Kombinaatorloogika

- *Väide:* Olgu $M(\vec{x})$ mingi λ-term mis sisaldab vabu muutujaid $\vec{x}$. Siis leidub term F selline, et $F\,\vec{x} = M(\vec{x})$.
- Kombinaatortermide süntaks

$$E \quad ::= \quad \mathbf{K} \mid \mathbf{S} \mid (E_1 E_2)$$

- Reduktsioonid

$$\begin{array}{lcl} \mathbf{I}\,x & \to & x \\ \mathbf{K}\,x\,y & \to & x \\ \mathbf{S}\,f\,g\,x & \to & f\,x\,(g\,x) \end{array}$$

- Identsuskombinaator on teistest defineeritav: $\mathbf{SKK} = \mathbf{I}$

Teooria *CL*

- Aksioomid

$$P = P \qquad \mathbf{K}\,P\,Q = P \qquad \mathbf{S}\,P\,Q\,R = P\,R\,(Q\,R)$$

- Tuletusreeglid

$$\frac{P = Q}{Q = P} \qquad \frac{P = Q}{PR = QR}$$

$$\frac{P = Q \quad Q = R}{P = R} \qquad \frac{P = Q}{RP = RQ}$$

Kombinaatorloogika ja λ-arvutus

- $CL \Rightarrow \Lambda$

$$\begin{array}{lcl} \mathbf{I} & \equiv & \lambda x.\, x \\ \mathbf{K} & \equiv & \lambda x\, y.\, x \\ \mathbf{S} & \equiv & \lambda f\, g\, x.\, f\, x\, (g\, x) \end{array}$$

- *Abstraktsioon:*

$$\begin{array}{lcl} [x]\, x & \equiv & \mathbf{I} \\ [x]\, y & \equiv & \mathbf{K}\, y \\ [x]\, c & \equiv & \mathbf{K}\, c \\ [x]\, (E_1\, E_2) & \equiv & \mathbf{S}\, ([x]\, E_1)\, ([x]\, E_2) \end{array}$$

Kombinaatorloogika ja λ-arvutus

- $\Lambda \Rightarrow CL$

$$\begin{array}{lcl} \mathcal{C}[\![x]\!] & = & x \\ \mathcal{C}[\![c]\!] & = & c \\ \mathcal{C}[\![(E_1\, E_2)]\!] & = & (\mathcal{C}[\![E_1]\!])\, (\mathcal{C}[\![E_2]\!]) \\ \mathcal{C}[\![(\lambda x.\, E)]\!] & = & [x]\, (\mathcal{C}[\![E]\!]) \end{array}$$

- Näide:

$$\begin{array}{l} \mathcal{C}[\![(\lambda x\, y.\, y\, x)]\!] \\ \quad = \quad [x]\, ([y]\, (y\, x)) \\ \quad = \quad [x]\, (\mathbf{S}\, ([y]\, y)\, ([y]\, x)) \\ \quad = \quad [x]\, (\mathbf{S}\, \mathbf{I}\, (\mathbf{K}\, x)) \\ \quad = \quad \mathbf{S}\, ([x]\, \mathbf{S}\, \mathbf{I})\, ([x]\, \mathbf{K}\, x) \\ \quad = \quad \mathbf{S}\, (\mathbf{S}\, ([x]\, \mathbf{S})\, ([x]\, \mathbf{I}))\, (\mathbf{S}\, ([x]\, \mathbf{K})\, ([x]\, x)) \\ \quad = \quad \mathbf{S}\, (\mathbf{S}\, (\mathbf{K}\, \mathbf{S})\, (\mathbf{K}\, \mathbf{I}))\, (\mathbf{S}\, (\mathbf{K}\, \mathbf{K})\, \mathbf{I}) \end{array}$$

Kombinaatorloogika ja λ-arvutus

- *Lemma:* $CL \vdash P = Q \implies \lambda \vdash P_\lambda = Q_\lambda$
- Vastupidine ei kehti!! (Näiteks: $\mathbf{SK} \neq \mathbf{KI}$)
- Curry aksioomid:

$$\mathbf{K} = \mathbf{S}(\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{KK})\mathbf{K}))(\mathbf{K}(\mathbf{SKK}))$$
$$\mathbf{S} = \mathbf{S}(\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{K}(\mathbf{S}(\mathbf{KS})))(\mathbf{S}(\mathbf{K}(\mathbf{S}(\mathbf{KK})))\mathbf{S})))(\mathbf{K}(\mathbf{K}(\mathbf{SKK})))$$
$$\mathbf{S}(\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{KK})(\mathbf{S}(\mathbf{KS})\mathbf{K})))(\mathbf{KK}) = \mathbf{S}(\mathbf{KK})$$
$$\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{KK})) = \mathbf{S}(\mathbf{KK})(\mathbf{S}(\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{KK})(\mathbf{SKK})))(\mathbf{K}(\mathbf{SKK})))$$
$$\mathbf{S}(\mathbf{K}(\mathbf{S}(\mathbf{KS})))(\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{KS}))) = \mathbf{S}(\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{KK})(\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{K}(\mathbf{S}(\mathbf{KS})))\mathbf{S}))))(\mathbf{KS})$$

Optimiseerimised

- Kombinaatortermi suurus on eksponentsiaalne esialgse termi argumentide arvust!
- Kombinaatorid **B** ja **C**

$$\begin{aligned} \mathbf{B} &\equiv \mathbf{S}(\mathbf{KS})\mathbf{K} \\ \mathbf{C} &\equiv \mathbf{S}(\mathbf{BS}(\mathbf{BKS}))(\mathbf{KK}) \end{aligned}$$

- Reduktsioonid

$$\begin{aligned} \mathbf{B}\ f\ g\ x &\to f\ (g\ x) \\ \mathbf{C}\ f\ g\ x &\to f\ x\ g \end{aligned}$$

Optimiseerimised

- Optimiseerimisreeglid

$$\begin{array}{lcl}
\mathbf{S}\ (\mathbf{K}\ p)\ (\mathbf{K}\ q) & = & \mathbf{K}\ (p\ q) \\
\mathbf{S}\ (\mathbf{K}\ p)\ \mathbf{I} & = & p \\
\mathbf{S}\ (\mathbf{K}\ p)\ q & = & \mathbf{B}\ p\ q \\
\mathbf{S}\ p\ (\mathbf{K}\ q) & = & \mathbf{C}\ p\ q
\end{array}$$

- Näide:

$$\begin{array}{lcl}
\mathcal{C}[\![(\lambda x\ y.\ y\ x)]\!] & = & [x]\ (\mathbf{S}\ \mathbf{I}\ (\mathbf{K}\ x)) \\
& = & [x]\ (\mathbf{C}\ \mathbf{I}\ x) \\
& = & \mathbf{S}\ ([x](\mathbf{C}\ \mathbf{I})\ ([x]x)) \\
& = & \mathbf{S}\ (\mathbf{S}\ (\mathbf{K}\ \mathbf{C})\ (\mathbf{K}\ \mathbf{I}))\ \mathbf{I} \\
& = & \mathbf{S}\ (\mathbf{K}\ (\mathbf{C}\ \mathbf{I}))\ \mathbf{I} \\
& = & \mathbf{C}\ \mathbf{I}
\end{array}$$

Optimiseerimised

- Optimiseeritud kombinaatortermi suurus on esialgse termi ruut!

$$\begin{array}{lcl}
\mathcal{C}[\![\lambda x_1.p\ q]\!] & = & \mathbf{S}\ p_1\ q_1 \\
\mathcal{C}[\![\lambda x_2 x_1.p\ q]\!] & = & \mathbf{S}\ (\mathbf{B}\ \mathbf{S}\ p_2)\ q_2 \\
\mathcal{C}[\![\lambda x_3 x_2 x_1.p\ q]\!] & = & \mathbf{S}\ (\mathbf{B}\ \mathbf{S}\ (\mathbf{B}\ (\mathbf{B}\ \mathbf{S})\ p_3))\ q_3 \\
\mathcal{C}[\![\lambda x_4 x_3 x_2 x_1.p\ q]\!] & = & \mathbf{S}\ (\mathbf{B}\ \mathbf{S}\ (\mathbf{B}\ (\mathbf{B}\ \mathbf{S})\ (\mathbf{B}\ (\mathbf{B}\ (\mathbf{B}\ \mathbf{S}))\ p_4)))\ q_4
\end{array}$$

- Turneri kombinaatorid

$$\begin{array}{lcl}
\mathbf{S}'\ c\ f\ g\ x & \rightarrow & c\ (f\ x)\ (g\ x) \\
\mathbf{B}'\ c\ f\ g\ x & \rightarrow & c\ f\ (g\ x) \\
\mathbf{C}'\ c\ f\ g\ x & \rightarrow & c\ (f\ x)\ g \\
\mathbf{B}^{\star}\ c\ f\ g\ x & \rightarrow & c\ (f\ (g\ x))
\end{array}$$

Optimiseerimised

- Optimiseerimisreeglid

$$
\begin{array}{lcl}
\mathbf{S}\,(\mathbf{K}\,p)\,(\mathbf{K}\,q) & = & \mathbf{K}\,(p\,q) \\
\mathbf{S}\,(\mathbf{K}\,p)\,\mathbf{I} & = & p \\
\mathbf{S}\,(\mathbf{K}\,p)\,(\mathbf{B}\,q\,r) & = & \mathbf{B}^{\star}\,p\,q\,r \\
\mathbf{S}\,(\mathbf{K}\,p)\,q & = & \mathbf{B}\,p\,q \\
\mathbf{S}\,(\mathbf{B}\,p\,q)\,(\mathbf{K}\,r) & = & \mathbf{C}'\,p\,q\,r \\
\mathbf{S}\,p\,(\mathbf{K}\,q) & = & \mathbf{C}\,p\,q \\
\mathbf{S}\,(\mathbf{B}\,p\,q)\,r & = & \mathbf{S}'\,p\,q\,r
\end{array}
$$