Order Preserving Embeddings

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Joint work with Peter Morris, original idea due to Altenkirch, Hofmann and Streicher. It was shown to me by Conor McBride

This work in context

- I am interested in:
  - Semantics of programming languages based on lambda-calculus.
  - Formalised in type theoretic theorem provers such as Agda, Epigram and Coq
    - In these systems:
      - Proofs and programs are the same
      - The types of programs can express their specifications

What’s the problem?

- Doing these things in detail, as you are forced to do in a theorem prover, makes it very important to use appropriate representations of things
- The greater the level of detail, the greater the choice of representation
- A general problem for formal mathematics and programming language semantics

Finite sets

```
data Fin : Nat -> Set where
  fzero : Fin (suc n)
  fsuc  : Fin n -> Fin (suc n)
```

Pictorial enumeration of the first three instances:

`Fin zero`

`Fin (suc zero)`

`Fin (suc (suc zero))`

Example usage:

```
safe_lookup : Fin n -> Array X n -> X
```

Untyped Lambda calculus

The well-scoped lambda terms

```
data Lam : Nat -> Set where
  var : Fin n -> Lam n
  λ   : Lam (suc n) -> Lam n
  app : Lam n -> Lam n -> Lam n
```

Example expressions:

```
λ (var fzero)                -- identity function
λ (λ (var (fsuc fzero))) -- 'K' combinator
```

What can you go from here?

- Define syntactic operations:
  - Weakening, substitution, etc.
- Implement an evaluator/normaliser
- Extend it:
  - Data types, effects, annotate with simple or dependent types, etc.

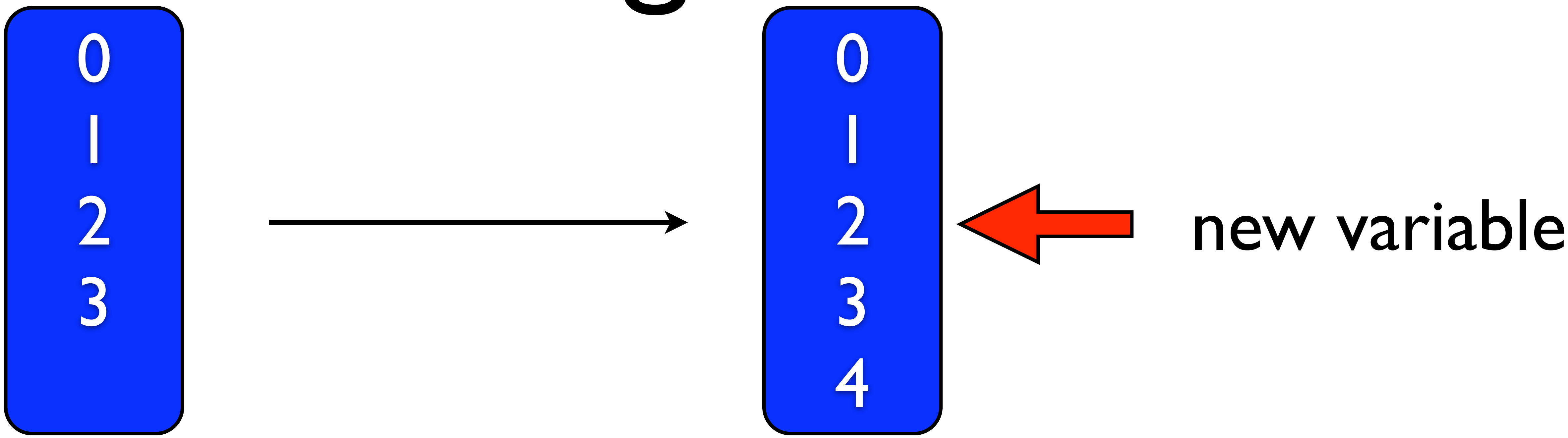
Implementing Weakening

Weakening adds a fresh variable at the 'zero' position and increments the rest

```
weak : Lam n -> Lam (suc n)
weak (var x)   = var (fsuc x)
weak (app t u) = app (weak t) (weak u)
weak (λ t)     = λ ? -- we're stuck
```

We need to generalise from adding a new variable at the end of the context to an arbitrary position

Thinning: A solution?



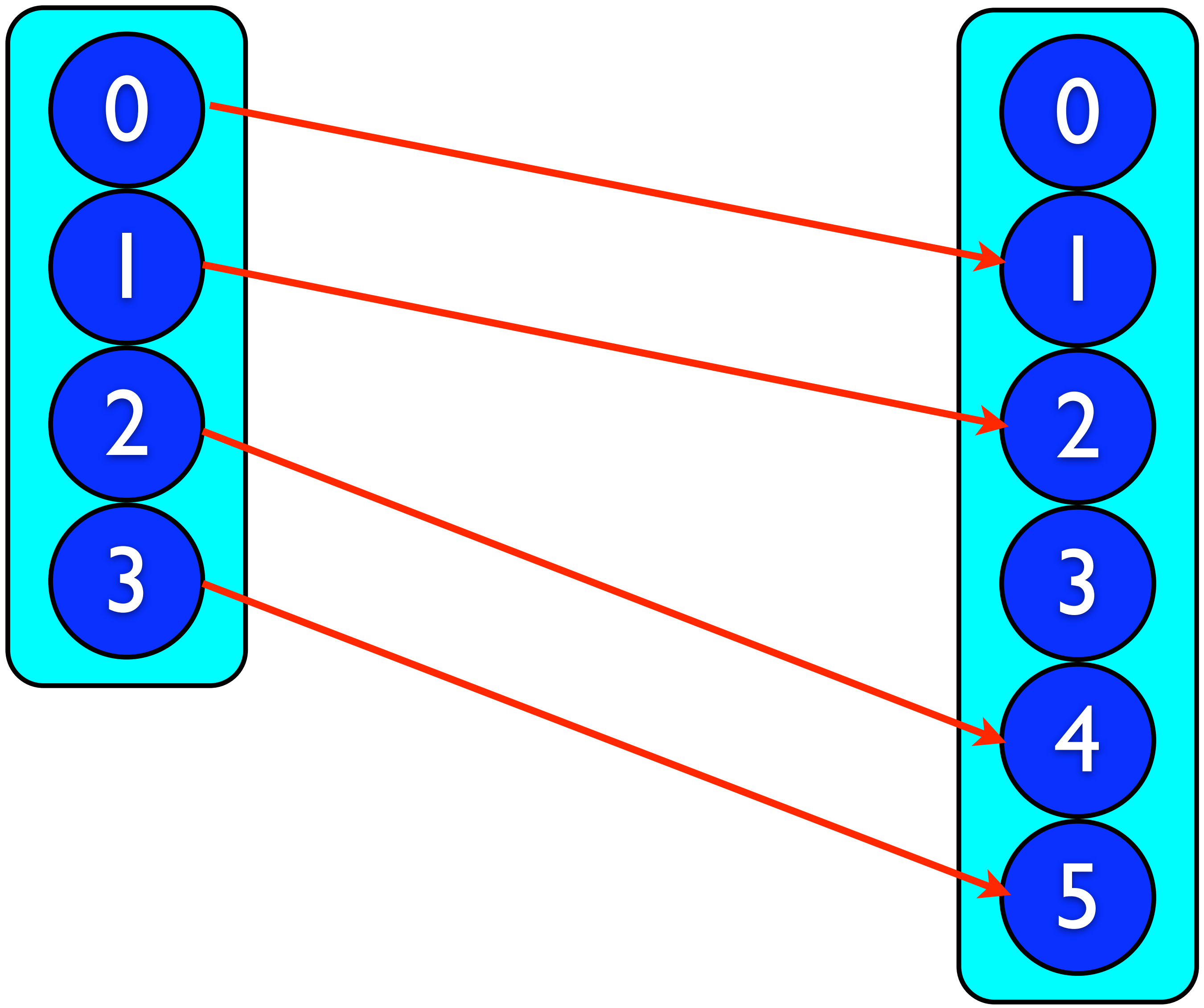
We implement it for terms as follows:

```
tlam : Fin (suc n) -> Lam n -> Lam (suc n)
tlam x (var y)   = var (tvar x y)
tlam x (app t u) = app (tlam x t) (tlam x u)
tlam x (λ t)     = λ (tlam (fsuc x) t)
```

Ordinary weakening for terms is now easy:

```
weak : Lam n -> Lam (suc n)
weak t = tlam fzero t
```

A better solution: Order Preserving Embeddings



OPEs

```
data OPE : Nat -> Nat -> Set where
  done : OPE zero zero
  skip : OPE m n -> OPE m (suc n)
  keep : OPE m n -> OPE (suc m) (suc n)
```

The identity OPE (`id : OPE n n`) is recursively defined

The weakening OPE is a now easy to define:

```
oweak : OPE n (suc n)
oweak = skip id
```

Action of OPEs

We can easily define the following operation lifting an OPE to a function on lambda expressions

```
olam : OPE m n -> (Lam m -> Lam n)
olam f (var x)   = var (ovar x)
olam f (app t u) = app (olam f t) (olam f u)
olam f (λ t)     = λ (olam (keep f) t)
```

Ordinary weakening for terms is now easy:

```
weak : Lam n -> Lam (suc n)
weak t = olam oweak t
```

OPEs form a category

- The objects are natural numbers: `0`, `1`, ...
- The morphisms are OPEs: `f`, `g`, ...
- For every object `n` an OPE `id : OPE n n`
- For every `f : OPE l m` and `g : OPE m n` there is an operation `●` such that `f ● g : OPE l n`
- and the following properties hold:
  - `f ● id = f` and `id ● f = f`
  - `f ● (g ● h) = (f ● g) ● h`

Why is this a good representation?

- Avoids reasoning about functions, first order structures are easier to deal with
- Avoids junk, captures only what we want
- Simple (elegant?) algebraic structure

Big-step Normalisation

- Central part of my thesis:
  - based on “Big-step Normalisation” by Altenkirch and Chapman
  - Published (soon!) in Special issue of Journal of Functional Programming. 2009. Eds. C. McBride and T. Uustalu
- Big win: simplified reasoning about weakenings; avoids problematic reasoning about context extensions.

Dependent types

- OPEs are helpful here for well-typed terms, as even defining variables requires reference to weakening.
- If we implement weakening by referring to variables we quickly get into a knot.
- OPEs avoid this and the fact the form a category become an integral part of the definition.