De Bruijn'i λ-arvutus

De Bruijn'i λ-termid

Muutujad on asendatud siduva λ kaugusega

$$\begin{array}{lcll} E & ::= & N & \text{muutuja} \\ & | & (E_1\ E_2) & \text{aplikatsioon} \\ & | & (\lambda\ E) & \text{abstraktsioon} \end{array}$$

Näited

$$\begin{array}{lcl} \lambda x.\ x & \leftrightarrow & \lambda 1 \\ \lambda x\ y.\ x\ y & \leftrightarrow & \lambda\lambda 21 \\ \lambda x\ y.\ x\ y\ (\lambda y.\ x\ y\ y) & \leftrightarrow & \lambda\lambda 21(\lambda 311) \\ \lambda x.\ x\ (\lambda y.\ x\ y\ y) & \leftrightarrow & \lambda 1(\lambda 211) \end{array}$$

De Bruijn'i λ-arvutus

β-konversioon

$$(\lambda M)\, N = M[N/1]$$

Substitutsiooni definitsioon

$$\begin{array}{rcll}
m[N/n] & = & m & \text{if } m < n \\
 & & m-1 & \text{if } m > n \\
 & & \mathit{shift}_{n,1}(N) & \text{if } m = n \\
(M_1\ M_2)[N/n] & = & (M_1[N/n])\ (M_2[N/n]) & \\
(\lambda\, M)[N/n] & = & \lambda(M[N/n+1]) & \\
 & & & \\
\mathit{shift}_{n,i}(j) & = & j & \text{if } j < i \\
 & & j+n-1 & \text{if } j \geqslant i \\
\mathit{shift}_{n,i}(N_1\ N_2) & = & \mathit{shift}_{n,i}(N_1)\ \mathit{shift}_{n,i}(N_2) & \\
\mathit{shift}_{n,i}(\lambda\, N) & = & \lambda\ \mathit{shift}_{n,i+1}(N) &
\end{array}$$

Kombinaatorloogika

Väide

Olgu $M(\vec{x})$ mingi λ-term mis sisaldab vabu muutujaid $\vec{x}$. Siis leidub term F selline, et $F\,\vec{x} = M(\vec{x})$.

Kombinaatortermide süntaks

$$E \quad ::= \quad \mathbf{I} \mid \mathbf{K} \mid \mathbf{S} \mid (E_1 E_2)$$

Reduktsioonireeglid

$$\begin{array}{lcl}
\mathbf{I}\ x & \longrightarrow & x \\
\mathbf{K}\ x\ y & \longrightarrow & x \\
\mathbf{S}\ f\ g\ x & \longrightarrow & f\ x\ (g\ x)
\end{array}$$

NB!

Identsuskombinaator on teistest defineeritav: $\mathbf{SKK} = \mathbf{I}$

Kombinaatorloogika

Teooria CL

- Aksioomid

$$P = P \qquad \mathbf{K}\ P\ Q = P \qquad \mathbf{S}\ P\ Q\ R = P\ R\ (Q\ R)$$

- Tuletusreeglid

$$\frac{P = Q}{Q = P} \qquad \frac{P = Q}{PR = QR}$$

$$\frac{P = Q \quad Q = R}{P = R} \qquad \frac{P = Q}{RP = RQ}$$

Kombinaatorloogika ja λ-arvutus

$CL \Rightarrow \Lambda$

$$
\begin{array}{lcl}
\mathbf{I} & \equiv & \lambda x.\ x \\
\mathbf{K} & \equiv & \lambda x\ y.\ x \\
\mathbf{S} & \equiv & \lambda f\ g\ x.\ f\ x\ (g\ x)
\end{array}
$$

Abstraktsioon

$$
\begin{array}{lcl}
[x]\ x & \equiv & \mathbf{I} \\
[x]\ y & \equiv & \mathbf{K}\ y \\
[x]\ c & \equiv & \mathbf{K}\ c \\
[x]\ (E_1\ \ E_2) & \equiv & \mathbf{S}\ ([x]\ \ E_1)\ ([x]\ \ E_2)
\end{array}
$$

Kombinaatorloogika ja λ-arvutus

$\Lambda \Rightarrow CL$

$$
\begin{array}{lcl}
\mathcal{C}[\![x]\!] & = & x \\
\mathcal{C}[\![c]\!] & = & c \\
\mathcal{C}[\![(E_1\ E_2)]\!] & = & (\mathcal{C}[\![E_1]\!])\ (\mathcal{C}[\![E_2]\!]) \\
\mathcal{C}[\![(\lambda x.\ E)]\!] & = & [x]\ (\mathcal{C}[\![E]\!])
\end{array}
$$

Näide

$$
\begin{array}{ll}
\multicolumn{2}{l}{\mathcal{C}[\![(\lambda x\ y.\ y\ x)]\!]} \\
= & [x]\ ([y]\ (y\ x)) \\
= & [x]\ (\mathbf{S}\ ([y]\ y)\ ([y]\ x)) \\
= & [x]\ (\mathbf{S}\ \mathbf{I}\ (\mathbf{K}\ x)) \\
= & \mathbf{S}\ ([x]\ \mathbf{S}\ \mathbf{I})\ ([x]\ \mathbf{K}\ x) \\
= & \mathbf{S}\ (\mathbf{S}\ ([x]\ \mathbf{S})\ ([x]\ \mathbf{I}))\ (\mathbf{S}\ ([x]\ \mathbf{K})\ ([x]\ x)) \\
= & \mathbf{S}\ (\mathbf{S}\ (\mathbf{K}\ \mathbf{S})\ (\mathbf{K}\ \mathbf{I}))\ (\mathbf{S}\ (\mathbf{K}\ \mathbf{K})\ \mathbf{I})
\end{array}
$$

Kombinaatorloogika ja λ-arvutus

Lemma

$CL \vdash P = Q \implies \lambda \vdash P_\lambda = Q_\lambda$

NB!

Vastupidine ei kehti!! (Näiteks: $\mathbf{SK} \neq \mathbf{KI}$)

Curry aksioomid

$$
\begin{aligned}
&\mathbf{K} = \mathbf{S}(\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{KK})\mathbf{K}))(\mathbf{K}(\mathbf{SKK}))\\
&\mathbf{S} = \mathbf{S}(\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{K}(\mathbf{S}(\mathbf{KS})))(\mathbf{S}(\mathbf{K}(\mathbf{S}(\mathbf{KK})))\mathbf{S})))(\mathbf{K}(\mathbf{K}(\mathbf{SKK})))\\
&\mathbf{S}(\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{KK})(\mathbf{S}(\mathbf{KS})\mathbf{K})))(\mathbf{KK}) = \mathbf{S}(\mathbf{KK})\\
&\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{KK})) = \mathbf{S}(\mathbf{KK})(\mathbf{S}(\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{KK})(\mathbf{SKK})))(\mathbf{K}(\mathbf{SKK})))\\
&\mathbf{S}(\mathbf{K}(\mathbf{S}(\mathbf{KS})))(\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{KS})))\\
&\qquad = \mathbf{S}(\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{KK})(\mathbf{S}(\mathbf{KS})(\mathbf{S}(\mathbf{K}(\mathbf{S}(\mathbf{KS})))\mathbf{S}))))(\mathbf{KS})
\end{aligned}
$$

Optimiseerimised

NB!

Kombinaatortermi suurus on eksponentsiaalne esialgse termi argumentide arvust!

Kombinaatorid B ja C

$$\begin{array}{lcl} \mathbf{B} & \equiv & \mathbf{S}(\mathbf{KS})\mathbf{K} \\ \mathbf{C} & \equiv & \mathbf{S}(\mathbf{BS}(\mathbf{BKS}))(\mathbf{KK}) \end{array}$$

Reduktsioonireeglid

$$\begin{array}{lcl} \mathbf{B}\ f\ g\ x & \longrightarrow & f\ (g\ x) \\ \mathbf{C}\ f\ g\ x & \longrightarrow & f\ x\ g \end{array}$$

Optimiseerimised

Optimiseerimisreeglid

$$\begin{array}{lcl} \mathbf{S}\ (\mathbf{K}\ p)\ (\mathbf{K}\ q) & = & \mathbf{K}\ (p\ q) \\ \mathbf{S}\ (\mathbf{K}\ p)\ \mathbf{I} & = & p \\ \mathbf{S}\ (\mathbf{K}\ p)\ q & = & \mathbf{B}\ p\ q \\ \mathbf{S}\ p\ (\mathbf{K}\ q) & = & \mathbf{C}\ p\ q \end{array}$$

Näide

$$\begin{array}{lcl} \mathcal{C}[\![(\lambda x\ y.\ y\ x)]\!] & = & [x]\ (\mathbf{S}\ \mathbf{I}\ (\mathbf{K}\ x)) \\ & = & [x]\ (\mathbf{C}\ \mathbf{I}\ x) \\ & = & \mathbf{S}\ ([x](\mathbf{C}\ \mathbf{I})\ ([x]x)) \\ & = & \mathbf{S}\ (\mathbf{S}\ (\mathbf{K}\ \mathbf{C})\ (\mathbf{K}\ \mathbf{I}))\ \mathbf{I} \\ & = & \mathbf{S}\ (\mathbf{K}\ (\mathbf{C}\ \mathbf{I}))\ \mathbf{I} \\ & = & \mathbf{C}\ \mathbf{I} \end{array}$$

Optimiseerimised

NB!

Optimiseeritud kombinaatortermi suurus on esialgse termi ruut!

$$
\begin{array}{lcl}
\mathcal{C}[\![\lambda x_1.p\ q]\!] & = & \mathbf{S}\ p_1\ q_1 \\
\mathcal{C}[\![\lambda x_2 x_1.p\ q]\!] & = & \mathbf{S}\ (\mathbf{B}\ \mathbf{S}\ p_2)\ q_2 \\
\mathcal{C}[\![\lambda x_3 x_2 x_1.p\ q]\!] & = & \mathbf{S}\ (\mathbf{B}\ \mathbf{S}\ (\mathbf{B}\ (\mathbf{B}\ \mathbf{S})\ p_3))\ q_3 \\
\mathcal{C}[\![\lambda x_4 x_3 x_2 x_1.p\ q]\!] & = & \mathbf{S}\ (\mathbf{B}\ \mathbf{S}\ (\mathbf{B}\ (\mathbf{B}\ \mathbf{S})\ (\mathbf{B}\ (\mathbf{B}\ (\mathbf{B}\ \mathbf{S}))\ p_4)))\ q_4
\end{array}
$$

Optimiseerimised

Turneri kombinaatorid

$$
\begin{array}{lcl}
\mathbf{S}'\ c\ f\ g\ x & \longrightarrow & c\ (f\ x)\ (g\ x) \\
\mathbf{B}'\ c\ f\ g\ x & \longrightarrow & c\ f\ (g\ x) \\
\mathbf{C}'\ c\ f\ g\ x & \longrightarrow & c\ (f\ x)\ g \\
\mathbf{B}^{\star}\ c\ f\ g\ x & \longrightarrow & c\ (f\ (g\ x))
\end{array}
$$

Optimiseerimisreeglid

$$
\begin{array}{lcl}
\mathbf{S}\ (\mathbf{K}\ p)\ (\mathbf{K}\ q) & = & \mathbf{K}\ (p\ q) \\
\mathbf{S}\ (\mathbf{K}\ p)\ \mathbf{I} & = & p \\
\mathbf{S}\ (\mathbf{K}\ p)\ (\mathbf{B}\ q\ r) & = & \mathbf{B}^{\star}\ p\ q\ r \\
\mathbf{S}\ (\mathbf{K}\ p)\ q & = & \mathbf{B}\ p\ q \\
\mathbf{S}\ (\mathbf{B}\ p\ q)\ (\mathbf{K}\ r) & = & \mathbf{C}'\ p\ q\ r \\
\mathbf{S}\ p\ (\mathbf{K}\ q) & = & \mathbf{C}\ p\ q \\
\mathbf{S}\ (\mathbf{B}\ p\ q)\ r & = & \mathbf{S}'\ p\ q\ r
\end{array}
$$