

# Selected Topics in the Theories of Gravity

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## 1 Speed of gravitational waves

In this lecture we discuss the properties of gravitational waves in alternative theories of gravity. Our discussion is based on the previous two lectures on gravitational waves in general relativity. We only consider the propagation of gravitational waves in vacuum, where the energy-momentum tensor  $T_{\mu\nu}$  vanishes. In particular we are interested in theories in which the flat Minkowski metric  $g_{\mu\nu} = \eta_{\mu\nu}$  solves the gravitational vacuum field equations. In addition to the metric there may be other gravitational fields present in the theory, such as scalars, vectors or additional tensor fields, which we collectively denote by  $X$ . We assume that these have some fixed, constant background value  $X_0$ , so that the full vacuum solution is given by  $(\eta_{\mu\nu}, X_0)$ . This allows us to consider small perturbations around this vacuum solution,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad X = X_0 + \chi, \quad (1.1)$$

and to linearize the field equations in the perturbations  $h_{\mu\nu}$  and  $\chi$ . The resulting field equations are then generically of the form

$$\mathcal{D}_A{}^B(h, \chi)_B = 0, \quad (1.2)$$

where  $(h, \chi)_A$  runs over all components of the metric perturbation  $h_{\mu\nu}$  and the auxiliary field perturbation  $\chi$ , and  $\mathcal{D}_A{}^B$  denotes some differential operator. This means that  $\mathcal{D}_A{}^B$  is a matrix whose components contain constants and partial derivatives  $\partial_\mu$  acting on the fields  $(h, \chi)_A$ . In order to determine solutions to these linearized field equations we consider the plane wave ansatz

$$h_{\mu\nu}(x) = \hat{h}_{\mu\nu} e^{ik_\rho x^\rho}, \quad \chi(x) = \hat{\chi} e^{ik_\rho x^\rho} \quad (1.3)$$

with constant wave covector  $k_\mu$  and complex amplitudes  $\hat{h}_{\mu\nu}$  and  $\hat{\chi}$ . This ansatz corresponds to a single mode in a Fourier decomposition of the gravitational wave - the physical gravitational wave must be real, of course. This ansatz is useful since the derivatives  $\partial_\mu$  from the differential operator  $\mathcal{D}_A{}^B$  act only on the exponential function and yield the well-known result

$$\partial_\mu e^{ik_\rho x^\rho} = ik_\mu e^{ik_\rho x^\rho}. \quad (1.4)$$

This means that we can replace the differential operator  $\mathcal{D}_A{}^B$  in the linearized field equations (1.2) by its Fourier transform  $\hat{\mathcal{D}}_A{}^B(k)$ , where we simply replace all derivatives  $\partial_\mu$  by  $ik_\mu$ . The resulting Fourier transformed field equations,

$$\hat{\mathcal{D}}_A{}^B(k)(\hat{h}, \hat{\chi})_B = 0, \quad (1.5)$$

are now simply a set of linear algebraic equations. These equations possess non-trivial solutions if and only if the matrix  $\hat{\mathcal{D}}_A^B(k)$  is degenerate, i.e., if its determinant vanishes. We can therefore determine all allowed wave covectors  $k_\mu$  by solving the equation  $\det \hat{\mathcal{D}}_A^B(k) = 0$ , which is a polynomial equation in  $k_\mu$ .

As an example consider the linearized vacuum field equations in scalar tensor gravity,

$$\square\psi = 0, \quad \square h_{\mu\nu} + h^\rho{}_{\rho,\mu\nu} - h_{\mu\rho,\nu}{}^\rho - h_{\nu\rho,\mu}{}^\rho = -2\frac{\psi_{,\mu\nu}}{\Psi_0}. \quad (1.6)$$

Choosing a gauge in which

$$h_{\mu\rho}{}^{,\rho} - \frac{1}{2}h^\rho{}_{\rho,\mu} - \frac{\psi_{,\mu}}{\Psi_0} = 0 \quad (1.7)$$

simplifies the second field equation to  $\square h_{\mu\nu} = 0$ . Now the differential operator is simply given by  $\mathcal{D}_A^B = \square\delta_A^B$  and its Fourier transform is  $\hat{\mathcal{D}}_A^B = -k_\mu k^\mu \delta_A^B$ . The indices run over the 10 independent components of  $h_{\mu\nu}$  and the scalar field, so that the determinant is given by  $(-k_\mu k^\mu)^{11}$ . This vanishes if and only if  $k_\mu$  is a null covector, from which we see that gravitational waves in scalar tensor gravity propagate at the speed of light.

Although there has been no direct observation of gravitational waves, there are indirect hints which indicate that they should propagate at the speed of light. These indirect hints come from the observation of ultrarelativistic cosmic particles. If the speed of gravitational waves would be lower than the speed of light, there would be an energy cut-off if the velocity of cosmic particles equals the speed of gravitational waves, since any particles which are faster would dissipate energy in the form of gravitational waves. In the following we will therefore restrict ourselves to gravitational waves which propagate at the speed of light.

## 2 Polarization of gravitational waves

Recall from the previous two lectures that the observable effect of gravitational waves on a set of test masses, whose trajectories are timelike geodesics, is given by the geodesic deviation. We have seen that a suitable coordinate system is given by Fermi coordinates around the trajectory of a given observer. Here the time coordinate  $t$  is given by the proper time along the observer geodesic and the spatial coordinates  $x^i$  are chosen so that on the observer trajectory  $x^i = 0$ , the metric is given by the Minkowski metric  $\eta_{\mu\nu}$  and the Christoffel symbols  $\Gamma^\mu{}_{\nu\rho}$  vanish. In these coordinates the acceleration of a test mass is given by

$$\frac{d^2 x^i}{dt^2} = -R_{0i0j} x^j. \quad (2.1)$$

The 6 components  $R_{0i0j}$  of the Riemann tensor are denoted electric components.

It is convenient to use a complex double null basis of the tangent spaces introduced by Newman and Penrose which is spanned by the vectors  $l^\mu, n^\mu, m^\mu, \bar{m}^\mu$  given by

$$l = \partial_0 + \partial_3, \quad n = \frac{1}{2}(\partial_0 - \partial_3), \quad m = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2), \quad \bar{m} = \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2), \quad (2.2)$$

and to express tensors in terms of this basis [3]. For example, the Minkowski metric in this basis takes the form

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (2.3)$$

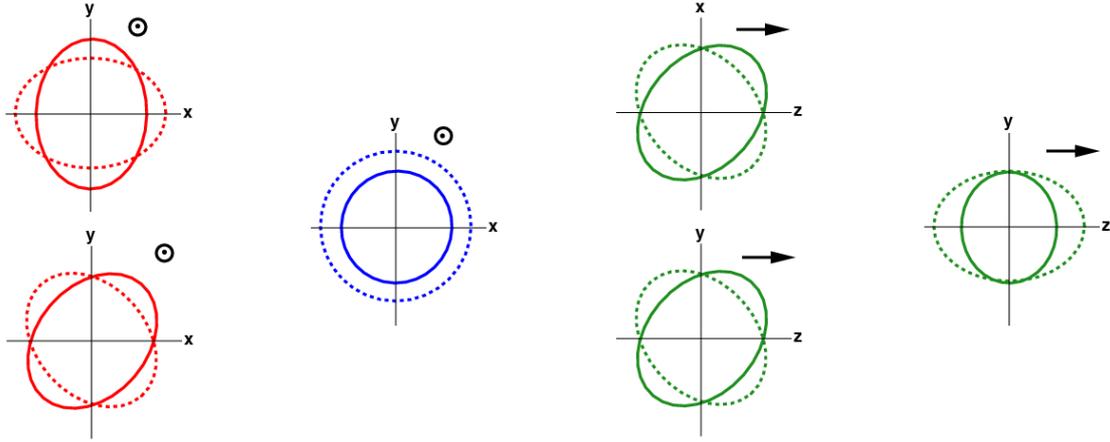


Figure 1: Effects of gravitational waves on a set of test masses, from left to right, for the wave components  $\Psi_4, \Phi_{22}, \Psi_3, \Psi_2$ . For the two complex components  $\Psi_4, \Psi_3$  the upper image shows the real part and the lower image shows the imaginary part.

The diagonal elements vanish since  $l^\mu, n^\mu, m^\mu, \bar{m}^\mu$  are null vectors, and the only non-vanishing scalar products are  $n_\mu l^\mu = -1$  and  $m_\mu \bar{m}^\mu = 1$ .

We now consider a plane wave propagating in the positive  $x^3$  direction, which is given by

$$h_{\mu\nu} = \hat{h}_{\mu\nu} e^{-i\omega l_\mu x^\mu} = \hat{h}_{\mu\nu} e^{i\omega(t-x^3)} = \hat{h}_{\mu\nu} e^{i\omega u}, \quad \chi = \hat{\chi} e^{i\omega u}, \quad (2.4)$$

where we have introduced the retarded time  $u = t - x^3$ . This means that all quantities which are constructed from the metric perturbation  $h_{\mu\nu}$  and the perturbations  $\chi$  of other gravitational fields depend only on  $u$ . In particular we can calculate the Riemann tensor for this plane wave and find that it is fully determined by the components in the Newman-Penrose basis given by

$$\begin{aligned} \Psi_2 &= -\frac{1}{6} R_{nlnl} = \frac{1}{12} \ddot{h}_{ll}, & \Psi_3 &= -\frac{1}{2} R_{nlm\bar{m}} = -\frac{1}{2} \overline{R_{nlm\bar{m}}} = \frac{1}{4} \ddot{h}_{l\bar{m}} = \frac{1}{4} \overline{\ddot{h}_{lm}}, \\ \Psi_4 &= -R_{n\bar{m}m\bar{m}} = -\overline{R_{nmnm}} = \frac{1}{2} \ddot{h}_{\bar{m}\bar{m}} = \frac{1}{2} \overline{\ddot{h}_{mm}}, & \Phi_{22} &= -R_{nmn\bar{m}} = \frac{1}{2} \ddot{h}_{m\bar{m}}, \end{aligned} \quad (2.5)$$

where dots denote derivatives with respect to  $u$ . Note that  $\Psi_3$  and  $\Psi_4$  are complex. From these one can easily calculate the electric components of the Riemann tensor and see its influence on a set of test masses. These are shown in figure 1.

We now consider two observers located at the same point  $x^\mu = x'^\mu = 0$  whose coordinate systems are related by a Lorentz transform, such that both agree on the observed frequency  $\omega$  and direction of the wave. This means that the Lorentz transform must leave the retarded time  $u = u'$  and thus the wave vector  $l^\mu = l'^\mu$  unchanged. In the Newman-Penrose basis this Lorentz transform can be parametrized in the form

$$\begin{pmatrix} l'^\mu \\ n'^\mu \\ m'^\mu \\ \bar{m}'^\mu \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha \bar{\alpha} & 1 & \bar{\alpha} e^{i\phi} & \alpha e^{-i\phi} \\ \alpha & 0 & e^{i\phi} & 0 \\ \bar{\alpha} & 0 & 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} l^\mu \\ n^\mu \\ m^\mu \\ \bar{m}^\mu \end{pmatrix} = \Lambda(\phi, \alpha) \begin{pmatrix} l^\mu \\ n^\mu \\ m^\mu \\ \bar{m}^\mu \end{pmatrix} \quad (2.6)$$

using parameters  $\phi \in [0, 2\pi)$  and  $\alpha \in \mathbb{C}$ . The product of two such matrices is given by

$$\Lambda(\phi', \alpha') \Lambda(\phi, \alpha) = \Lambda(\phi + \phi', \alpha' + \alpha e^{i\phi'}), \quad (2.7)$$

which shows that these Lorentz transforms form a group isomorphic to  $U(1) \times \mathbb{C} \cong SO(2) \times \mathbb{R}^2 = E(2)$ , the two-dimensional Euclidean group spanned by rotations with angle  $\phi$  and translations by  $(\Re\alpha, \Im\alpha)$ .

If we apply the Lorentz transform shown above to the components of the Riemann tensor we see that they transform as

$$\begin{pmatrix} \Psi'_2 \\ \Psi'_3 \\ \bar{\Psi}'_3 \\ \Psi'_4 \\ \bar{\Psi}'_4 \\ \Phi'_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 3\bar{\alpha} & e^{-i\phi} & 0 & 0 & 0 & 0 \\ 3\alpha & 0 & e^{i\phi} & 0 & 0 & 0 \\ 6\bar{\alpha}^2 & 4\bar{\alpha}e^{i\phi} & 0 & e^{-2i\phi} & 0 & 0 \\ 6\alpha^2 & 0 & 4\alpha e^{-i\phi} & 0 & e^{2i\phi} & 0 \\ 6\alpha\bar{\alpha} & 2\alpha e^{-i\phi} & 2\bar{\alpha}e^{i\phi} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Psi_2 \\ \Psi_3 \\ \bar{\Psi}_3 \\ \Psi_4 \\ \bar{\Psi}_4 \\ \Phi_{22} \end{pmatrix} \quad (2.8)$$

From these transformations we can read off some properties of the corresponding waves. Under pure rotations  $\alpha = 0$  the matrix becomes diagonal and we see that  $\Psi_2$  and  $\Phi_{22}$  have helicity 0, so they are scalar modes;  $\Psi_3$  and  $\bar{\Psi}_3$  have helicity  $\pm 1$ , so they are vector modes;  $\Psi_4$  and  $\bar{\Psi}_4$  have helicity  $\pm 2$ , so they are tensor modes; the latter are the two polarizations found in general relativity.

Decomposing the matrix above into its real and imaginary parts we see that we have constructed a six-dimensional, real representation of the Euclidean group. This representation is not irreducible, since it contains subspaces which are invariant under the group action. These subspaces allow a decomposition of the full six-dimensional representation space into subsets which is invariant under Lorentz transformations. This means that if a two observers measure a gravitational wave, they may not agree on the individual components  $\Psi_2, \Psi_3, \Psi_4, \Phi_{22}$ , but they agree on the subset in which this wave is located. The subsets are as follows, and are labeled by the Petrov type of the non-vanishing Weyl tensor and by the dimension of the corresponding representation of  $E(2)$ :

- $O_0$ :  $\Psi_2 = \Psi_3 = \Psi_4 = \Phi_{22} = 0$ .
- $O_1$ :  $\Psi_2 = \Psi_3 = \Psi_4 = 0, \Phi_{22} \neq 0$ .
- $N_2$ :  $\Psi_2 = \Psi_3 = \Phi_{22} = 0, \Psi_4 \neq 0$ .
- $N_3$ :  $\Psi_2 = \Psi_3 = 0, \Psi_4 \neq 0, \Phi_{22} \neq 0$ .
- $III_5$ :  $\Psi_2 = 0, \Psi_3 \neq 0$ .
- $II_6$ :  $\Psi_2 \neq 0$ .

In any theory of gravity some of these wave types may be allowed, while other may be prohibited by the gravitational field equations. The largest set of allowed waves determines the  $E(2)$  class of the theory. As an example, consider the linearized vacuum field equations of scalar tensor gravity given by

$$\square\psi = 0, \quad R_{\mu\nu} = \frac{\psi_{,\mu\nu}}{\Psi_0}, \quad (2.9)$$

together with the wave ansatz

$$h_{\mu\nu} = \hat{h}_{\mu\nu} e^{i\omega u}, \quad \psi = \hat{\psi} e^{i\omega u}. \quad (2.10)$$

This ansatz already solves the field equation for the scalar field. The field equation for the metric yields

$$R_{\mu\nu} = \frac{\psi_{,\mu\nu}}{\Psi_0} = -\frac{\omega^2 \hat{\psi}}{\Psi_0} e^{i\omega u} l_\mu l_\nu, \quad (2.11)$$

so that the only non-vanishing component of the Ricci tensor is  $R_{nn}$ . From the wave ansatz we find the Ricci tensor

$$R_{nn} = -\ddot{h}_{m\bar{m}}, \quad R_{nm} = -\frac{1}{2}\ddot{h}_{lm}, \quad R_{n\bar{m}} = -\frac{1}{2}\ddot{h}_{l\bar{m}}, \quad R_{m\bar{m}} = -\frac{1}{2}\ddot{h}_{ll} \quad (2.12)$$

and all other components vanish identically. The field equations thus yield

$$\hat{h}_{lm} = \hat{h}_{l\bar{m}} = \hat{h}_{ll} = 0, \quad (2.13)$$

so that  $\Psi_2 = \Psi_3 = 0$ . Gravitational waves with  $\Psi_4 \neq 0$  and  $\Phi_{22} \neq 0$  solve the field equations, so that the E(2) class of scalar tensor gravity is  $N_3$ .

## References

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