Problem 1: Qubits

(a) Which of the following are valid quantum states:

\[ |1\rangle, \ |0\rangle + |1\rangle, \ \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle), \ \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |1\rangle), \ \sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle. \]

(b) For each of the valid quantum states from (a), answer the following: You perform a measurement in basis \(|0\rangle, |1\rangle\) (i.e., you ask “whether the state is \(|0\rangle\) or \(|1\rangle\)). What is the probability of answer 0 (i.e., yes), what is the probability of answer 1 (i.e., no)? What is the state after the measurement in each of those cases?

(c) For each of the valid quantum states from (a), answer the following: You perform a measurement in basis \(|+\rangle, |-\rangle\). What is the probability of answer + (i.e., yes), what is the probability of answer – (i.e., no)? What is the state after the measurement in each of those cases?

(d) Let a quantum state \(|\Psi\rangle \in \mathbb{C}^2\) and an (orthonormal) measurement basis \(|\text{yes}\rangle, |\text{no}\rangle \in \mathbb{C}^2\) be given. Measure \(|\Psi\rangle\) in that measurement basis. Let \(P_{\text{yes}}\) be the probability of outcome yes, and \(P_{\text{no}}\) the probability of outcome no. Show that \(P_{\text{yes}} + P_{\text{no}} = 1\).

(e) Show that by applying a unitary transformation to a quantum state, no information is ever lost. More exactly, assume that a unitary transformation \(U\) is applied to a given quantum state \(|\Psi\rangle\), resulting in a state \(|\Phi\rangle\). Then show that there is another unitary transformation \(V\) (not depending on \(|\Psi\rangle\) or \(|\Phi\rangle\)) such that applying \(V\) to \(|\Phi\rangle\) gives \(|\Psi\rangle\) again.
(f) Assume that a photon is in the state $|\Psi\rangle = \alpha |\uparrow\rangle + \beta |\rightarrow\rangle$. Let $R$ be a rotation of angle $\theta = \frac{\pi}{3}$. Let $F$ denote a polarisation filter that lets only vertically polarised light through ($|\uparrow\rangle$). Assume that the photon $|\Psi\rangle$ is first sent through $R$ and then through $F$. It turns out that in this setting, the photon is absorbed by $F$ with probability 1. Given these informations, what do you know about $\alpha$? (I.e., what are the possible values of $\alpha$?)

(g) What is wrong with the following approach:

Alice has a qubit $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. She wants to initialise the qubit to $|0\rangle$. She knows that when measuring $|\Psi\rangle$ in the computational basis $|0\rangle, |1\rangle$, with probability $\frac{1}{2}$ she get the measurement outcome 0 and the qubit will be in state $|0\rangle$. Thus she repeatedly measures the qubit in the computational basis until she gets the outcome 0. Since the probability is $\frac{1}{2}$ each time, the expected number of measurements until she gets her $|0\rangle$-initialised qubit is 2.

(h) Which of the following are valid (unitary) transformations:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}. $$