alpha\beta} k_\alpha k_\beta = 0 \).

\Rightarrow\quad \text{Terms involving } c_1 \text{ and } c_3 \text{ do not contribute for } \Box h_{\mu\nu} = 0.

\Rightarrow\quad \text{Field equations expressed in Newman-Penrose basis:}

\begin{align*}
0 &= E_{nn} = -(c_2 + c_4 + c_5) \ddot{h}_{ln} + c_5 \ddot{h}_{m\bar{m}}, \\
0 &= E_{mn} = E_{nm} = -(c_2 + c_4) \ddot{h}_{lm}, \\
0 &= E_{m\bar{m}} = E_{\bar{m}m} = c_5 \ddot{h}_{l\bar{l}}, \\
0 &= E_{nl} = E_{ln} = -(c_2 + c_4) \ddot{h}_{l\bar{l}}.
\end{align*}
Assume plane null wave \( h_{\mu\nu} = \hat{h}_{\mu\nu} e^{ik_\mu x^\mu} \) with \( \eta^{\alpha\beta} k_\alpha k_\beta = 0 \).

\( \Rightarrow \) Terms involving \( c_1 \) and \( c_3 \) do not contribute for \( \Box h_{\mu\nu} = 0 \).

\( \Rightarrow \) Field equations expressed in Newman-Penrose basis:

\[
0 = E_{nn} = -(c_2 + c_4 + c_5) \ddot{h}_{ln} + c_5 \ddot{h}_{m\bar{m}},
\]

\[
0 = E_{mn} = E_{nm} = -(c_2 + c_4) \ddot{h}_{lm},
\]

\[
0 = E_{m\bar{m}} = E_{\bar{m}m} = c_5 \ddot{h}_{ll},
\]

\[
0 = E_{nl} = E_{ln} = -(c_2 + c_4) \ddot{h}_{ll}.
\]

Possible \( E(2) \) classes:

- \( c_2 + c_4 = c_5 = 0 \): all six modes are allowed \( \Rightarrow \text{II}_6 \).
Assume plane null wave $h_{\mu\nu} = \hat{h}_{\mu\nu} e^{ik_{\mu}x^{\mu}}$ with $\eta^{\alpha\beta} k_{\alpha} k_{\beta} = 0$.

Terms involving $c_1$ and $c_3$ do not contribute for $\Box h_{\mu\nu} = 0$.

Field equations expressed in Newman-Penrose basis:

$$0 = E_{nn} = -(c_2 + c_4 + c_5)\ddot{h}_{ln} + c_5\ddot{h}_{m\bar{m}},$$
$$0 = E_{mn} = E_{nm} = -(c_2 + c_4)\ddot{h}_{lm},$$
$$0 = E_{m\bar{m}} = E_{\bar{m}m} = c_5\ddot{h}_{ll},$$
$$0 = E_{nl} = E_{ln} = -(c_2 + c_4)\ddot{h}_{ll}.$$

Possible $E(2)$ classes:

- $c_2 + c_4 = c_5 = 0$: all six modes are allowed $\Rightarrow \text{II}_6$.
- $c_2 + c_4 = 0$, $c_5 \neq 0$: only scalar $\Psi_2 \sim \ddot{h}_{ll}$ prohibited $\Rightarrow \text{III}_5$. 
Assume plane null wave $h_{\mu\nu} = \hat{h}_{\mu\nu} e^{ik_{\mu}x^{\mu}}$ with $\eta^{\alpha\beta} k_{\alpha} k_{\beta} = 0$.

⇒ Terms involving $c_1$ and $c_3$ do not contribute for $\Box h_{\mu\nu} = 0$.

⇒ Field equations expressed in Newman-Penrose basis:

\[ 0 = E_{nn} = -(c_2 + c_4 + c_5) \ddot{h}_{ln} + c_5 \ddot{h}_{m\bar{m}}, \]
\[ 0 = E_{mn} = E_{nm} = -(c_2 + c_4) \ddot{h}_{lm}, \]
\[ 0 = E_{m\bar{m}} = E_{\bar{m}m} = c_5 \ddot{h}_{ll}, \]
\[ 0 = E_{nl} = E_{ln} = -(c_2 + c_4) \ddot{h}_{ll}. \]

Possible E(2) classes:

- $c_2 + c_4 = c_5 = 0$: all six modes are allowed ⇒ $\text{II}_6$.
- $c_2 + c_4 = 0$, $c_5 \neq 0$: only scalar $\Psi_2 \sim \ddot{h}_{ll}$ prohibited ⇒ $\text{III}_5$.
- $c_2 + c_4 \neq 0$, $c_2 + c_4 + c_5 \neq 0$: also vector $\Psi_3 \sim \ddot{h}_{lm}$ prohibited ⇒ $\text{N}_3$. 
Newman-Penrose formalism

- Assume plane null wave \( h_{\mu\nu} = \hat{h}_{\mu\nu} e^{ik_\mu x^\mu} \) with \( \eta^{\alpha\beta} k_\alpha k_\beta = 0 \).
  \[ \Rightarrow \] Terms involving \( c_1 \) and \( c_3 \) do not contribute for \( \Box h_{\mu\nu} = 0 \).
  \[ \Rightarrow \] Field equations expressed in Newman-Penrose basis:

\[
0 = E_{nn} = -(c_2 + c_4 + c_5) \ddot{h}_{ln} + c_5 \ddot{h}_{m\bar{m}},
\]
\[
0 = E_{mn} = E_{nm} = -(c_2 + c_4) \ddot{h}_{lm},
\]
\[
0 = E_{m\bar{m}} = E_{\bar{m}m} = c_5 \ddot{h}_{ll},
\]
\[
0 = E_{nl} = E_{ln} = -(c_2 + c_4) \ddot{h}_{ll}.
\]

- Possible E(2) classes:
  - \( c_2 + c_4 = c_5 = 0 \): all six modes are allowed \( \Rightarrow \) II_6.
  - \( c_2 + c_4 = 0, c_5 \neq 0 \): only scalar \( \Psi_2 \sim \ddot{h}_{ll} \) prohibited \( \Rightarrow \) III_5.
  - \( c_2 + c_4 \neq 0, c_2 + c_4 + c_5 \neq 0 \): also vector \( \Psi_3 \sim \ddot{h}_{lm} \) prohibited \( \Rightarrow \) N_3.
  - \( c_2 + c_4 + c_5 = 0, c_5 \neq 0 \): also scalar \( \Phi_2 \sim \ddot{h}_{m\bar{m}} \) prohibited \( \Rightarrow \) N_2.
\[ c_2 = \sin \theta \cos \phi \]
\[ c_4 = \sin \theta \sin \phi \]
\[ c_5 = \cos \theta \]
Field content and geometry

- Fundamental fields in the gravity sector:
  - Coframe field $\theta^a = \theta^a_\mu \, dx^\mu$.
  - Flat spin connection $\omega^a_b = \omega^a_{b\mu} \, dx^\mu$.
Field content and geometry

- **Fundamental fields in the gravity sector:**
  - Coframe field \( \theta^a = \theta^a_{\mu} dx^\mu \).
  - Flat spin connection \( \omega^a_b = \omega^a_{b\mu} dx^\mu \).

- **Derived quantities:**
  - Frame field \( e_a = e^a_{\mu} \partial_\mu \) with \( \iota^a \theta^b = \delta^b_a \).
  - Metric \( g_{\mu\nu} = \eta_{ab} \theta^a_{\mu} \theta^b_{\nu} \).
  - Volume form \( \theta dx^4 = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \).
  - Levi-Civita connection
    \[
    \omega'_{ab} = -\frac{1}{2}(\iota^b \iota^c d\theta_a + \iota^c \iota^a d\theta_b - \iota^a \iota^b d\theta_c) \theta^c.
    \]
  - Torsion \( T^a = d\theta^a + \omega^a_b \wedge \theta^b \).
Field content and geometry

- **Fundamental fields in the gravity sector:**
  - Coframe field \( \theta^a = \theta^a_\mu dx^\mu \).
  - Flat spin connection \( \dot{\omega}^a_b = \dot{\omega}^a_{b\mu} dx^\mu \).

- **Derived quantities:**
  - Frame field \( e_a = e_a^\mu \partial_\mu \) with \( \iota_{e_a} \theta^b = \delta^b_a \).
  - Metric \( g_{\mu \nu} = \eta_{ab} \theta^a_\mu \theta^b_\nu \).
  - Volume form \( \theta d^4x = \theta^0 \land \theta^1 \land \theta^2 \land \theta^3 \).
  - Levi-Civita connection
    \[
    \ddot{\omega}^{ab} = -\frac{1}{2} (\iota_{e_b} \iota_{e_c} d\theta^a + \iota_{e_c} \iota_{e_a} d\theta^b - \iota_{e_a} \iota_{e_b} d\theta^c) \theta^c .
    \]
  - Torsion \( T^a = d\theta^a + \dot{\omega}^a_b \land \theta^b \).
  - Gauge fixing
    - Perform local Lorentz transformation:
      \[
      \theta'^a = \Lambda^a_b \theta^b , \quad \dot{\omega}'^a_b = \Lambda^a_c \dot{\omega}^c_d \Lambda^d_b + \Lambda^a_c d \Lambda^b_c .
      \]
    - Weitzenböck gauge: set \( \dot{\omega}^a_b \equiv 0 \).
Most general action and corresponding field equations

Most general action:

\[ S = \frac{1}{2\kappa^2} \int d^4x \, e \left( c_1 T^{\mu\nu\rho} T_{\mu\nu\rho} + c_2 T^{\mu\nu\rho} T_{\rho\nu\mu} + c_3 T^{\mu}_{\mu\rho} T^\nu_{\nu\rho} \right). \]
Most general action and corresponding field equations

- Most general action:
  \[
  S = \frac{1}{2\kappa^2} \int d^4 x \, e \left( c_1 T^{\mu\nu\rho} T_{\mu\nu\rho} + c_2 T^{\mu\nu\rho} T_{\rho\nu\mu} + c_3 T^{\mu}_{\ \mu\rho} T^{\nu}_{\ \nu\rho} \right).
  \]

- Linear perturbation:
  \[
  \theta^a_{\mu} = \Delta^a_{\mu} + \Delta^a_{\rho} \eta^{\rho\sigma} T_{\sigma\mu}.
  \]
Most general action and corresponding field equations

Most general action:

\[ S = \frac{1}{2\kappa^2} \int d^4 x \ e (c_1 T_{\mu\nu\rho} T_{\mu\nu\rho} + c_2 T_{\mu\nu\rho} T_{\rho\nu\mu} + c_3 T_{\mu\rho\nu} T_{\nu\rho\mu}) . \]

Linear perturbation:

\[ \theta^a_{\mu} = \Delta^a_{\mu} + \Delta^a_{\rho} \eta^{\rho\sigma} T_{\sigma\mu} . \]

Linearized vacuum field equations:

\[ \partial_\sigma (F^{\mu\rho\sigma} + B^{\mu\rho\sigma}) = 0 . \]
Most general action and corresponding field equations

- **Most general action:**
  \[ S = \frac{1}{2\kappa^2} \int d^4 x \ e (c_1 T^{\mu\nu\rho} T_{\mu\nu\rho} + c_2 T^{\mu\nu\rho} T_{\rho\nu\mu} + c_3 T^{\mu}_{\mu\rho} T^{\nu\rho}) . \]

- **Linear perturbation:**
  \[ \theta^a_\mu = \Delta^a_\mu + \Delta^a_\rho \eta^{\rho\sigma} T_{\sigma\mu} . \]

- **Linearized vacuum field equations:**
  \[ \partial_\sigma (F^{\mu\rho\sigma} + B^{\mu\rho\sigma}) = 0 . \]

- **Symmetric perturbation part,** \( \phi_{\mu\nu} = \tau_{(\mu\nu)} = \frac{1}{2} h_{\mu\nu} : \)

  \[ F^{\mu\rho\sigma} = (2c_1 + c_2) (\partial^\sigma \phi^{\mu\rho} - \partial^\rho \phi^{\mu\sigma}) \]

  \[ + c_3 \left[ (\partial^\sigma \phi^{\alpha}_\alpha - \partial_\alpha \phi^{\alpha\sigma}) \eta^{\mu\rho} - (\partial^\rho \phi^{\alpha}_\alpha - \partial_\alpha \phi^{\alpha\rho}) \eta^{\mu\sigma} \right] . \]
Most general action and corresponding field equations

- **Most general action:**
  \[
  S = \frac{1}{2\kappa^2} \int d^4x \, e (c_1 T^{\mu\nu\rho} T_{\mu\nu\rho} + c_2 T^{\mu\nu\rho} T_{\rho\nu\mu} + c_3 T^\mu_{\mu\rho} T^{\nu\rho}) .
  \]

- **Linear perturbation:**
  \[
  \theta^a_{\mu} = \Delta^a_{\mu} + \Delta^a_{\rho} \eta^{\rho\sigma} T_{\sigma\mu} .
  \]

- **Linearized vacuum field equations:**
  \[
  \partial_\sigma (F^{\mu\rho\sigma} + B^{\mu\rho\sigma}) = 0 .
  \]

- **Symmetric perturbation part,** \( \phi_{\mu\nu} = \tau(\mu\nu) = \frac{1}{2} h_{\mu\nu} : \)
  \[
  F^{\mu\rho\sigma} = (2c_1 + c_2) (\partial^\sigma \phi^{\mu\rho} - \partial^\rho \phi^{\mu\sigma})
  + c_3 \left[ (\partial^\sigma \phi^\alpha_{\alpha} - \partial_\alpha \phi^{\alpha\sigma}) \eta^{\mu\rho} - (\partial^\rho \phi^\alpha_{\alpha} - \partial_\alpha \phi^{\alpha\rho}) \eta^{\mu\sigma} \right] .
  \]

- **Antisymmetric perturbation part,** \( a_{\mu\nu} = \tau[\mu\nu] : \)
  \[
  B^{\mu\rho\sigma} = (2c_1 - c_2) (\partial^\sigma a^{\mu\rho} - \partial^\rho a^{\mu\sigma}) + (2c_2 + c_3) \partial^\mu a^{\sigma\rho} .
  \]
Decomposition of amplitude $\hat{\tau}_{\lambda\rho}$ relative to wave vector:

$$\hat{\tau}_{\beta\sigma} = k_\beta k_\sigma U + V_\beta k_\sigma + k_\beta W_\sigma + Q_{\beta\sigma}.$$
Decomposition of amplitude $\hat{\tau}_{\lambda\rho}$ relative to wave vector:

$$\hat{\tau}_{\beta\sigma} = k_\beta k_\sigma U + V_\beta k_\sigma + k_\beta W_\sigma + Q_{\beta\sigma}.$$ 

Conditions imposed on projected components:

$$k_\alpha V^\alpha = 0, \quad k_\alpha W^\alpha = 0, \quad k_\alpha Q^\alpha_\beta = 0, \quad k_\alpha Q_\beta^\alpha = 0.$$
Decomposition of amplitude $\hat{r}_{\lambda\rho}$ relative to wave vector:

$$\hat{r}_{\beta\sigma} = k_\beta k_\sigma U + V_\beta k_\sigma + k_\beta W_\sigma + Q_{\beta\sigma}.$$ 

Conditions imposed on projected components:

$$k_\alpha V^\alpha = 0, \quad k_\alpha W^\alpha = 0, \quad k_\alpha Q^\alpha_\beta = 0, \quad k_\alpha Q^\alpha_\beta = 0.$$ 

$\Rightarrow$ $U$ and $V_\alpha$ cancel in field equations - pure gauge fields.
Decomposition of amplitude $\hat{\tau}_{\lambda\rho}$ relative to wave vector:

$$\hat{\tau}_{\beta\sigma} = k_{\beta} k_{\sigma} U + V_{\beta} k_{\sigma} + k_{\beta} W_{\sigma} + Q_{\beta\sigma}.$$ 

Conditions imposed on projected components:

$$k_\alpha V_\alpha = 0, \quad k_\alpha W_\alpha = 0, \quad k_\alpha Q^\alpha_{\beta} = 0, \quad k_\alpha Q_{\beta}^\alpha = 0.$$

$\Rightarrow$ $U$ and $V_\alpha$ cancel in field equations - pure gauge fields.

Write $Q_{\alpha\beta}$ in trace, symmetric traceless and antisymmetric part:

$$Q^{\tau\kappa} = S^{\tau\kappa} + A^{\tau\kappa} + \frac{1}{3} \left( \eta^{\tau\kappa} - \frac{k^\tau k^\kappa}{\eta^{\mu\nu} k_\mu k_\nu} \right) Q^{\sigma\sigma}.$$
Principal polynomial and speed of propagation

- Decomposition of amplitude $\hat{\tau}_{\lambda\rho}$ relative to wave vector:
  \[ \hat{\tau}_{\beta\sigma} = k_\beta k_\sigma U + V_\beta k_\sigma + k_\beta W_\sigma + Q_{\beta\sigma}. \]

- Conditions imposed on projected components:
  \[ k_\alpha V^\alpha = 0, \quad k_\alpha W^\alpha = 0, \quad k_\alpha Q^{\alpha\beta} = 0, \quad k_\alpha Q^\alpha_{\beta\alpha} = 0. \]

  $\Rightarrow$ $U$ and $V_\alpha$ cancel in field equations - pure gauge fields.

- Write $Q_{\alpha\beta}$ in trace, symmetric traceless and antisymmetric part:
  \[ Q^{\tau\kappa} = S^{\tau\kappa} + A^{\tau\kappa} + \frac{1}{3} \left( \eta^{\tau\kappa} - \frac{k^\tau k^\kappa}{\eta^{\mu\nu} k_\mu k_\nu} \right) Q^\sigma_\sigma. \]

- Decomposed field equations:
  \[ 0 = (2c_1 + c_2 + c_3)(\eta^{\alpha\beta} k_\alpha k_\beta)^2 W^\kappa, \quad 0 = (2c_1 - c_2)\eta^{\alpha\beta} k_\alpha k_\beta A^{\tau\kappa}, \]
  \[ 0 = (2c_1 + c_2 + 3c_3)\eta^{\alpha\beta} k_\alpha k_\beta Q^{\tau\tau}, \quad 0 = (2c_1 + c_2)\eta^{\alpha\beta} k_\alpha k_\beta S^{\tau\kappa}. \]
Principal polynomial and speed of propagation

- Decomposition of amplitude $\hat{t}_{\lambda\rho}$ relative to wave vector:
  \[
  \hat{t}_{\beta\sigma} = k_\beta k_\sigma U + V_\beta k_\sigma + k_\beta W_\sigma + Q_{\beta\sigma}.
  \]
- Conditions imposed on projected components:
  \[
  k_\alpha V^\alpha = 0, \quad k_\alpha W^\alpha = 0, \quad k_\alpha Q_{\alpha\beta} = 0, \quad k_\alpha Q_{\beta\alpha} = 0.
  \]
  \[\Rightarrow\]
  $U$ and $V_\alpha$ cancel in field equations - pure gauge fields.

- Write $Q_{\alpha\beta}$ in trace, symmetric traceless and antisymmetric part:
  \[
  Q^{\tau\kappa} = S^{\tau\kappa} + A^{\tau\kappa} + \frac{1}{3} \left( \eta^{\tau\kappa} - \frac{k^\tau k^\kappa}{\eta^{\mu\nu} k_\mu k_\nu} \right) Q^{\sigma\sigma}.
  \]
- Decomposed field equations:
  \[
  0 = \left( 2c_1 + c_2 + c_3 \right) \left( \eta^{\alpha\beta} k_\alpha k_\beta \right)^2 W_\kappa, \quad 0 = \left( 2c_1 - c_2 \right) \eta^{\alpha\beta} k_\alpha k_\beta A^{\tau\kappa},
  \]
  \[
  0 = \left( 2c_1 + c_2 + 3c_3 \right) \eta^{\alpha\beta} k_\alpha k_\beta Q^{\tau\tau}, \quad 0 = \left( 2c_1 + c_2 \right) \eta^{\alpha\beta} k_\alpha k_\beta S^{\tau\kappa}.
  \]
- Principal polynomial $\bar{p}(x, k) = \text{const.} \cdot \left( \eta^{\alpha\beta} k_\alpha k_\beta \right)^{15}$. 
Principal polynomial and speed of propagation

- Decomposition of amplitude $\hat{\tau}_{\lambda \rho}$ relative to wave vector:
  \[ \hat{\tau}_{\beta \sigma} = k_\beta k_\sigma U + V_\beta k_\sigma + k_\beta W_\sigma + Q_{\beta \sigma}. \]

- Conditions imposed on projected components:
  \[ k_\alpha V^\alpha = 0, \quad k_\alpha W^\alpha = 0, \quad k_\alpha Q^{\alpha \beta} = 0, \quad k_\alpha Q_{\beta \alpha} = 0. \]

  $\Rightarrow$ $U$ and $V_\alpha$ cancel in field equations - pure gauge fields.

- Write $Q_{\alpha \beta}$ in trace, symmetric traceless and antisymmetric part:
  \[ Q^{\tau \kappa} = S^{\tau \kappa} + A^{\tau \kappa} + \frac{1}{3} \left( \eta^{\tau \kappa} - \frac{k^\tau k^\kappa}{\eta^{\mu \nu} k_\mu k_\nu} \right) Q^{\sigma \sigma}. \]

- Decomposed field equations:
  \[ 0 = (2c_1 + c_2 + c_3)(\eta^{\alpha \beta} k_\alpha k_\beta)^2 W_\kappa, \quad 0 = (2c_1 - c_2)\eta^{\alpha \beta} k_\alpha k_\beta A^{\tau \kappa}, \]
  \[ 0 = (2c_1 + c_2 + 3c_3)\eta^{\alpha \beta} k_\alpha k_\beta Q^{\tau \tau}, \quad 0 = (2c_1 + c_2)\eta^{\alpha \beta} k_\alpha k_\beta S^{\tau \kappa}. \]

- Principal polynomial $\bar{p}(x, k) = \text{const.} \cdot (\eta^{\alpha \beta} k_\alpha k_\beta)^{15}.$

- $\eta^{\alpha \beta} k_\alpha k_\beta = 0 \iff$ propagation at the speed of light.
Newman-Penrose formalism

- Assume plane null wave $\tau_{\mu\nu} = \hat{\tau}_{\mu\nu} e^{ik_{\mu}x^\mu}$ with $\eta^{\alpha\beta} k_\alpha k_\beta = 0$. 
Newman-Penrose formalism

- Assume plane null wave \( \tau_{\mu\nu} = \hat{\tau}_{\mu\nu} e^{ik_{\mu}x^{\mu}} \) with \( \eta^{\alpha\beta} k_{\alpha} k_{\beta} = 0 \).

\[ \Rightarrow \text{Field equations expressed in Newman-Penrose basis:} \]

\[ 0 = E_{nn} = (2c_1 + c_2 + c_3)\ddot{\phi}_{nl} + 2c_3\ddot{\phi}_{m\bar{m}} + (2c_1 + c_2 + c_3)\dddot{a}_{nl}, \]

\[ 0 = E_{mn} = (2c_1 + c_2)\ddot{\phi}_{ml} + (2c_1 - c_2)\dddot{a}_{ml}, \]

\[ 0 = E_{nm} = -c_3\dddot{\phi}_{lm} - (2c_2 + c_3)\dddot{a}_{lm}, \]

\[ 0 = E_{m\bar{m}} = -c_3\dddot{\phi}_{l}, \]

\[ 0 = E_{ln} = (2c_1 + c_2)\dddot{\phi}_{l}, \]
Newman-Penrose formalism

Assume plane null wave \( \tau_{\mu\nu} = \hat{\tau}_{\mu\nu} e^{ik_{\mu}x^{\mu}} \) with \( \eta^{\alpha\beta} k_{\alpha} k_{\beta} = 0 \).

\[ 0 = E_{nn} = (2c_1 + c_2 + c_3) \ddot{\phi}_{nl} + 2c_3 \ddot{\phi}_{m\bar{m}} + (2c_1 + c_2 + c_3) \ddot{a}_{nl} , \]
\[ 0 = E_{mn} = (2c_1 + c_2) \ddot{\phi}_{ml} + (2c_1 - c_2) \ddot{a}_{ml} , \]
\[ 0 = E_{nm} = -c_3 \ddot{\phi}_{lm} - (2c_2 + c_3) \ddot{a}_{lm} , \]
\[ 0 = E_{m\bar{m}} = -c_3 \ddot{\phi}_{l\bar{l}} , \]
\[ 0 = E_{ln} = (2c_1 + c_2) \ddot{\phi}_{l\bar{l}} , \]

Possible \( E(2) \) classes:

- 2\( c_1 + c_2 = c_3 = 0 \): all six modes are allowed \( \Rightarrow \text{II}_6 \).
Newman-Penrose formalism

Assume plane null wave \( \tau_{\mu\nu} = \hat{\tau}_{\mu\nu} e^{ik_{\mu}x^{\mu}} \) with \( \eta^{\alpha\beta} k_{\alpha} k_{\beta} = 0 \).

\( \Rightarrow \) Field equations expressed in Newman-Penrose basis:

\[
\begin{align*}
0 &= E_{nn} = (2c_1 + c_2 + c_3) \ddot{\phi}_{nl} + 2c_3 \ddot{\phi}_{m\bar{m}} + (2c_1 + c_2 + c_3) \ddot{a}_{nl}, \\
0 &= E_{mn} = (2c_1 + c_2) \ddot{\phi}_{ml} + (2c_1 - c_2) \ddot{a}_{ml}, \\
0 &= E_{nm} = -c_3 \ddot{\phi}_{lm} - (2c_2 + c_3) \ddot{a}_{lm}, \\
0 &= E_{m\bar{m}} = -c_3 \ddot{\phi}_{l}, \\
0 &= E_{ln} = (2c_1 + c_2) \ddot{\phi}_{ll}, .
\end{align*}
\]

Possible E(2) classes:

- \( 2c_1 + c_2 = c_3 = 0 \): all six modes are allowed \( \Rightarrow \) II\(_6\).
- \( 2c_1(c_2 + c_3) + c_2^2 = 0, 2c_1 + c_2 + c_3 \neq 0 \): only scalar \( \Psi_2 \sim \dot{h}_{ll} = 0 \) \( \Rightarrow \) III\(_5\).
Newman-Penrose formalism

- Assume plane null wave $\tau_{\mu\nu} = \hat{\tau}_{\mu\nu} e^{ik_{\mu}x^{\mu}}$ with $\eta^{\alpha\beta} k_{\alpha} k_{\beta} = 0$.

⇒ Field equations expressed in Newman-Penrose basis:

\[
\begin{align*}
0 &= E_{nn} = (2c_1 + c_2 + c_3) \ddot{\phi}_{nl} + 2c_3 \ddot{\phi}_{m\bar{m}} + (2c_1 + c_2 + c_3) \ddot{a}_{nl}, \\
0 &= E_{mn} = (2c_1 + c_2) \ddot{\phi}_{ml} + (2c_1 - c_2) \ddot{a}_{ml}, \\
0 &= E_{nm} = -c_3 \ddot{\phi}_{lm} - (2c_2 + c_3) \ddot{a}_{lm}, \\
0 &= E_{m\bar{m}} = -c_3 \ddot{\phi}_{\|}, \\
0 &= E_{l\bar{n}} = (2c_1 + c_2) \ddot{\phi}_{\|}. 
\end{align*}
\]

- Possible $E(2)$ classes:
  - $2c_1 + c_2 = c_3 = 0$: all six modes are allowed $\Rightarrow \text{II}_{6}$.
  - $2c_1(c_2 + c_3) + c_2^2 = 0$, $2c_1 + c_2 + c_3 \neq 0$: only scalar $\Psi_2 \sim \ddot{h}_{\|} = 0$ $\Rightarrow \text{III}_{5}$.
  - $2c_1(c_2 + c_3) + c_2^2 \neq 0$, $2c_1 + c_2 + c_3 \neq 0$: also vector $\Psi_3 \sim \ddot{h}_{lm} = 0$ $\Rightarrow \text{N}_3$. 

Manuel Hohmann (University of Tartu)  Waves in teleparallel gravity  Athens - 25. January 2019  24/29
Newman-Penrose formalism

Assume plane null wave \( \tau_{\mu\nu} = \hat{\tau}_{\mu\nu} e^{ik_{\mu}x^{\mu}} \) with \( \eta^{\alpha\beta} k_\alpha k_\beta = 0 \).

Field equations expressed in Newman-Penrose basis:

\[
0 = E_{nn} = (2c_1 + c_2 + c_3) \ddot{\phi}_{nl} + 2c_3 \ddot{\phi}_{m\bar{m}} + (2c_1 + c_2 + c_3) \ddot{a}_{nl},
\]
\[
0 = E_{mn} = (2c_1 + c_2) \ddot{\phi}_{ml} + (2c_1 - c_2) \ddot{a}_{ml},
\]
\[
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\]
\[
0 = E_{m\bar{m}} = -c_3 \ddot{\phi}_{l\bar{l}},
\]
\[
0 = E_{ln} = (2c_1 + c_2) \ddot{\phi}_{l\bar{l}}.
\]

Possible \( E(2) \) classes:

- \( 2c_1 + c_2 = c_3 = 0 \): all six modes are allowed \( \Rightarrow \) II_6.
- \( 2c_1(2c_2 + c_3) + c_2^2 = 0 \), \( 2c_1 + c_2 + c_3 \neq 0 \): only scalar \( \Psi_2 \sim \ddot{h}_{ll} = 0 \) \( \Rightarrow \) III_5.
- \( 2c_1(2c_2 + c_3) + c_2^2 \neq 0 \), \( 2c_1 + c_2 + c_3 \neq 0 \): also vector \( \Psi_3 \sim \ddot{h}_{lm} = 0 \) \( \Rightarrow \) N_3.
- \( 2c_1 + c_2 + c_3 = 0 \), \( c_3 \neq 0 \): also scalar \( \Phi_{22} \sim \ddot{h}_{m\bar{m}} = 0 \) \( \Rightarrow \) N_2.
Gravitational wave polarisations

\[ c_1 = \sin \theta \cos \phi \]
\[ c_2 = \sin \theta \sin \phi \]
\[ c_3 = \cos \theta \]
Teleparallel gravity:
- Fields are tetrad and flat spin connection.
- Only torsion, no curvature or non-metricity.
- Most general theory needs 3 parameters at linearized level.

Results:
- Gravitational waves propagate at the speed of light.
- Polarisation classes $N_2$, $N_3$, $III_5$, $II_6$: tensor + maybe more.
Summary

Teleparallel gravity:
- Fields are tetrad and flat spin connection.
- Only torsion, no curvature or non-metricity.
- Most general theory needs 3 parameters at linearized level.

Symmetric teleparallel gravity:
- Fields are metric and flat, symmetric affine connection.
- Only non-metricity, no curvature or torsion.
- Most general theory needs 5 parameters at linearized level.

Results:
Gravitational waves propagate at the speed of light.
Polarisation classes $N_2$, $N_3$, III$_5$, II$_6$: tensor + maybe more.
Teleparallel gravity:
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