Andmete esitamine λ-arvutuses

Baaskombinaatorid

$$
\begin{array}{lcl}
\mathbf{I} & \equiv & \lambda x.\ x \\
\mathbf{K} & \equiv & \lambda x\ y.\ x \\
\mathbf{S} & \equiv & \lambda f\ g\ x.\ f\ x\ (g\ x)
\end{array}
$$

"Astendamine"

$$
\begin{array}{lcl}
E^0\,E' & \equiv & E' \\
E^n\,E' & \equiv & \underbrace{E(E(\ldots(E}_{n\ \text{tükki}}\ E')\ldots))
\end{array}
$$

NB!

$$
E^n(E\ E') \equiv E^{n+1}E' \equiv E\ (E^n\,E')
$$

Tõeväärtused

Spetsifikatsioon

$$\begin{array}{lcl} \textbf{not true} & = & \textbf{false} \\ \textbf{not false} & = & \textbf{true} \end{array}$$

Definitsioon

$$\begin{array}{lcll} \textbf{true} & \equiv & \lambda xy.\ x & \quad (\equiv \quad \mathbf{K}) \\ \textbf{false} & \equiv & \lambda xy.\ y & \\ \textbf{not} & \equiv & \lambda t.\ t\ \textbf{false}\ \textbf{true} & \end{array}$$

Näide

$$\begin{array}{lcl} \textbf{not true} & \equiv & (\lambda t.\ t\ \textbf{false}\ \textbf{true})\ \textbf{true} \\ & \longrightarrow & \textbf{true}\ \textbf{false}\ \textbf{true} \\ & \equiv & (\lambda x.\ \lambda y.\ x)\ \textbf{false}\ \textbf{true} \\ & \longrightarrow & (\lambda y.\ \textbf{false})\ \textbf{true} \\ & \longrightarrow & \textbf{false} \end{array}$$

Tingimuslause

Spetsifikatsioon

$$\begin{aligned}\textbf{cond true } E_1\ E_2 &= E_1\\ \textbf{cond false } E_1\ E_2 &= E_2\end{aligned}$$

Definitsioon

$$\textbf{cond} \equiv \lambda t\ x\ y.\ t\ x\ y$$

Näide

$$\begin{aligned}\textbf{cond false } E_1\ E_2 &\equiv (\lambda t\ x\ y.\ t\ x\ y)\ \textbf{false } E_1\ E_2\\ &\Longrightarrow \textbf{false } E_1\ E_2\\ &\equiv (\lambda x\ y.\ y)\ E_1\ E_2\\ &\Longrightarrow E_2\end{aligned}$$

Paaride ja ennikud

Paarid

$$
\begin{aligned}
\mathbf{fst} &\equiv \lambda p.\ p\ \mathbf{true} \\
\mathbf{snd} &\equiv \lambda p.\ p\ \mathbf{false} \\
(E_1, E_2) &\equiv \lambda f.\ f\ E_1\ E_2
\end{aligned}
$$

Ennikud

$$
\begin{aligned}
(E_1, \ldots, E_n) &\equiv (E_1, (\ldots (E_{n-1}, E_n) \ldots)) \\
E \overset{n}{\downarrow} 1 &\equiv \mathbf{fst}\ E \\
E \overset{n}{\downarrow} 2 &\equiv \mathbf{fst}\ (\mathbf{snd}\ E) \\
&\ldots \\
E \overset{n}{\downarrow} i &\equiv \mathbf{fst}\ (\mathbf{snd}^{i-1} E) \\
&\ldots \\
E \overset{n}{\downarrow} n &\equiv \mathbf{snd}^{n-1}\ E
\end{aligned}
$$

Naturaalarvud

Standardnumbrid

$$
\begin{array}{lcll}
\ulcorner 0 \urcorner & \equiv & \lambda x.\ x & (\equiv \quad \mathbf{I}) \\
\ulcorner n{+}1 \urcorner & \equiv & (\mathbf{false}, \ulcorner n \urcorner) & \\
\mathbf{succ} & \equiv & \lambda n.\ (\mathbf{false}, n) & \\
\mathbf{pred} & \equiv & \lambda n.\ n\ \mathbf{false} & (\equiv \quad \mathbf{snd}) \\
\mathbf{iszero} & \equiv & \lambda n.\ n\ \mathbf{true} & (\equiv \quad \mathbf{fst})
\end{array}
$$

Liitmine (?!)

$$\mathbf{add} \quad = \quad \lambda x\ y.\ \mathbf{cond}(\mathbf{iszero}\ x)\ y\ (\mathbf{add}(\mathbf{pred}\ x)(\mathbf{succ}\ y))$$

Naturaalarvud

Church'i numbrid

$$\begin{array}{lcl}
\underline{n} & \equiv & \lambda f\ x.\ f^n\ x \\
\mathbf{succ} & \equiv & \lambda n.\ \lambda f\ x.\ n\ f\ (f\ x) \\
\mathbf{iszero} & \equiv & \lambda n.\ n\ (\lambda x.\ \mathbf{false})\ \mathbf{true} \\
\mathbf{add} & \equiv & \lambda m\ n.\ \lambda f\ x.\ m\ f\ (n\ f\ x)
\end{array}$$

Näide

$$\begin{array}{rcl}
\mathbf{add}\ \underline{2}\ \underline{1} & \equiv & (\lambda m\ n.\ \lambda f\ x.\ m\ f\ (n\ f\ x))\ \underline{2}\ \underline{1} \\
 & \Longrightarrow & \lambda f\ x.\ \underline{2}\ f\ (\underline{1}\ f\ x) \\
 & \Longrightarrow & \lambda f\ x.\ f\ (f\ (\underline{1}\ f\ x)) \\
 & \Longrightarrow & \lambda f\ x.\ f\ (f\ (f\ x)) \\
 & \equiv & \underline{3}
\end{array}$$

Korrutamine ja astendamine

$$\begin{array}{lcl}
\mathbf{mul} & \equiv & \lambda m\ n.\ \lambda f\ x.\ m\ (n\ f)\ x \\
\mathbf{exp} & \equiv & \lambda m\ n.\ \lambda f\ x.\ n\ m\ f\ x
\end{array}$$

Naturaalarvud

Ühe lahutamine — abifunktsiooni spetsifikatsioon

$$
\begin{array}{lcl}
\mathbf{prefn}\ f\ (\mathbf{true}, x) & = & (\mathbf{false}, x) \\
\mathbf{prefn}\ f\ (\mathbf{false}, x) & = & (\mathbf{false}, f\ x) \\
(\mathbf{prefn}\ f)^n\ (\mathbf{false}, x) & = & (\mathbf{false}, f^n\ x) \\
(\mathbf{prefn}\ f)^n\ (\mathbf{true}, x) & = & (\mathbf{false}, f^{n-1}\ x)
\end{array}
$$

Ühe lahutamine — definitsioon

$$
\begin{array}{lcl}
\mathbf{prefn} & \equiv & \lambda f\, p.\, (\mathbf{false}, (\mathbf{cond}\, (\mathbf{fst}\, p)\, (\mathbf{snd}\, p)\, (f\, (\mathbf{snd}\, p)))) \\
\mathbf{pred} & \equiv & \lambda n.\, \lambda f\, x.\, \mathbf{snd}\, (n\, (\mathbf{prefn}\, f)\, (\mathbf{true}, x))
\end{array}
$$

Näide

$$
\begin{array}{lcl}
\mathbf{pred}\ \underline{n}\ f\ x & = & \mathbf{snd}\ (\underline{n}\ (\mathbf{prefn}\ f)\ (\mathbf{true}, x)) \\
 & = & \mathbf{snd}\ ((\mathbf{prefn}\ f)^n\ (\mathbf{true}, x)) \\
 & = & \mathbf{snd}\ (\mathbf{false}, f^{n-1}\ x) \\
 & = & f^{n-1}\ x
\end{array}
$$

Listid

Definitsioon

$$
\begin{array}{lcll}
\textbf{nil} & \equiv & \lambda z.\ z & (\equiv\ \ \mathbf{I}) \\
\textbf{cons} & \equiv & \lambda x\ y.\ (\textbf{false}, (x, y)) & \\
\textbf{null} & \equiv & \lambda z.\ z\ \textbf{true} & (\equiv\ \ \textbf{fst}) \\
\textbf{hd} & \equiv & \lambda z.\ \textbf{fst}\ (\textbf{snd}\ z)) & \\
\textbf{tl} & \equiv & \lambda z.\ \textbf{snd}\ (\textbf{snd}\ z)) &
\end{array}
$$

Näide

$$
\begin{array}{rcl}
\textbf{null nil} & \equiv & \textbf{fst}\ (\lambda z.\ z) \\
 & \equiv & (\lambda p.\ p\ \textbf{true})\ (\lambda z.\ z) \\
 & \Longrightarrow & \textbf{true}
\end{array}
$$

Püsipunktid

Püsipunktiteoreem

$$\forall F.\ \exists X.\ X = F\,X$$

Mõisted

- Termi M nimetatakse püsipunkti kombinaatoriks kui

$$\forall F.\ M\,F = F\,(M\,F)$$

- Curry "paradoksaalne" kombinaator

$$\mathbf{Y} \equiv \lambda f.(\lambda x.f(x\,x))\,(\lambda x.f(x\,x))$$

- "Tugev" püsipunkti kombinaator

$$\boldsymbol{\Theta} \equiv (\lambda xy.\ y(xxy))\,(\lambda xy.\ y(xxy))$$

Püsipunktid

Lemma

Olgu $G \equiv \lambda y f . f(y f) \quad (\equiv \mathbf{SI})$

$$M \text{ on püsipunkti kombinaator} \iff M = G M$$

Tõestus

($\Leftarrow$) Kui $M = G M$, siis:

$$\begin{array}{rcl} \forall F.\ M\ F & = & G\ M\ F \\ & \equiv & (\lambda y f . f(y f))\ M\ F \\ & = & F(M\ F) \end{array}$$

($\Rightarrow$) Kui M on püsipunkti kombinaator, siis:

$$\begin{array}{rcl} G\ M & = & \lambda f . f(M f) \\ & = & \lambda f . M f \\ & = & M \end{array}$$

Püsipunktid

Lemma

Kõik alljärgneva jada elemendid on püsipunkti kombinaatorid

$$\begin{aligned} \mathbf{Y}^0 &\equiv \mathbf{Y} \\ \mathbf{Y}^{n+1} &\equiv \mathbf{Y}^n\, G \end{aligned}$$

Tõestus

$$\begin{aligned} \mathbf{Y}^{n+1} &\equiv \mathbf{Y}^n\, G \\ &= G\,(\mathbf{Y}^n\, G) \\ &\equiv G\,\mathbf{Y}^{n+1} \end{aligned}$$

NB!

$$\mathbf{Y}^1 \Longrightarrow \mathbf{\Theta}$$

Rekursioon

Lemma

Olgu $C \equiv C(f, \vec{x})$, siis

- $\exists F.\ \forall \vec{N}.\ F\ \vec{N} = C(F, \vec{N})$
- $\exists F.\ \forall \vec{N}.\ F\ \vec{N} \Longrightarrow C(F, \vec{N})$

Tõestus

Võtame $F \equiv \boldsymbol{\Theta}(\lambda f \vec{x}.C(f, \vec{x}))$

Standardnumbrite liitmine

$$\begin{array}{lcl} \textbf{add} & = & \lambda x\ y.\ \textbf{cond}\,(\textbf{iszero}\ x)\ y\ (\textbf{add}(\textbf{pred}\ x)(\textbf{succ}\ y)) \\ \textbf{add} & \equiv & \mathbf{Y}\,(\lambda f\ x\ y.\ \textbf{cond}\,(\textbf{iszero}\ x)\ y\ (f\,(\textbf{pred}\ x)(\textbf{succ}\ y))) \end{array}$$

Mitmekohalised funktsioonid

"Currymine"

$$\begin{array}{lcl} \mathbf{curry}_n & \equiv & \lambda f\ x_1\ \ldots\ x_n.\ f\ (x_1,\ldots,x_n) \\ \mathbf{uncurry}_n & \equiv & \lambda f\ p.\ f\ (p \overset{n}{\downarrow} 1)\ \ldots\ (p \overset{n}{\downarrow} n) \end{array}$$

NB!

$\mathbf{curry}_n(\mathbf{uncurry}_n\ N) = N \qquad \mathbf{uncurry}_n(\mathbf{curry}_n\ M) = M$

Üldistatud λ-abstraktsioon

$$\lambda(V_1,\ldots,V_n).\ E \ \equiv\ \mathbf{uncurry}_n\ (\lambda V_1\ \ldots\ V_n.\ E)$$

Üldistatud β-konversioon

$$(\lambda(V_1,\ldots,V_n).\ E)(E_1,\ldots,E_n) \ \longrightarrow_\beta\ E[E_1\ldots E_n/V_1\ldots V_n]$$

Rekursioon

Vastastikune rekursioon

$$\begin{array}{rcl} f_1 & = & F_1\ f_1 \ldots f_n \\ f_2 & = & F_2\ f_1 \ldots f_n \\ & \ldots & \\ f_n & = & F_n\ f_1 \ldots f_n \end{array}$$

Definitsioon

$$\begin{array}{rcl} f & \equiv & \mathbf{Y}(\lambda(f_1, \ldots, f_n).\ (F_1 f_1 \ldots f_n,\ \ldots,\ F_n f_1 \ldots f_n)) \\ f_1 & \equiv & f \overset{n}{\downarrow} 1 \\ f_2 & \equiv & f \overset{n}{\downarrow} 2 \\ & \ldots & \\ f_n & \equiv & f \overset{n}{\downarrow} n \end{array}$$

Arvutatavus

Church'i tees

Kõik arvutatavad funktsioonid on esitatavad λ-arvutuses!

Lihtrekursiivsed funktsioonid

(i) 0

(ii) $S(x) = x + 1$

(iii) $U_n^i(x_1, x_2, \ldots, x_n) = x_i$

(iv) $f(x_1, \ldots, x_n) = g(h_1(x_1, \ldots, x_n), \ldots, h_r(x_1, \ldots, x_n))$

(v)
$$\begin{aligned} f(0, x_2, \ldots, x_n) &= g(x_2, \ldots, x_n) \\ f(S(x_1), x_2, \ldots, x_n) &= h(f(x_1, x_2, \ldots, x_n), x_2, \ldots, x_n) \end{aligned}$$

Osaliselt rekursiivsed funktsioonid

(vi) $f(x_1, \ldots, x_n) = \min \{\, y \mid g(y, x_2, \ldots, x_n) = x_1 \,\}$

Arvutatavus

Minimiseerimine

$$
\begin{aligned}
\mathbf{min}\, x\, f\, (x_1, \ldots, x_n) \quad = \quad & \mathbf{cond}\, (\mathbf{eq}\, (f(x, x_2, \ldots, x_n)\, x_1) \\
& \quad x \\
& \quad (\mathbf{min}\, (\mathbf{succ}\, x)\, f\, (x_1, \ldots, x_n)))
\end{aligned}
$$

Definitsioon

$$
\begin{aligned}
\mathbf{min} \quad \equiv \quad & \mathbf{Y}\ (\lambda m\, x\, f\, (x_1, \ldots, x_n). \\
& \qquad \mathbf{cond}\ (\mathbf{eq}\ (f(x, x_2, \ldots, x_n))\ x_1) \\
& \qquad\qquad x \\
& \qquad\qquad (m(\mathbf{succ}\ x)\ f\ (x_1, \ldots, x_n)))
\end{aligned}
$$