Naturaaldeduktsioon ($ND(\to\wedge\vee)$)

Tuletusreeglid

$$\frac{\begin{array}{c}\overline{\gamma^{(i)}}\\ \vdots\\ \varphi\end{array}}{\gamma \to \varphi^{(i)}}\ \to\mathrm{I} \qquad\qquad \frac{\begin{array}{cc}\vdots & \vdots\\ \gamma \to \varphi & \gamma\end{array}}{\varphi}\ \to\mathrm{E}$$

$$\frac{\begin{array}{cc}\vdots & \vdots\\ \varphi_1 & \varphi_2\end{array}}{\varphi_1 \wedge \varphi_2}\ \wedge\mathrm{I} \qquad\qquad \frac{\begin{array}{c}\vdots\\ \varphi_1 \wedge \varphi_2\end{array}}{\varphi_1}\ \wedge\mathrm{E_L} \qquad \frac{\begin{array}{c}\vdots\\ \varphi_1 \wedge \varphi_2\end{array}}{\varphi_2}\ \wedge\mathrm{E_R}$$

$$\frac{\begin{array}{c}\vdots\\ \varphi_1\end{array}}{\varphi_1 \vee \varphi_2}\ \vee\mathrm{I_L} \qquad \frac{\begin{array}{c}\vdots\\ \varphi_2\end{array}}{\varphi_1 \vee \varphi_2}\ \vee\mathrm{I_R} \qquad\qquad \frac{\begin{array}{ccc} & \overline{\varphi_1^{(i)}} & \overline{\varphi_2^{(i)}}\\ \vdots & \vdots & \vdots\\ \varphi_1 \vee \varphi_2 & \gamma & \gamma\end{array}}{\gamma^{(i)}}\ \vee\mathrm{E}$$

Naturaaldeduktsioon $(ND(\to\wedge\vee))$

Näide

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\varphi^{(2)}} \qquad \dfrac{\overline{(\varphi\to\psi)\wedge(\varphi\to\rho)^{(1)}}}{\varphi\to\psi}\,\wedge\mathrm{E}}{\psi}\,\to\mathrm{E} \qquad \dfrac{\overline{\varphi^{(2)}} \qquad \dfrac{\overline{(\varphi\to\psi)\wedge(\varphi\to\rho)^{(1)}}}{\varphi\to\rho}\,\wedge\mathrm{E}}{\rho}\,\to\mathrm{E}}{\psi\wedge\rho}\,\wedge\mathrm{I}}{\varphi\to\psi\wedge\rho^{(2)}}\,\to\mathrm{I}}{(\varphi\to\psi)\wedge(\varphi\to\rho)\to\varphi\to\psi\wedge\rho^{(1)}}\,\to\mathrm{I}$$

Naturaaldeduktsioon ($ND(\to\wedge\vee)$)

Normaalkujud

$$
\cfrac{\cfrac{\overline{\varphi^{(1)}}}{\varphi\to\varphi^{(1)}}\to\mathsf{I} \qquad \cfrac{\cfrac{\overline{\varphi\to\varphi^{(2)}}}{\psi\to\varphi\to\varphi^{(3)}}\to\mathsf{I}}{(\varphi\to\varphi)\to\psi\to\varphi\to\varphi^{(2)}}\to\mathsf{I}}{\psi\to\varphi\to\varphi}\to\mathsf{E}
\qquad\qquad
\cfrac{\cfrac{\overline{\varphi^{(1)}}\to\mathsf{I}}{\varphi\to\varphi^{(1)}}\to\mathsf{I}}{\psi\to\varphi\to\varphi^{(2)}}\to\mathsf{I}
$$

Teoreem

Iga tõestatava valemi korral leidub normaalkujuline tõestus.

Naturaaldeduktsioon ($ND(\rightarrow \wedge \vee)$)

Normaliseerimisreeglid

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \Sigma\\ \varphi_1\end{array} \qquad \begin{array}{c}\vdots\ \Pi\\ \varphi_2\end{array}}{\varphi_1 \wedge \varphi_2}}{\varphi_1} \longrightarrow \begin{array}{c}\vdots\ \Sigma\\ \varphi_1\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \Theta\\ \varphi_1\end{array}}{\varphi_1 \vee \varphi_2} \qquad \begin{array}{c}\overline{\varphi_1}\\ \vdots\ \Sigma\\ \gamma\end{array} \qquad \begin{array}{c}\overline{\varphi_2}\\ \vdots\ \Pi\\ \gamma\end{array}}{\gamma} \longrightarrow \begin{array}{c}\vdots\ \Theta\\ \varphi_1\\ \vdots\ \Sigma\\ \gamma\end{array}$$

$$\dfrac{\begin{array}{c}\vdots\ \Sigma\\ \psi\end{array} \qquad \dfrac{\begin{array}{c}\overline{\psi}\\ \vdots\ \Pi\\ \varphi\end{array}}{\psi \rightarrow \varphi}}{\varphi} \longrightarrow \begin{array}{c}\vdots\ \Sigma\\ \psi\\ \vdots\ \Pi\\ \varphi\end{array}$$

Curry-Howard isomorfism

Teoreem

(i) Kui $\Gamma \vdash M : \varphi$, siis $|\Gamma| \vdash_{ND(\to)} \varphi$, kus $|\Gamma| = \{\varphi \mid (x : \varphi) \in \Gamma\}$.

(ii) Kui $\Gamma \vdash_{ND(\to)} \varphi$, siis leidub term M selline et $\Delta \vdash M : \varphi$, kus $\Delta = \{x_\varphi : \varphi \mid \varphi \in \Gamma\}$.

Curry-Howard vastavus

$\lambda \to$	$ND(\to)$
tüüp	valem
tüübimuutuja	lausemuutuja
term	tõestus
termimuutuja	eeldus
tüübikonstruktor	loogiline konnektiiv
asustatavus	tõestatavus
reduktsioon	normaliseerimine

Paaridja summad ($\lambda(\to \times +)$)

Tüüpimisreeglid

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \tau}{\Gamma \vdash (M, N) : \sigma \times \tau} \qquad \frac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \texttt{fst}\ M : \sigma} \qquad \frac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \texttt{snd}\ M : \tau}$$

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \texttt{inl}\ M : \sigma + \tau} \qquad \frac{\Gamma \vdash M : \tau}{\Gamma \vdash \texttt{inr}\ M : \sigma + \tau}$$

$$\frac{\Gamma \vdash L : \sigma + \tau \quad \Gamma, \{x : \sigma\} \vdash M : \rho \quad \Gamma, \{y : \tau\} \vdash N : \rho}{\Gamma \vdash \mathsf{case}(L; x.M; y.N) : \rho}$$

Paarid ja variandid ($\lambda(\rightarrow \times +)$)

Reduktsioonireeglid

$$
\begin{array}{lcl}
\texttt{fst}\ (M, N) & \longrightarrow & M \\
\texttt{snd}\ (M, N) & \longrightarrow & N \\
\\
\texttt{case}(\texttt{inl}\, L; x.M; y.N) & \longrightarrow & M[L/x] \\
\texttt{case}(\texttt{inr}\, L; x.M; y.N) & \longrightarrow & N[L/y]
\end{array}
$$

NB!

Kehtib Curry-Howardi isomorfism $\lambda(\rightarrow \times +)$ ja $ND(\rightarrow \wedge \vee)$ vahel.

Polümorfne λ-arvutus (F)

Tüübid

$$\tau \;:=\; \alpha \mid (\tau \to \tau) \mid (\forall\alpha.\ \tau)$$

Termid

$$M := x \mid (\lambda x{:}\tau.M) \mid (M\ M) \mid (\Lambda\alpha.M) \mid (M\ \tau)$$

Tüüpimisreeglid

$$\frac{\Gamma \Vdash M : \sigma}{\Gamma \Vdash (\Lambda\alpha.M) : \forall\alpha.\sigma}\ (\alpha \notin \mathrm{FV}(\Gamma)) \qquad \frac{\Gamma \Vdash M : \forall\alpha.\sigma}{\Gamma \Vdash M\,\tau : \sigma[\tau/\alpha]}$$

β-reduktsioon

$$(\lambda x{:}\tau.M)\,N \longrightarrow_\beta M[N/x]$$
$$(\Lambda\alpha.M)\,\tau \longrightarrow_\beta M[\tau/\alpha]$$

Polümorfne λ-arvutus (F)

Näited

$$\vdash \Lambda\alpha.\lambda x^{\forall\alpha.\alpha\to\alpha}.x(\alpha\to\alpha)(x\alpha) : \forall\alpha.(\forall\alpha.\alpha\to\alpha)\to\alpha\to\alpha$$
$$\vdash \Lambda\alpha.\lambda f^{\alpha\to\alpha}.\lambda x^{\alpha}.f(fx) : \forall\alpha.(\alpha\to\alpha)\to(\alpha\to\alpha)$$

Teoreem

Süsteem **F** on rangelt normaliseeruv.

NB!

Kehtib Curry-Howardi isomorfism süsteemi **F** ja teist järku intuitsionistliku predikaatarvutuse $\{\forall,\to\}$-fragmendi vahel.

Järeldus

Tüübi asustatavuse probleem süsteemis **F** ei ole lahenduv.

Polümorfne λ-arvutus (F)

- *Absurd:* $\bot \equiv \forall\alpha.\alpha$
- *Konjunktsioon:* $\sigma \times \tau \equiv \forall\alpha.((\sigma \to \tau \to \alpha) \to \alpha)$

$$
\begin{array}{lcl}
(P, Q) & \equiv & \Lambda\alpha.\lambda z^{\sigma\to\tau\to\alpha}.zPQ \\
\mathtt{fst}(M^{\sigma\times\tau}) & \equiv & M\,\sigma\,(\lambda x^{\sigma}.\lambda y^{\tau}.x) \\
\mathtt{snd}(M^{\sigma\times\tau}) & \equiv & M\,\tau\,(\lambda x^{\sigma}.\lambda y^{\tau}.x)
\end{array}
$$

- *Disjunktsioon:* $\sigma + \tau \equiv \forall\alpha.((\sigma \to \alpha) \to (\tau \to \alpha) \to \alpha)$

$$
\begin{array}{lcl}
\mathtt{inl}(M^{\sigma}) & \equiv & \Lambda\alpha.\lambda u^{\sigma\to\alpha}.\lambda v^{\tau\to\alpha}.uM \\
\mathtt{inr}(M^{\tau}) & \equiv & \Lambda\alpha.\lambda u^{\sigma\to\alpha}.\lambda v^{\tau\to\alpha}.vM \\
\mathtt{case}(L^{\sigma+\tau}; x^{\sigma}.M^{\rho}; y^{\tau}.N^{\rho}) & \equiv & L\,\rho\,(\lambda x^{\sigma}.M)\,(\lambda y^{\tau}.N)
\end{array}
$$

- *Churchi numbrid:* $\omega \equiv \forall\alpha.(\alpha \to \alpha) \to \alpha \to \alpha$

$$
\begin{array}{lcl}
\underline{n} & \equiv & \Lambda\alpha.\lambda f^{\alpha\to\alpha}x^{\alpha}.f^{n}x \\
\mathtt{succ} & \equiv & \lambda n^{\omega}.\Lambda\alpha.\lambda f^{\alpha\to\alpha}x^{\alpha}.f(n\alpha f x)
\end{array}
$$

Polümorfne λ-arvutus (F)

NB!

Leidub rangelt normaliseeruvaid terme, mis ei ole süsteemis **F** tüübitavad

$$(\lambda zy.y(z\mathtt{I})(z\mathtt{K}))(\lambda x.xx)$$

Teoreem

Süsteemis **F** defineeritavate funktsioonide klass langeb kokku teist järku Peano aritmeetikas tõestatavalt rekursiivsete funktsioonide klassiga.

Teoreem

Polümorfse λ-arvutuse tüübituletamine ja tüübikontroll on ekvivalentsed ja mittelahenduvad probleemid.