**Symmetries under group actions**

**Definition.** A symmetry of a tetrad field equation is a group action \( \phi : \mathbb{G} \times M \rightarrow M \) such that the induced metric (4) and affine connection (5) are invariant, i.e., \( \phi^*g = g \) and \( \phi^*\omega = \omega \) for all \( \phi \in \mathbb{G} \), where

\[
(\phi^*g)^{\mu\nu} = g^{\phi^\mu\phi^\nu} = g^{\mu\nu}, \quad (\phi^*\omega)^{\mu\nu\rho} = \omega^{\phi^\mu\phi^\nu\phi^\rho} = \omega^{\mu\nu\rho}
\]

and

\[
(\phi^*\mathcal{L})_g = \phi^*\mathcal{L}_g = \frac{1}{2} \frac{d}{d\tau} g_{\phi^\mu\phi^\nu} = \frac{1}{2} \frac{d}{d\tau} g^{\mu\nu}
\]

for all \( \phi \in \mathbb{G} \), where

\[
(\phi^*f)_{\phi^\mu\phi^\nu} = f^{\phi^\mu\phi^\nu}, \quad (\phi^*\omega)_{\phi^\mu\phi^\nu\phi^\rho} = \omega^{\phi^\mu\phi^\nu\phi^\rho}, \quad (\phi^*\mathcal{L})_{\phi^\mu\phi^\nu} = \mathcal{L}_{\phi^\mu\phi^\nu}
\]

for all \( \phi \in \mathbb{G} \).

**Tetrad and spin connections as cosmological symmetries**

We call a tetrad/field geometry the triple

\[ (\mathfrak{M}, \mathfrak{G}, \mathfrak{F}) \]

where we used the total covariant derivative

\[ D_{\mathfrak{G}} \mathfrak{F} = D_{\mathfrak{G}}^{\mathfrak{G}} + D_{\mathfrak{G}}^{\mathfrak{F}} = D_{\mathfrak{G}}^{\mathfrak{G}} + D_{\mathfrak{G}}^{\mathfrak{F}} \]

and

\[
(\mathfrak{G}^{\mathfrak{G}})^{\phi^\mu\phi^\nu} = \mathfrak{G}^{\phi^\mu\phi^\nu} = \mathfrak{G}^{\phi^\mu\phi^\nu}, \quad (\mathfrak{G}^{\mathfrak{F}})^{\phi^\mu\phi^\nu} = \mathfrak{F}^{\phi^\mu\phi^\nu} = \mathfrak{F}^{\phi^\mu\phi^\nu}
\]

Solving the antisymmetric field equation

There are different ways to solve the antisymmetric field equations (22):

1. For the solutions with \( f = 0 \) and \( f = 0 \), so that \((\hat{T}, \hat{F}) = (\mathcal{L}, -\mathcal{L})\), the equations (22) are solved identically for any field configuration.

2. Field configurations with \( f = 0 \) and \( f = 0 \), i.e., constant torsion scalar and constant scalar fields, always solve the equations (22), independently of the function \( f \). The remaining field equations (21) reduce to general relativity with cosmological constant.

3. Field configurations where \( \Gamma^\nu_{\mu\nu} \) depend only on a single coordinate \( y \) satisfy \( \theta^\nu_{\mu\nu} \) and \( \eta^\nu_{\mu\nu} \) solve the equations (22) if the six vector fields, which are defined by the terms in brackets (22) for the six values of \( \eta (y) \), are tangent to the hypersurfaces of constant \( y \), independently of the function \( f \).

4. In the general case, the solutions depend on \( f \). A general field configuration solves the equations (22) if the six vector fields mentioned above are tangent to the hypersurfaces of constant \( f \).

**Cosmological dynamics**

**Scalar field equation**

Scalar field equation for the shown cosmological tetrads (single scalar field case):

\[
\mathcal{E}_s = -2f_s + 2(\mathcal{E}_A^s - \mathcal{E}_B^s) = 0
\]

**Tetrad field equations:**

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1. With the Weitzenböck tetrad (34) or diagonal tetrad (29) and spin connection (35) we have

\[
\frac{1}{f_s^2} + 2f_s^2 + (f_s^2 - 1) = 2f_s^2 + 2f_s^2 = 0
\]

Tetrad field equations: \( k = 1 \), real tetrad

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Tetrad field equations: \( k = 1 \), complex tetrad

With the Weitzenböck tetrad (34) or diagonal tetrad (29) and spin connection (35) we have

\[
\frac{1}{f_s^2} + 2f_s^2 + (f_s^2 - 1) = 2f_s^2 + 2f_s^2 = 0
\]