A Brief Introduction to Bicategories

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1 Definition

1. MOTIVATION. Elements of a set are either equal or different. In a category, however, two objects can be different but still essentially the same, in a very precise way. Technically, they are said to be *isomorphic* and the construction witnessing this fact can be made explicit and used in calculations. For example, in Set, $A \times B$ and $B \times A$ are made isomorphic by the (explicitly introduced) bijection $\mathsf{s} = \langle \pi_2, \pi_1 \rangle$. Moreover, objects can be essentially the same in more than one way, as such 'sameness' may be witnessed by different isomorphisms.

In order to extend this view to the level of morphisms, the category has to be equipped with further structure. In particular, this requires arrows between arrows and a corresponding notion of composition. To avoid name clashing, such arrows are usually called 2-cells (objects and object morphisms being similarly named 0-cells and 1-cells, respectively).

In this context, the space of morphisms between any given pair of objects, usually referred to as a (hom-)set, acquires itself the structure of a category. Therefore the basic (1-cell) composition and unit laws become functorial, since they transform both objects (1-cells) and arrows (2-cells) of each (hom-)category in an uniform way. In consequence, 2-cells are composable by two different, but intrinsically related (see §4 below) ways.

Such a structure has the ability to refine the notion of 'sameness' by distinguishing isomorphisms further than the equality level. A further step, in the same spirit, consists of weakening the degree of strictness up to which the usual associative and unit laws for 1-cell composition are supposed to hold. If, in particular, equality is relaxed to isomorphism one ends up with a *bicategory*. Of course, the witnessing isomorphisms have to be explicitly included in the definition as particular, invertible 2-cells. However, there is a price to be paid for this increased expressiveness: such isomorphisms should themselves obey some (coherence) laws to be used in calculations as if they were proper equalities.

The following paragraphs introduce some basic definitions and a few examples of bicategories. The basic reference on bicategories is Bénabou's original paper [Ben67]. Comprehensive accounts can be found in, *e.g.*, [Bor94] and [Str96].

2. DEFINITION. The underlying structure of a *bicategory* B consists of

- a class of objects A, B, C, ...
- for each pair $\langle A, B \rangle$ of objects, a small (hom-)category $\mathsf{B}(A, B)$ with arrows p, q, r, ... from A to B as objects and arrows $h, k, l, \ldots$ between them denoted as in, *e.g.*, $h : p \Longrightarrow q$, and referred to as 2-cells. Composition in $\mathsf{B}(A, B)$ is denoted by $\cdot$ and the identity on p, for each $p : A \longrightarrow B$, by $1_p : p \Rightarrow p$.
- for each triple $\langle A, B, C \rangle$ of objects, a composition law given by a (bi)functor

$$;_{A,B,C} : \mathsf{B}(A, B) \times \mathsf{B}(B, C) \longrightarrow \mathsf{B}(A, C)$$

- for each object A, an identity functor

$$I_A : \mathsf{1} \longrightarrow \mathsf{B}(A, A)$$

where $\mathsf{1}$ stands for the final object in the category Cat of small categories.

3. REMARK. From the definition above, 2-cells in B come equipped with two forms of composition, called, respectively, *vertical* and *horizontal* after the respective diagrammatic presentation. Vertical composition is given by the composition law in each hom category. On the other hand, horizontal composition corresponds to the action of functor ; on 2-cells. The action of I_A on the unique object of 1 is the 1-cell $I_A : A \longrightarrow A$, the identity on object A (wrt ; on 1-cells), whereas its action on the unique arrow of 1 is the 2-cell $1_{I_A} : I_A \Rightarrow I_A$, the identity on the 1-cell $I_A : A \longrightarrow A$ (wrt ; on 2-cells, *i.e.*, horizontal 2-cell composition). This is sketched as follows:

$$\begin{tikzcd}
A \arrow[r, bend left, "I_A"] \arrow[r, bend right, "I_A"'] \arrow[r, phantom, "\Downarrow 1_{I_A}"] & A
\end{tikzcd}$$

4. INTERCHANGE LAW. The two forms of composition mentioned above are related by the equality

$$(k\,;k') \cdot (h\,;h') \;=\; (k \cdot h)\,;(k' \cdot h')$$

which gives an unambiguous meaning to the diagram

$$\begin{tikzcd}[row sep=large]
A \arrow[r, bend left=50, "p"] \arrow[r, "q" description] \arrow[r, bend right=50, "r"'] \arrow[r, phantom, shift left=3, "\Downarrow h"] \arrow[r, phantom, shift right=3, "\Downarrow k"] & B \arrow[r, bend left=50, "p'"] \arrow[r, "q'" description] \arrow[r, bend right=50, "r'"'] \arrow[r, phantom, shift left=3, "\Downarrow h'"] \arrow[r, phantom, shift right=3, "\Downarrow k'"] & C
\end{tikzcd}$$

The equation arises simply because $;_{A,B,C}$, for each triple $\langle A, B, C\rangle$, is a bifunctor. This is widely used in calculations involving natural transformations.

5. ASSOCIATIVITY AND UNIT LAWS. The motivation in §1 has hopefully shed some light on why associativity and unit axioms for ; are supposed to hold only up to isomorphism. Therefore, such axioms are introduced explicitly in the definition of a bicategory as natural isomorphisms. Moreover, they are subject to some suitable coherence laws integrating the definition as well.

6. DEFINITION. A *bicategory* B is defined by the structure introduced in §2 plus the following natural isomorphisms, for each A, B, C and D[1]:

$$\begin{aligned}
\mathsf{a}_{A,B,C,D} &: \; ;_{A,B,D} \circ (\mathsf{Id} \times ;_{B,C,D}) \Longrightarrow \; ;_{A,C,D} \circ (;_{A,B,C} \times \mathsf{Id}) \\
\mathsf{r}_{A,B} &: \; ;_{A,A,B} \circ (I_A \times \mathsf{Id}) \Longrightarrow \mathsf{Id} \\
\mathsf{l}_{A,B} &: \; ;_{A,B,B} \circ (\mathsf{Id} \times I_B) \Longrightarrow \mathsf{Id}
\end{aligned}$$

diagrammatically,

$$\begin{tikzcd}
\mathsf{B}(A,B) \times \mathsf{B}(B,C) \times \mathsf{B}(C,D) \arrow[r, ";_{A,B,C} \times \mathsf{Id}"] \arrow[d, "\mathsf{Id} \times ;_{B,C,D}"'] & \mathsf{B}(A,C) \times \mathsf{B}(C,D) \arrow[d, ";_{A,C,D}"] \\
\mathsf{B}(A,B) \times \mathsf{B}(B,D) \arrow[r, ";_{A,B,D}"'] \arrow[ur, Rightarrow, "\mathsf{a}_{A,B,C,D}" description] & \mathsf{B}(A,D)
\end{tikzcd}$$

and

$$\begin{tikzcd}
1 \times \mathsf{B}(A,B) \arrow[d, "I_A \times \mathsf{Id}"'] & \mathsf{B}(A,B) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, equal] & \mathsf{B}(A,B) \times 1 \arrow[d, "\mathsf{Id} \times I_B"] \\
\mathsf{B}(A,A) \times \mathsf{B}(A,B) \arrow[r, ";_{A,A,B}"'] \arrow[ur, Rightarrow, "\mathsf{r}_{A,B}" description] & \mathsf{B}(A,B) & \mathsf{B}(A,B) \times \mathsf{B}(B,B) \arrow[l, ";_{A,B,B}"] \arrow[ul, Rightarrow, "\mathsf{l}_{A,B}" description]
\end{tikzcd}$$

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[1] Recall that $\circ$ (often abbreviated by juxtaposition) denotes functor composition and Id is the identity functor on any category.

The components of these isomorphisms, for $p : A \longrightarrow B$, $q : B \longrightarrow C$, $r : C \longrightarrow D$ and $t : D \longrightarrow E$, are, as expected, the invertible 2-cells

$$\begin{aligned} \mathsf{a}_{p,q,r} &: p \,;(q \,; r) \overset{\cong}{\Longrightarrow} (p \,; q) \,; r \\ \mathsf{r}_p &: I_A \,; p \overset{\cong}{\Longrightarrow} p \\ \mathsf{l}_p &: p \,; I_B \overset{\cong}{\Longrightarrow} p \end{aligned}$$

subject to the coherence laws expressed by the commutativity of the following diagrams:

$$\begin{tikzcd}
p\,;(q\,;(r\,;t)) \arrow[rr, "1_p;\mathsf{a}"] \arrow[d, "\mathsf{a}"'] & & p\,;((q\,;r)\,;t) \arrow[d, "\mathsf{a}"] \\
(p\,;q)\,;(r\,;t) \arrow[dr, "\mathsf{a}"'] & & (p\,;(q\,;r))\,;t \arrow[dl, "\mathsf{a};1_t"] \\
& ((p\,;q)\,;r)\,;t &
\end{tikzcd}$$

and

$$\begin{tikzcd}
p\,;(I_B\,;q) \arrow[rr, "\mathsf{a}"] \arrow[dr, "1_p;\mathsf{r}"'] & & (p\,;I_B)\,;q \arrow[dl, "\mathsf{l};1_q"] \\
& p\,;q &
\end{tikzcd}$$

7. 2-CATEGORIES. The structure arising by taking the families of natural isomorphisms a, l and r as mere identities, is called a *2-category*. In this stricter setting the coherence axioms hold automatically.

2 Examples

8. EXAMPLES. A typical example of a 2-category is Cat, with small categories, functors and natural transformations as 0, 1 and 2-cells respectively.

Another trivial example of a bicategory arises by duality. The dual B^{op} of a bicategory B is still a bicategory, formed by reversing the 1-cells. The 2-cells of B, however, remain unchanged (just think on $\mathsf{Cat}^{\mathsf{op}}$).

Finally, any category C can be seen as a special case of a 2-category, and hence of a bicategory, by regarding each homset $\mathsf{C}(A, B)$ as a discrete category.

9. SPANS. A more interesting example of a bicategorical structure is given by the spans of a category C with pullbacks. The construction is as follows: take the objects of C as 0-cells and define 1-cell (f, g) from A to B as a span $\langle X, f, g\rangle$, *i.e.*, a pair of C-arrows, $f : X \longrightarrow A$ and $g : X \longrightarrow B$, with a common domain. Spans compose by pullbacking, *i.e.*, $(f, g)\,;(f', g') = (f \cdot x, g' \cdot y)$, where (x, y) is an arbitrarily specified pullback of (g, f') as in the following diagram:

$$\begin{tikzcd}
& & P \arrow[dl, "x"'] \arrow[dr, "y"] & & \\
& X \arrow[dl, "f"'] \arrow[dr, "g"] & & X' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\
A & & B & & C
\end{tikzcd}$$

The identity on object A is, of course, $(\mathsf{id}_A, \mathsf{id}_A)$, where id_A is the identity on A in C. Now define a 2-cell as a morphism between spans on the same objects, *i.e.*, a C-arrow $k : X \longrightarrow Y$ making the following diagram to commute.

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, Rightarrow, "k"] & \\
A & & B \\
& Y \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

Vertical composition of such morphisms is simply inherited from C. On the other hand, horizontal composition is less obvious. Given $k : (f, g) \Rightarrow (f', g')$ and $l : (m, n) \Rightarrow (m', n')$, as below, their composite $k\,;l$ is the 2-cell $h : (f \cdot x, n \cdot y) \Rightarrow (f' \cdot x', n' \cdot y')$ defined as the unique factorisation through the pullback $\langle P', x', y' \rangle$ of $k \cdot x$ and $l \cdot y$, as illustrated in the following diagram.

$$\begin{tikzcd}
& & P \arrow[dl, "x"'] \arrow[dr, "y"] \arrow[dddd, Rightarrow, "h"] & & \\
& X \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, Rightarrow, "k"] & & Y \arrow[dl, "m"'] \arrow[dr, "n"] \arrow[dd, Rightarrow, "l"] & \\
A & & B & & C \\
& X' \arrow[ul, "f'"] \arrow[ur, "g'"'] & & Y' \arrow[ul, "m'"] \arrow[ur, "n'"'] & \\
& & P' \arrow[ul, "x'"] \arrow[ur, "y'"'] & &
\end{tikzcd}$$

Note that

$$\begin{aligned}
& g' \cdot k \cdot x \\
= \quad & \{\ k \text{ is a 2-cell}\ \} \\
& g \cdot x \\
= \quad & \{\ P \text{ is a pullback}\ \} \\
& m \cdot y \\
= \quad & \{\ l \text{ is a 2-cell}\ \} \\
& m' \cdot l \cdot y
\end{aligned}$$

and h is actually a 2-cell:

$$\begin{aligned}
& f' \cdot x' \cdot h \\
= \quad & \{\ h \text{ factorizes } k \cdot x\ \} \\
& f' \cdot k \cdot x \\
= \quad & \{\ k \text{ is a 2-cell}\ \} \\
& f \cdot x
\end{aligned}$$

and similarly one can prove $n' \cdot y' \cdot h \ = \ n \cdot y$. Spans are a useful device to generalise relations to an arbitrary category. As pullbacks are defined up to isomorphism and in the definition of ; they are arbitrarily chosen, the corresponding associativity also holds only up to isomorphism. Having chosen the A-identity span as the identity on A, the isomorphisms corresponding to the left or right unit laws, become simply the identity natural transformations.

10. Partial Maps. Another typical use of spans arises in the definition of sets and partial maps. In fact a partial map from a set A to a set B may be regarded as an isomorphism class of spans on A and B, where the first component (dom) is a monomorphism, *i.e.*,

$$\begin{tikzcd}
& U \arrow[dl, hook', "\mathrm{dom}"'] \arrow[dr, "f"] & \\
A & & B
\end{tikzcd}$$

Let $[\langle U, \mathrm{dom}, f\rangle]_{\simeq}$ stand for the isomorphism class of $\langle U, \mathrm{dom}, f\rangle$. A morphism between two partial maps $[\langle U, \mathrm{dom}, f\rangle]_{\simeq}$ and $[\langle V, \mathrm{dom}', f'\rangle]_{\simeq}$ from A to B is just a 2-cell between the corresponding spans, *i.e.*,

$$\begin{tikzcd}
& U \arrow[dl, hook', "\mathrm{dom}"'] \arrow[dr, "f"] \arrow[dd, Rightarrow, "p"'] & \\
A & & B \\
& V \arrow[ul, hook, "\mathrm{dom}'"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

Wherever it exists, this arrow is unique, which makes Par, the category of partial maps, a locally-ordered bicategory (see [Car87] for details).

3 Further Structure

11. Stepping Down. Cell-reindexing is a simple way to *step down* from a bicategory to a category. One forms the new category by taking the original 1-cells as objects and the 2-cells as arrows. This new category still carries the 'genetic inheritance' of the original one, in the form of some additional structure. Here is a particularly simple, but illustrative example.

Think, first, in an ordinary (1-)category with an unique object A. One may form a new category taking as objects the (auto)morphisms on A and as arrows the morphisms between them. As we have begun with an ordinary (1-)category the latter simply do not exist and, therefore, the result is just the *set* (*i.e.*, the discrete or 0-category) of automorphisms. Their composition appears now as a binary associative operation, with identity, on the set, which, thanks to this "inheritance", becomes a *monoid*.

If the same procedure is applied to a bicategory also with just one object, the original 2-cells become the arrows of an ordinary category, and composition amounts to the original vertical composition. Notice that, again, the obtained category inherits some additional structure: since its objects are arrows of the original bicategory, it gets for free a 'multiplication' at the object level. The result is, of course, a *monoidal* category.

12. Homomorphisms of Bicategories. Just as bicategories generalise categories, a bicategory homomorphism arises as a generalisation of the notion of a functor. The cornerstone of such an extension is compatibility with the 2-cell structure.

Again the functoriality axioms can be required to hold at different levels of strictness. Therefore, they are built into the definition as particular natural transformations obeying some coherence laws.

In the definition below nothing but naturality is assumed about such transformations. The kind of homomorphism defined is called a *lax functor* and is the standard notion of bicategory homomorphism, consistent with the weak approach to n-categories (see §17 further on).

Requiring $\mathsf{m}_{A,B,C}$ and u_A below to be isomorphisms amounts to the definition of a *pseudo-functor*, *i.e.*, a functor being functorial up to isomorphism. Hence, $\mathsf{T}f \;;' \mathsf{T}g \;\cong\; \mathsf{T}(f \;;\, g)$ and $\mathsf{T}I \;\cong\; \mathsf{T}I'$. Pseudo-functors of the form $\mathsf{I} : C^{\mathsf{op}} \longrightarrow \mathsf{Cat}$ are well-known in applications of category theory to computer science to model *indexing* situations. In such a context they are named *indexed categories* and shown to be equivalent to *fibrations* [Gro70] a more convenient tool to achieve the same (see the recent book by B. Jacobs [Jac99] for systematic applications to logic and type theory or [Str99] for a tutorial).

Finally, a stricter approach will enforce $\mathsf{m}_{A,B,C}$ and u_A as effective identities, making all the axioms to hold on the nose. This defines a *2-functor*, which is the usual notion of a homomorphism of 2-categories.

13. DEFINITION. Let B and B' be two bicategories. A *homomorphism* T from B to B', is called a *lax functor* and consists of

- a function T mapping objects of C into objects of C',
- for each pair $\langle A, B\rangle$ of objects, a functor $\mathsf{T}_{A,B} : \mathsf{B}(A,B) \longrightarrow \mathsf{B}'(\mathsf{T}A, \mathsf{T}B)$
- for each triple $\langle A, B, C\rangle$ of objects, a natural transformation $\mathsf{m} : ;' \circ (\mathsf{T}_{A,B} \times \mathsf{T}_{B,C}) \Longrightarrow \mathsf{T}_{A,C} \circ ;$ whose components are 2-cells $\mathsf{m}_{p,q} : \mathsf{T}p ;' \mathsf{T}q \longrightarrow \mathsf{T}(p ; q)$, for each $p : A \longrightarrow B$ and $q : B \longrightarrow C$. In a diagram:

$$\begin{tikzcd}
\mathsf{B}(A,B) \times \mathsf{B}(B,C) \arrow[r, ";_{A,B,C}"] \arrow[d, "\mathsf{T}_{A,B} \times \mathsf{T}_{B,C}"'] & \mathsf{B}(A,C) \arrow[d, "\mathsf{T}_{A,C}"] \\
\mathsf{B}'(\mathsf{T}A,\mathsf{T}B) \times \mathsf{B}'(\mathsf{T}B,\mathsf{T}C) \arrow[r, ";'_{\mathsf{T}A,\mathsf{T}B,\mathsf{T}C}"'] \arrow[ur, Rightarrow, "\mathsf{m}_{A,B,C}"] & \mathsf{B}'(\mathsf{T}A,\mathsf{T}C)
\end{tikzcd}$$

- for each object A, a natural transformation $\mathsf{u} : I'_{\mathsf{T}A} \Longrightarrow \mathsf{T}_{A,A} \circ I_A$ as in diagram

$$\begin{tikzcd}
1 \arrow[r, "I_A"] \arrow[d, equal] & \mathsf{B}(A,A) \arrow[d, "\mathsf{T}_{A,A}"] \\
1 \arrow[r, "I'_{\mathsf{T}A}"'] \arrow[ur, Rightarrow, "\mathsf{u}"] & \mathsf{B}'(\mathsf{T}A,\mathsf{T}A)
\end{tikzcd}$$

subject to the coherence laws expressed by the commutativity of the following diagrams:

$$\begin{tikzcd}
\mathsf{T}p ;' (\mathsf{T}q ;' \mathsf{T}r) \arrow[r, "\mathsf{Id};'\mathsf{m}"] \arrow[d, "\mathsf{a}'"'] & \mathsf{T}p ;' \mathsf{T}(q ; r) \arrow[r, "\mathsf{m}"] & \mathsf{T}(p ; (q ; r)) \arrow[d, "\mathsf{Ta}"] \\
(\mathsf{T}p ;' \mathsf{T}q) ;' \mathsf{T}r \arrow[r, "\mathsf{m};'\mathsf{Id}"'] & \mathsf{T}(p ; q) ;' \mathsf{T}r \arrow[r, "\mathsf{m}"'] & \mathsf{T}((p ; q) ; r)
\end{tikzcd}$$

and

$$\begin{tikzcd}
I'_{\mathsf{T}A} ;' \mathsf{T}p \arrow[r, "\mathsf{r}'"] \arrow[d, "\mathsf{u};'\mathsf{Id}"'] & \mathsf{T}p \arrow[dd, equal] & \mathsf{T}p ;' I'_{\mathsf{T}B} \arrow[l, "\mathsf{l}'"'] \arrow[d, "\mathsf{Id};'\mathsf{u}"] \\
\mathsf{T}I_A ;' \mathsf{T}p \arrow[d, "\mathsf{m}"'] & & \mathsf{T}p ;' \mathsf{T}I_B \arrow[d, "\mathsf{m}"] \\
\mathsf{T}(I_A ; p) \arrow[r, "\mathsf{Tr}"'] & \mathsf{T}p & \mathsf{T}(p ; I_B) \arrow[l, "\mathsf{Tl}"]
\end{tikzcd}$$

14. AN CHEAP EXAMPLE. Take the singleton set $\mathbf{1}$ as a discrete (bi)category and let B be any bicategory. Then any lax-functor $\mathsf{M} : \mathbf{1} \longrightarrow \mathsf{B}$ is nothing more than a *monad* in B. The good old friend of functional programmers is around the corner, just by instantiating B with Cat.

15. FURTHER STRUCTURE. A similar construction yields corresponding notions of natural transformation in a bicategorical setting. Components of a natural transformation τ between, say, (lax-)functors T and R

are, of course, 1-cells indexed by the objects. However, the naturality requirement is, again, introduced as the following family of natural transformations $\mathsf{n}_{A,B}$

$$\begin{tikzcd}
\mathsf{B}(A,B) \arrow[r, "\mathsf{T}_{A,B}"] \arrow[d, "\mathsf{R}_{A,B}"'] & \mathsf{B}'(\mathsf{T}A,\mathsf{T}B) \arrow[d, "\mathsf{B}'(\mathsf{T}A,\tau_B)"] \\
\mathsf{B}'(\mathsf{R}A,\mathsf{R}B) \arrow[r, "\mathsf{B}'(\tau_A,\mathsf{R}B)"'] \arrow[ur, Rightarrow, "\mathsf{n}_{A,B}"] & \mathsf{B}'(\mathsf{T}A,\mathsf{R}B)
\end{tikzcd}$$

where $\mathsf{B}(X,p) : \mathsf{B}(X,A) \longrightarrow \mathsf{B}(X,B)$ is the functor induced by the 1-cell $p : A \longrightarrow B$, for a given X, and, contravariantly, $\mathsf{B}(p,X) : \mathsf{B}(B,X) \longrightarrow \mathsf{B}(A,X)$.

One can go further and define a morphism between this kind of transformations

$$\begin{tikzcd}
\mathsf{T} \arrow[r, bend left, "\tau"] \arrow[r, bend right, "\tau'"'] \arrow[r, phantom, "\Downarrow \sigma"] & \mathsf{R}
\end{tikzcd}$$

whose components are 2-cells connecting, for each object A, τ_A to τ'_A. Such morphisms are known as *modifications* and, in particular, allow for the definition of the analogue of a functor category between two bicategories. Given B and B', a *functor bicategory* arises taking (lax-)functors, transformations and modifications as 0, 1 and 2-cells, respectively.

Having built all this structure, one can define what adjunctions are and, consequently, what limits mean, in it. There is also a notion of *representable* and an analogue of Yoneda lemma, which is used in the proof of the *coherence theorem* [Str96] asserting the possibility of reducing any bicategory to a (bi)equivalent 2-category. Although this will not be pursued here, we should remark that suitable generalisations of familiar categorical constructions emerge as expected, respecting the 2-cell structure and eventually relaxing the conditions up to which axioms are verified. Coherence requirements, however, may become rather heavy to state and prove.

Finally, notice that the fact that Cat is a (particular case of a) bicategory allows one to borrow common categorical constructions and have them interpreted in an arbitrary bicategory. For example, a pair of 1-cells $p : A \longrightarrow B$ and $q : B \longrightarrow A$ equipped with a 2-cell isomorphism $i : I_A \Rightarrow p \,;\, q$ in the hom-category $\mathsf{B}(A,A)$, and another one $j : I_B \Rightarrow q \,;\, p$ in the hom-category $\mathsf{B}(B,B)$, define an *equivalence* between objects A and B. In fact an equivalence of categories is just an instantiation of this notion in Cat.

16. COHERENCE. A final remark on *coherence* is in order. Coherence laws arise in the definition of a bicategory as well as in other related structures. A particularly familiar example in computer science is the case of monoidal categories, which, as mentioned above, can be thought of as born out of bicategories.

In a sense, coherence axioms are a price to be paid for the increased expressive power originated by weakening the defining structural properties. Think, for example, of the coherence diagram for associativity in definition §6. It identifies the basic ways in which the composition of 4 arrows can be parenthesised and relates them through a. To ensure that all such ways are unique is precisely the reason to enforce the commutativity of the diagram. By such a coherence axiom one knows that, in a bicategory, any two natural isomorphisms built out of a, r and l, by the composition and unit operations, actually coincide. That is to say, weakening has caused no special (calculational) damage.

The difficult question is: why is this so? Of course there are standard results asserting the fact (*e.g.*, Mac Lane's coherence theorem for monoidal categories [Mac71] or Bénabou's result for bicategories [Ben67]) but it would help to have a deeper understanding of the origins of coherence axioms.

Something that may help to build up intuition, is the observation, due to J. Dolan and J. Baez, among others, that an operation automatically satisfies all the (relevant) coherence laws if defined by an universal property. For example, Set has all finite products which are defined by an universal property. Moreover they are unique up to isomorphism. If one takes the product $A \times B$ for every pair of sets, making cartesian product an operation, three natural isomorphisms, expressing associativity, left and right units, get defined canonnically. Such isomorphisms verify the coherence axioms for a tensor product turning, therefore, $\langle \mathsf{Set}, \times \rangle$ into a monoidal category. One may therefore conclude that the usual definition of a monoidal category, with the explicit coherence axioms, amounts to the fact that any monoidal structure defined by universal properties automatically satisfies such axioms.

17. n-CATEGORIES. The whole subject of bicategories is much wider than we have been able to glimpse in this appendix. In fact, both generalisations embodied in the notion of a bicategory (*i.e.*, the introduction of arrows between arrows and the weakening of the degree of strictness up to which axioms hold) can be pursued further. Recall that the justification for the introduction of 2-cells is the possibility of having arrows around that are isomorphic rather than merely equal. Once all arrows become objects, the sentence will still make sense, as a justification for the introduction of categories themselves in the first place. One can easily imagine this process going on: considering 3-cells as 2-cell morphisms and so on. On the other hand, the relevant equations may be taken to hold up to isomorphism in the immediately lower level or, even more generally, up to an arbitrary arrow.

This is the broader context of *n-categories* [Bae97], whose basic claim is that equations should hold *on the nose* (*i.e.*, up to equality) only at the top level, *i.e.*, between n-cells. Therefore laws concerning k-cells, for $k < n$, should always be expressed as $(k+1)$-equivalences. In this context an equivalence between $(n-1)$-cells is an invertible n-cell whereas an equivalence between k-cells, for $k < n$, is just a $(k+1)$-cell invertible up to equivalence.

The framework is very expressive — in practice often things are only true up to (a suitable notion of) isomorphism, and sometimes only up to other things. But some care is needed to avoid getting puzzled by coherence conditions. The expressive power of *n-categories* is well illustrated by noting that a $(n+1)$-category with only one object can always be regarded as a special kind of a n-category. This was exactly what we have seen, for $n = 1$ and $n = 2$, in §11.

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