Problem 1: Trace Distance

(a) Let $E_1$ and $E_2$ be quantum state probability distributions. Let $\rho_1$ and $\rho_2$ be the corresponding density operators. Assume that $E_1$ and $E_2$ are physically indistinguishable. What is $TD(\rho_1, \rho_2)$?

(b) Let $E_1 := \{(|+, \frac{1}{2}) , (|-, \frac{1}{2})\}$ and $E_2 := \{(|0\rangle, 1\rangle\}$ be quantum state probability distributions. Let $\rho_1$ and $\rho_2$ be the corresponding density operators. What is $TD(\rho_1, \rho_2)$?

Note: You may use computer algebra software (e.g., SageMath) to compute the eigenvectors of matrices if you wish.

(c) Let $\rho = p\tau + q\rho'$ and $\sigma = p\tau' + q\sigma'$ where $\tau, \rho', \sigma'$ are density operators, and $p, q \geq 0$, $p + q = 1$. Show that $TD(\sigma, \rho) = q \cdot TD(\sigma', \rho')$.

Note: Do not use Lemma 9 in the lecture notes.

(d) Let $E_1 := \{(|+, \frac{1}{2}) , (|-, \frac{1}{2}) , (|\Psi\rangle, \frac{1}{2})\}$. Let $E_2 := \{(|0\rangle, \frac{1}{2}) , (|\Psi\rangle, \frac{1}{2})\}$. Here $|\Psi\rangle := \frac{1}{\sqrt{3}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$. Let $\rho_1$ and $\rho_2$ be the corresponding density operators. What is $TD(\rho_1, \rho_2)$?

Hint: Consider (c).

(e) Consider the following setup: Alice has a secret bit $b \in \{0, 1\}$. Then she chooses randomly $r \in \{0, 1\}$. If $r = 0$, she encodes $b$ in the $|0\rangle$ basis (i.e., she sends $|0\rangle$ for $b = 0$ and $|1\rangle$ for $b = 1$). If $r = 1$, she encodes $b$ in the $|+\rangle, |-\rangle$ basis. Then she sends the resulting state $|\Psi_b\rangle$ to Eve. Show that the trace distance between the mixed states $\rho_0$ and $\rho_1$ corresponding to $b = 0$ and $b = 1$, respectively, is $TD(\rho_0, \rho_1) = \frac{1}{\sqrt{2}}$.

Hint: The eigenvalues of $\left(\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{array}\right)$ are $\frac{1}{\sqrt{2}}$ and $-\frac{1}{\sqrt{2}}$. Note that this is not the toy protocol from the lecture, in the toy protocol $b$ selected the basis, not $r$.

(f) In the experiment described in (e), assume that the bit $b$ is chosen uniformly at random. Show that Eve cannot guess $b$ with probability larger than $\frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 85\%$.

Hint: Try to express the probability that Eve guesses correctly in terms of $Pr[G = x|b = y]$ for various $x, y \in \{0, 1\}$ (here $G$ denotes Eve’s guess) and then use (e).
Problem 2: Purification (Bonus problem)

Compute purifications of the following density operators:

\[ \rho_1 := \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_2 := \frac{1}{2} |00\rangle\langle 00| + \frac{1}{2} |11\rangle\langle 11|, \quad \rho_3 := \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix}. \]

Hint: Eigenvectors of \( \rho_3 \) are \( |+\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \) and \( |-\rangle = \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \).