On the relationship between Priestley and stably compact spaces

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Stone duality
Marshall Harvey Stone (1936)

Totally disconnected compact spaces
(Stone spaces)

$$\Updownarrow$$

Boolean algebras.

This was the starting point of a whole area of research known as Stone duality.

Dualities are generally good for translating problems form one space to another where it could be easier to solve.

Stone duality
Marshall Harvey Stone (1937)
Hillary Priestley (1970)

spectral spaces (T_0)

$\Updownarrow$ 1937

bounded distributive lattices.

$\Updownarrow$ 1970

Priestley spaces (Hausdorff)

Definition. *A* **Priestley space** *is a compact ordered space* $\langle X; \mathcal{T}, \leq\rangle$ *such that for every* $x, y \in X$*, if* $x \not\geq y$ *then there exists a clopen upper set* U *such that* $y \in U$ *and* $x \notin U$*.*

A **spectral space** is a stably compact space with a basis of compact open sets.

Semantics of programming languages:

is about developing techniques for designing and describing programming languages.

Semantics approaches include:

- axiomatic (the program logic) – an example is Hoare logic.

- operational – an example is Java Abstract Machine.

- denotational – gives mathematical meaning of language constructs.

Denotational semantics:

uses a category to interpret programming language constructs;

- data types $\Longleftrightarrow$ objects,

- programs $\Longleftrightarrow$ morphisms.

Domains – Dana Scott (1969):

Sets, topological spaces, vectors spaces, and groups are <u>not</u> a good choice for denotational semantics.

Domains = ordered sets + certain conditions.

<u>From now on:</u>

- data types $\Longleftrightarrow$ domains,
- programs $\Longleftrightarrow$ functions between domains.

<u>Scott topologies</u> on domains to measure computability.

Stone duality and computer science
Samson Abramsky(1991)

Logical representation for bifinite domains (a particular Cartesian-closed category of domains).

In this framework,

- bifinite domains $\Longleftrightarrow$ propositional theories,
- functions $\Longleftrightarrow$ program logic axiomatising the properties of domains.

The domain interpretation via bifinite domains and the logical interpretation are Stone duals to each other and specify each other up to isomorphism.

Stably compact spaces

Abramsky's work was extended by Achim Jung et al to a class of topological spaces, stably compact spaces defined as follows.

Definition. *A stably compact space is a topological space which is sober, compact, locally compact, and for which the collection of compact saturated subsets is closed under finite intersections, where a saturated set is an intersection of open sets.*

These spaces contains coherent domains in their Scott topologies.

Coherent domains include bifinite domains and other interesting Cartesian-closed categories of domains such as FS.

Achim Jung's work in more detail

If $\langle X, \mathcal{T}\rangle$ is a stably compact space then its lattice $\mathcal{B}_X$ of observable properties is defined as follows:

$$\mathcal{B}_X = \{\langle O, K\rangle \mid O \in \mathcal{T}, K \in \mathcal{K}_X \text{ and } O \subseteq K\},$$

where $\mathcal{K}_X$ is the set of compact saturated subsets of X.

The computational interpretation is as follows. For a point $x \in X$ and a property $\langle O, K\rangle \in \mathcal{B}_X$:

- $x \in O \iff x$ *satisfies* the property $\langle O, K\rangle$,

- $x \in X \setminus K \iff x$ does not satisfy the property $\langle O, K\rangle$, and

- $x \in K \setminus O \iff$ the property $\langle O, K\rangle$ is *unobservable for* x.

Proximity relation

On the lattice $\mathcal{B}_X$ of observable properties a binary relation (*strong proximity relation*)was defined as:

$$\langle O, K\rangle \prec \langle O', K'\rangle \overset{\text{def}}{\Longleftrightarrow} K \subseteq O'.$$

The computational interpretation of the strong proximity relation $\prec$ can be stated as follows:

$$\langle O, K\rangle \prec \langle O', K'\rangle$$

$$\Updownarrow$$

$(\forall x \in X)$ either $\langle O', K'\rangle$ is observably satisfied for x

or $\langle O, K\rangle$ is (observably) not satisfied for x.

Thus we can say that $\prec$ behaves like a classical implication.

$\mathcal{B}_X$ and $\prec$ abstractly:

Definition. *A binary relation $\prec$ on a bounded distributive lattice $\langle L; \vee, \wedge, 0, 1\rangle$ is called a* proximity *if, for every $a, x, y \in L$ and $M \subseteq_{fin} L$,*

$$
\begin{array}{ll}
(\prec\prec) & \prec \circ \prec = \prec, \\
(\vee\!-\prec) & M \prec a \iff \bigvee M \prec a, \\
(\prec-\wedge) & a \prec M \iff a \prec \bigwedge M, \\
(\prec-\vee) & a \prec x \vee y \Longrightarrow (\exists\ x', y' \in L)\ x' \prec x,\ y' \prec y \\
& \quad \text{and } a \prec x' \vee y', \\
(\wedge\!-\prec) & x \wedge y \prec a \Longrightarrow (\exists\ x', y' \in L)\ x \prec x',\ y \prec y' \\
& \quad \text{and } x' \wedge y' \prec a.
\end{array}
$$

A **strong proximity lattice** *is a bounded distributive lattice $\langle L; \vee, \wedge, 0, 1\rangle$ together with a proximity relation $\prec$ on L.*

The lattice order is always a proximity relation.

Approximable relations: Capturing continuous maps between stably compact spaces

Definition. *Let* $\langle L_1; \vee, \wedge, 0, 1; \prec_1\rangle$ *and* $\langle L_2; \vee, \wedge, 0, 1; \prec_2\rangle$ *be strong proximity lattices and let* $\vdash$ *be a binary relation from* L_1 *to* L_2*. The relation* $\vdash$ *is called* approximable *if for every* $a \in L_1, b \in L_2$*,* $M_1 \subseteq_{fin} L_1$ *and* $M_2 \subseteq_{fin} L_2$*,*

$$
\begin{array}{ll}
(\vdash - \prec_2) & \vdash \circ \prec_2 = \vdash, \\
(\prec_1 - \vdash) & \prec_1 \circ \vdash = \vdash, \\
(\vee - \vdash) & M_1 \vdash b \Longleftrightarrow \bigvee M_1 \vdash b, \\
(\vdash - \wedge) & a \vdash M_2 \Longleftrightarrow a \vdash \bigwedge M_2, \\
(\vdash - \vee) & a \vdash \bigvee M_2 \Longrightarrow (\exists\ N \subseteq_{fin} L_1)\ a \prec_1 \bigvee N \\
 & \quad \text{and } (\forall n \in N)(\exists\ m \in M_2)\ n \vdash m.
\end{array}
$$

Basic aim of this work

The primary aim is to introduce Priestley spaces to the world of semantics of programming languages.

This can be done by answering the following question:

How can Priestley duality for bounded distributive lattices be extended to strong proximity lattices?

Logically the answer is interesting because *theories* (or *models*) of $\mathcal{B}_X$ are represented by prime filters, which are the points of the Priestley dual space of $\mathcal{B}_X$ as a bounded distributive lattice.

Apartness relations:

To answer the question (MFPS 2006) we equip Priestley spaces with the following relation:

Definition. *A binary relation* $\propto$ *on a Priestley space* $\langle X; \leq, \mathfrak{T}\rangle$ *is called an* apartness *if, for every* $a, c, d, e \in X$,

$(\propto\mathfrak{T})$	$\propto$ *is open in* $\langle X; \mathfrak{T}\rangle \times \langle X; \mathfrak{T}\rangle$
$(\downarrow\propto\uparrow)$	$a \leq c \propto d \leq e \Longrightarrow a \propto e$,
$(\propto\forall)$	$a \propto c \Longleftrightarrow (\forall b \in X)\ a \propto b$ *or* $b \propto c$,
$(\propto\uparrow\uparrow)$	$a \propto ({\uparrow}c \cap {\uparrow}d) \Longrightarrow (\forall b \in X)\ a \propto b,\ b \propto c$ *or* $b \propto d$,
$(\downarrow\downarrow\propto)$	$({\downarrow}c \cap {\downarrow}d) \propto a \Longrightarrow (\forall b \in X)\ d \propto b,\ c \propto b$ *or* $b \propto a$.

The relation $\not\geq$ is always an apartness.

The answer is:

The dual of a strong proximity lattice L is the corresponding Priestley space of prime filters, equipped with the apartness,

$$F \propto_{\prec} G \overset{\text{def}}{\iff} (\exists\ x \in F)(\exists\ y \notin G)\ x \prec y.$$

Vice versa, the dual of a Priestley space X with apartness $\propto$ is the lattice of clopen upper sets equipped with the strong proximity,

$$A \prec_{\propto} B \overset{\text{def}}{\iff} A \propto (X \setminus B).$$

Up to isomorphism, the correspondence is one-to-one.

Concerning the morphisms...

We proof that:

Continuous order-preserving maps that reflect the apartness relation are in one-to-one correspondence with lattice homomorphisms that preserve the strong proximity relation.

Let X_1 *and* X_2 *be Priestley spaces with apartness relation. Then (weakly) separating relations from* X_1 *to* X_2 *are in one-to-one correspondence with (weakly) approximable relations from the dual of* X_1 *to the dual of* X_2*.*

Separating relations:

Definition. *Let* $\langle X_1; \leq_1; \mathcal{T}_1\rangle$ *and* $\langle X_2; \leq_2, \mathcal{T}_2\rangle$ *be Priestley spaces with apartness relations* $\propto_1$ *and* $\propto_2$*, respectively, and let* $\ltimes$ *be a binary relation from* X_1 *to* X_2*. The relation* $\ltimes$ *is called* separating *(or a* separator*) if it is open in* $\mathcal{T}_1 \times \mathcal{T}_2$ *and if, for every* $a, b \in X_1, d, e \in X_2$ *and* $\{d_i \mid 1 \leq i \leq n\} \subseteq X_2$*,*

$$
\begin{aligned}
(\downarrow_1\!\ltimes\!\uparrow_2) \quad & a \geq_1 b \ltimes d \geq_2 e \Longrightarrow a \ltimes e, \\
(\forall\ltimes) \quad & b \ltimes d \Longleftrightarrow (\forall c \in X_1)\ b \propto_1 c \text{ or } c \ltimes d, \\
(\ltimes\forall) \quad & b \ltimes d \Longleftrightarrow (\forall c \in X_2)\ b \ltimes c \text{ or } c \propto_2 d, \\
(\ltimes n\uparrow) \quad & b \ltimes \bigcap {\downarrow} d_i \Longrightarrow (\forall c \in X_1)\ b \propto_1 c \\
& \quad \text{or } (\exists\ i)\ c \ltimes d_i.
\end{aligned}
$$

The relation $\ltimes$ *is called* weakly separating (or weak separator) *if it satisfies all of the above conditions, but not necessarily* $(\ltimes n\uparrow)$*.*

Priestley and stably compact spaces

What is the direct relationship between the Priestley spaces equipped with apartness relations stably compact spaces?

The answer is the following:

Theorem. *Let* $\langle X; \leq, \mathfrak{T}\rangle$ *be a Priestley space with apartness* $\propto$. *Then* $\langle core(X), \mathfrak{T}'\rangle$, *where*

$$core(X) = \{x \in X \mid \{y \in X \mid x \propto y\} = X \setminus {\downarrow}x\}$$

and

$$\mathfrak{T}' = \{O \cap core(X) \mid O \text{ is an open lower subset of } X\},$$

is a stably compact space.

Moreover, every stably compact space can be obtained in this way and is a retract of a Priestley space with apartness.

Concerning morphisms again ...

We show that continuous maps between stably compact spaces are equivalent to separators between Priestley spaces equipped with apartness.

Thanks for your attention!