Multi-structure frameworks as adhesive fibrations

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Graph rewriting as a framework for concurrency

- Systems or states of a computation are represented by graphs.
- Semantics is defined by means of transformations on graphs:
  - match a subgraph with a lhs part of a rule
  - and then replacing it with the rhs.
- A categorical framework which is very suited for defining graphical (and more) models is *adhesive categories*. Indeed they support for theory of double pushout rewriting and of relative pushouts.

Aim of the talk

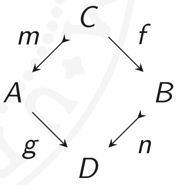
Investigate how adhesive graph-like categories can be modularly constructed.

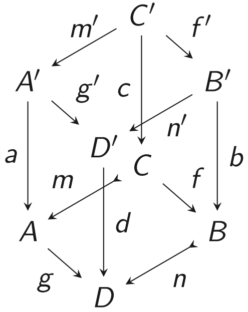
Adhesive categories

Definition

A category **C** is *adhesive* if

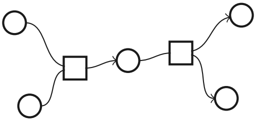
- **C** has pullbacks;
- **C** has pushouts along monomorphisms;
- pushouts along monomorphisms are Van Kampen squares.



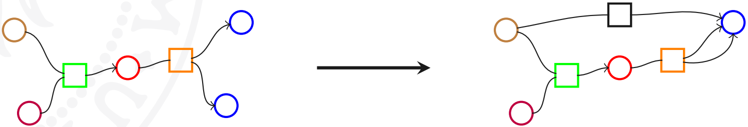


Example: hypergraph category (HGraph)

Objects: hypergraphs



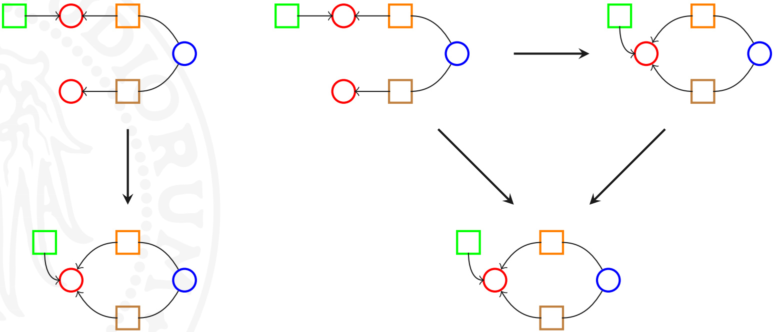
Morphisms: embeddings between hypergraphs



Embeddings must preserve the source and targets of edges.

Typed hypergraphs as slice category

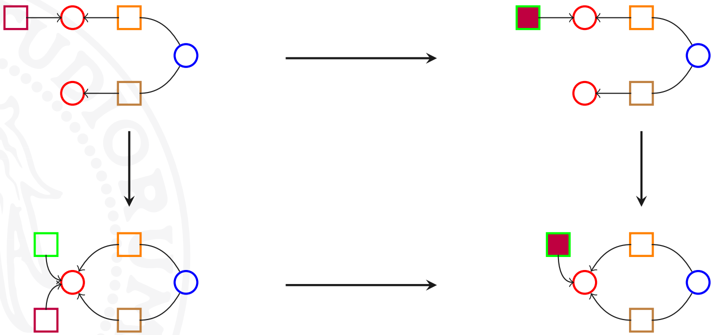
In order to impose a discipline on nodes and edges, often hypergraphs will be typed.



Slice category

We consider **HGraph**$/G$, where the object G defines the type. For every C, $\mathbf{C}/C$ is adhesive if $\mathbf{C}$ is so.

Typed hypergraphs as arrow category



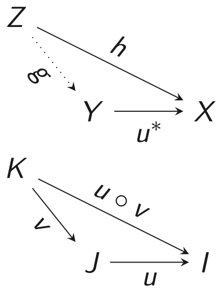
Arrow category

HGraph$^{\rightarrow}$ allows morphisms to change also the hypergraph type.
C$^{\rightarrow}$ is adhesive if **C** is so.

Fibrations

$\begin{array}{c}\mathbf{D}\\ \downarrow P\\ \mathbf{C}\end{array}$ is a *fibration* if and only if

- $P : \mathbf{D} \to \mathbf{C}$ is a functor;
- for all $u : J \to I$ in $\mathbf{C}$ and $X \in P(I)$ there is a *cartesian* arrow $u^* : Y \to X$ over u.



Let $P(I)$ be the subcategory of $\mathbf{C}$ consisting of those morphisms f such that $P(f) = \mathrm{id}_I$.

Codomain fibration

Let $\mathbf{C}$ be a category with pullbacks.

The codomain fibration $\begin{array}{c}\mathbf{C}^{\to}\\ \downarrow\\ \mathbf{C}\end{array}$ cod is defined as follows:

- an object $f : J \to I$ is mapped to its codomain I
- a morphism is mapped to its below morphism

$$\begin{array}{ccc} H & \longrightarrow & K \\ \downarrow & & \downarrow \\ J & \xrightarrow{f} & I \end{array} \quad \longmapsto \quad J \xrightarrow{f} I$$

- a map is *cartesian* if and only if it is a pullback square

$$\begin{array}{ccc} \bullet & \xrightarrow{u^*} & K \\ \downarrow & \lrcorner & \downarrow \\ J & \xrightarrow{u} & I \end{array}$$

also known as “change of base”

Notice that $\mathrm{cod}(C) \cong \mathbf{C}/C$.

Adhesive fibration

A fibration $\begin{array}{c}\mathbf{E}\\ \downarrow\\ \mathbf{B}\end{array}$ is *fibred adhesive*

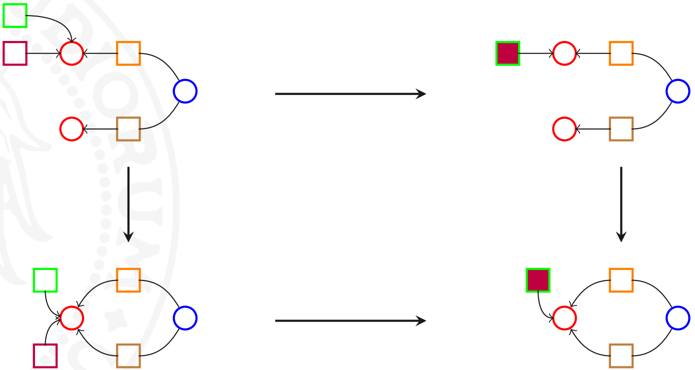
- if for every object B in $\mathbf{B}$ the fibre $\mathbf{E}_B$ is adhesive and
- the *reindexing functor* preserves adhesivity, i.e., all pullbacks, pushouts along monomorphisms and VK-squares.

By "adhesive fibration" we mean a fibred category which is fibred adhesive.

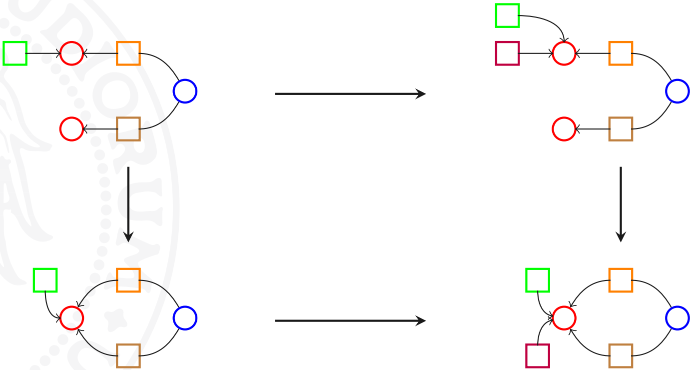
Theorem

The codomain fibration $\begin{array}{c}\mathbf{C}^{\to}\\ \downarrow\\ \mathbf{C}\end{array}\mathrm{cod}$ *is fibred adhesive if the underlying category* $\mathbf{C}$ *is adhesive and local cartesian closed.*

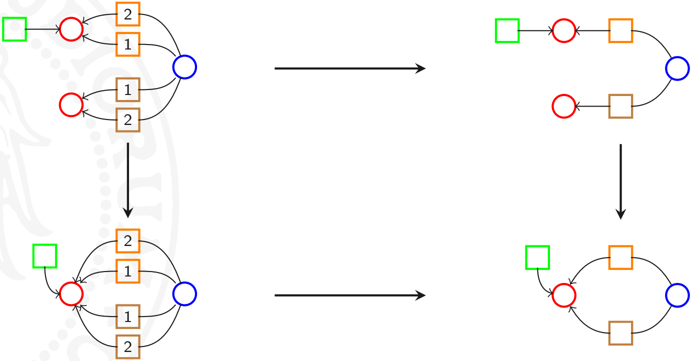
Change of base on hypergraphs - I



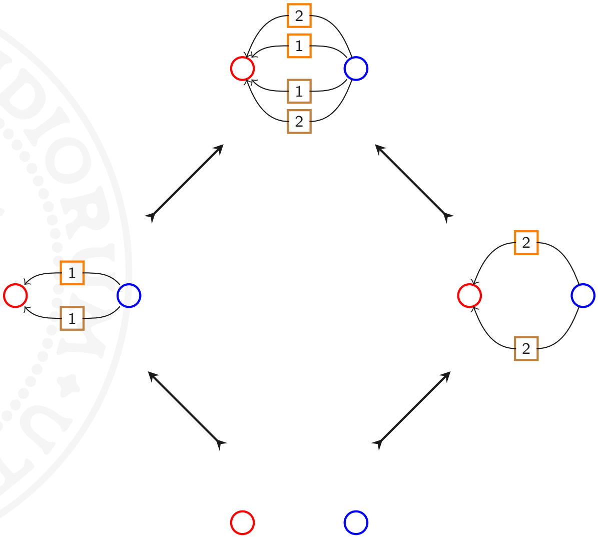
Change of base on hypergraphs - II



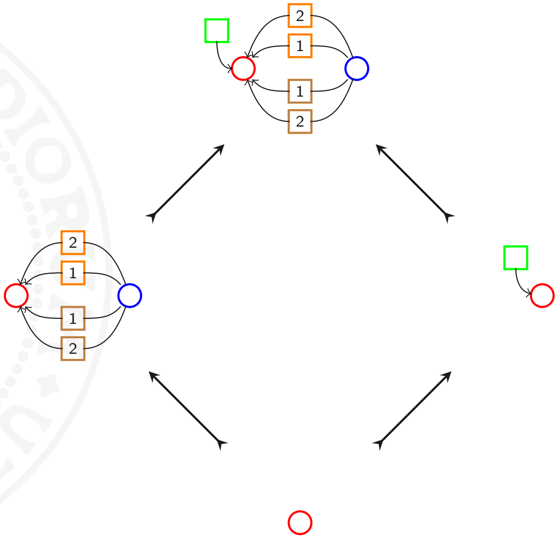
Change of base on hypergraphs - III



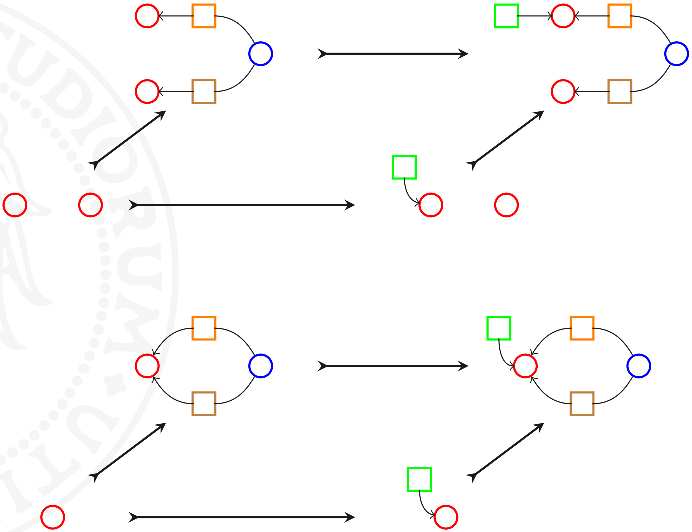
Modular composition of types - I



Modular composition of types - II

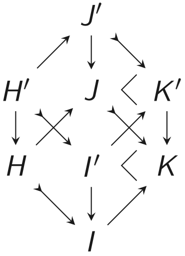


What about systems?



Pushout fibre is complete

Let $\mathbf{C}$ be adhesive, and let $f : H \rightarrowtail I$, $g : H \rightarrow J$ be a span in $\mathbf{C}$. Let $(K, f' : I \rightarrowtail K, g' J \rightarrow K)$ be the triple constructed as the pushout of f, g.



The *pushout fibre* of f, g, denoted as $POF(f, g)$, is the full subcategory of $\mathbf{C}/K$ whose objects are obtained by pushout of cleavages of objects over H, as above.

Proposition

$POF(f, g) \cong \mathbf{C}/K$. where K is the pushout object of types.

How adding interfaces to systems?

- The standard way: cospans!
- But cospan on what?

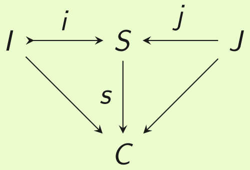
  ...we do not have just a category, but a fibration!
- There are two possible approaches:
- consider cospans on the single fibres
- defining somehow a fibration for cospans

Cospan over single fibres

Recall that $\mathrm{cod}(C) \cong \mathbf{C}/C$, where C in $\mathbf{C}$.

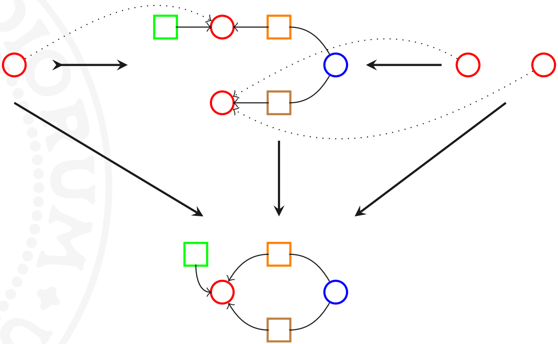
Input linear cospans on $\mathbf{C}/C$ (ilc($\mathbf{C}/C$))

Objects: the objects of $\mathbf{C}/C$, i.e., all the morphisms in $\mathbf{C}$ with codomain equal to C.
Morphisms: cospans on $\mathbf{C}/C$, i.e.,



- The system S and its input and output interfaces I, J are all typed over C.
- Composition is defined by pushouts as usual.

An example of cospan in typed hypergraphs

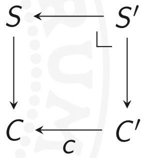


Change of base on cospans?

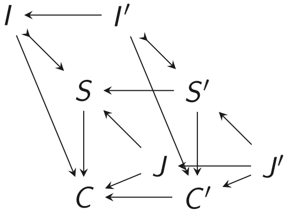
Given a morphisms $c : C' \to C$, it induces a (contravariant) functor

$$F(c' : C' \to C) : \mathrm{ilc}(\mathbf{C}/C) \to \mathrm{ilc}(\mathbf{C}/C').$$

Objects:



Morphisms:



Theorem

F preserves finite colimits (and hence pushouts = cospan composition) if $\mathbf{C}$ is a local cartesian closed category.

Interface types = system types?

- In the previous solution interface types and system types coincide.
- It is reasonable, i.e., interface of a Petri nets should be also typed as a Petri net. . .
- but in many cases, system interfaces are much simpler than systems.
- Particularly, one would like to expose just a "little" part of a system, e.g., just places for Petri nets.

Idea: add more information into types

A possible solution: take $\mathrm{ilc}(\mathbf{C}/C)$ as based category for defining a new fibration: $\begin{array}{c}\mathrm{ilc}((\mathbf{C}/C)^{\to}) \\ \downarrow \\ \mathrm{ilc}(\mathbf{C}/C)\end{array}$ icod.

Sub-typing system interfaces

The fibration $\begin{array}{c} \mathrm{ilc}(\mathbf{C}/C)^{\to} \\ \downarrow \\ \mathrm{ilc}(\mathbf{C}/C) \end{array}$ icod can be described as follows.

Objects:

$$\begin{array}{ccc} A & \longleftarrow & B \\ \downarrow & \swarrow & \\ C & & \end{array} \qquad \longmapsto \qquad \begin{array}{c} A \\ \downarrow \\ C \end{array}$$

Morphisms:

$$\begin{array}{ccccc} I & \rightarrowtail & S & \longleftarrow & J \\ \downarrow & & \downarrow & & \downarrow \\ H & \rightarrowtail & T & \longleftarrow & K \\ & \searrow & \downarrow & \swarrow & \\ & & C & & \end{array} \qquad \longmapsto \qquad \begin{array}{ccccc} H & \rightarrowtail & T & \longleftarrow & K \\ & \searrow & \downarrow & \swarrow & \\ & & C & & \end{array}$$

Conclusion

Summary

- Adhesive codomain fibration for composing categories of graph-like structures.
- Systems can be mapped or transported among fibres, allowing for a modular and incremental engineering.
- Typed hypergraphs as adhesive fibrations.

Future work

- Improve the interface typing mechanism and analyze what the change of base induces in this case.
- Investigate if reactive systems (and derived RPO lts) can be modularized w.r.t. the type composition.