Compactness properties of acts over semigroups

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Two partly overlapping communities

Although the objects of study - automata and acts over semigroups - are similar, communities studying them are quite different.
The plan

1. Represent the finite deterministic (FD) automata in algebraic form.

2. Formulate a classification problem of FD automata/input monoids.

3. Generalize the representation to acts over arbitrary monoids.

4. Demonstrate the classification of monoids by example of equational compactness.

5. List some open problems.
A finite (deterministic) automaton:

\[ M = M(Q, \Sigma, \delta, q_0, F), \ \delta : Q \times \Sigma \rightarrow Q, \ q_0 \in Q, \ F \subseteq Q, \ |Q|, |\Sigma| < \infty. \]

Mapping \( \delta \) can be iteratively extended to \( \Sigma^* \), the set of all words over the alphabet \( \Sigma \):

\[ \delta(q, \epsilon) = q, \ \delta(q, wa) = \delta(\delta(q, w), a), \ q \in Q, \ w \in \Sigma^*, \ a \in \Sigma, \ \epsilon \ - \ empty \ word. \]

Language accepted by \( M \):

\[ L = L(M) = \{ w \in \Sigma^* \mid \delta(q_0, w) \in F \}. \]
Action congruences on $\Sigma^*$

Define a binary operation $\bullet$ on $\Sigma^*$: $w_1 \bullet w_2 = w_1 w_2$, $\forall w_1, w_2 \in \Sigma^*$.

$(\Sigma^*, \bullet)$ is a free semigroup with the generating set $\Sigma$ and empty word $\varepsilon$ as the identity element.

Define binary relations $\rho$ and $\sigma$ on $\Sigma^*$:

$$w_1 \rho w_2 \iff \delta(q, w_1) = \delta(q, w_2) \text{ for all } q \in Q,$$

$$w_1 \sigma w_2 \iff \delta(q_0, w_1) = \delta(q_0, w_2).$$

$\rho$ is a congruence and $\sigma$ is a right congruence on $\Sigma^*$ with $\rho \subseteq \sigma$.

Therefore $\Sigma^*/\rho$ is a semigroup $S$ with identity (a monoid).
Assuming that all elements of $Q$ can be reached from $q_0$ we get that $Q$ and the factor set $\Sigma^*/\sigma$ can be identified: $Q = \Sigma^*/\sigma$.

Instead of input alphabet $\Sigma$, we can consider the input monoid $S=\Sigma^*/\rho$.

The automaton $M(Q,\Sigma,\delta,q_0,F)$ can be written in the form

$$M=M(\Sigma^*/\sigma,S,\delta',\varepsilon,L).$$

In the language of mathematics/algebra:

$M$ is a cyclic (generated by a single element $\varepsilon$) act over monoid $S$ together with a distinguished subset $N(=L)$.

Here both $M$ and $S$ are finite.
Back to automata: classification

Any automaton \( M(Q, \Sigma, \delta, q_0, F) \) determines uniquely a congruence \( \rho \) on \( \Sigma^* \) by \( w_1 \rho w_2 \iff \delta(q, w_1) = \delta(q, w_2) \) for all \( q \in Q \):

\[
M \sqsupseteq \rho = \rho(M).
\]

Consider the class \( \textbf{Aut-S} \), \( S = \Sigma^*/\rho \), of all automata that have \( S \) as its input monoid; assign \( M \sqsupseteq \textbf{Aut-S} \).

In fact \( \textbf{Aut-S} = \{X \mid \rho \subseteq \rho(X)\} \).

A classification problem: having fixed a monoid \( S \), describe the elements of \( \textbf{Aut-S} \) (find characterizing properties of automata from \( \textbf{Aut-S} \)).

The opposite problem: having fixed a property of automata in \( \textbf{Aut-S} \), find out what properties should have the input monoid \( S \) of these automata.
Generalization: an act over a monoid

**Definition.** Let $S$ be an arbitrary monoid and let $M$ be a set. $S$ is called a *right action on* $M$ (equivalently, $M$ is a right $S$-act) if there exists a mapping $M \times S \rightarrow M$ such that $m \varepsilon = m$ and $m(st) = (ms)t$ for all $m \in M$ and $s, t \in S$.

Differences from classical automata case:

1. $S$ is not necessarily finite nor finitely generated: the input alphabet of corresponding automaton can be infinite.
2. $M$ is not necessarily finite nor finitely generated: the corresponding automaton can have more than one initial state - generators of $M$.

Any subset of $M$ of elements having certain property can be considered as a set of final states.
Some typical problems


Homological characterization of monoids: having fixed a certain property $P$ of $S$-acts, find conditions for monoid $S$ so that all $S$-acts would have the property $P$. Example: equational compactness.

Decomposition of $S$-acts: find conditions under which an $S$-act that has certain property $P$, can be presented as a composition of “smaller” $S$-acts each of which has property $P$. 
Finite versus infinite


A possible solution: compactness properties.

A property is called *a compactness property* if from the fact that certain conditions are fulfilled for arbitrary finite subsets of certain set, it follows that these conditions are fulfilled for the whole set.

Examples: equational compactness and its generalizations, congruence compactness.
Equational compactness

(Polynomial) equations on a right $S$-act $M$:

$$xs = yt, \; xs = xt, \; xs = a,$$
where $s, t \in S, \; a \in M, \; x, y$ are variables.

Definition. An $S$-act $M$ is called *equationally compact* if every system of polynomial equations (with constants from $M$) has a solution in $M$ provided that every finite subsystem has a solution in $M$.

Example: All finite acts are equationally compact.

Negative example: $M$ - a set of natural numbers; $S = (N, \cdot)$ - monoid of natural numbers; $\{x_1 = x_2 \cdot 2, \; x_2 = x_3 \cdot 2, \; x_3 = x_4 \cdot 2, \; \ldots\}$ is not solvable.
Generalizations of equational compactness

<table>
<thead>
<tr>
<th>Type of system of equations</th>
<th>Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite number of variables</td>
<td><em>f</em>-equational compactness</td>
</tr>
<tr>
<td>Only one variable</td>
<td><em>1</em>-equational compactness</td>
</tr>
<tr>
<td>No constants</td>
<td><em>Weak</em> equational compactness</td>
</tr>
<tr>
<td>Every maximal connected subsystem contains constants</td>
<td><em>c</em>-equational compactness</td>
</tr>
</tbody>
</table>

Example: \( M \) - a set of natural numbers; \( S = (\mathbb{N}, \cdot) \) - monoid of natural numbers; \( M \) is both *f*-equationally compact and *1*-equationally compact.
Injectivity

An S-act $M$ is called injective if for any homomorphism $\varphi:X \to M$ and any monomorphism $\iota:X \to Y$ there exists a unique homomorphism $\psi:Y \to M$ such that the following diagram is commutative.

Non-formal description of injectivity: if $M$ simulates an act $X$ then it simulates any bigger act containing $X$ as well.

**Theorem.** An act is equationally compact $\iff$ it is injective in relation to pure embeddings.
Homological description of monoids

**Definition.** A monoid $S$ is called right absolutely $f$-equationally compact if all right $S$-acts are $f$-equationally compact.

**Theorem.** The following conditions on a monoid $S$ are equivalent:
1) $S$ is right absolutely $f$-equationally compact;
2) $S$ is right absolutely 1-equationally compact;
3) All right ideals of $S$ are finitely generated and for every right congruence $\rho$ of $S$ and every finite set $\{s_1, \ldots, s_k\} \subseteq S$ there exist an element $u \in S$ and a finitely generated subcongruence $\psi \subseteq \rho$ such that $(s_i, su) \in \rho$ for every $i$ and $\rho \subseteq \psi'^u \cup \rho'^u | J$, where $J$ is the right ideal of $S$ generated by the set $\{s_1, \ldots, s_k\}$.

Here $s \psi'^u \iff su \psi tu$. 
Dependences between properties

Injectivity

Eq-Comp

f-Eq-Comp ↓

1-Eq-Comp ↑

W-Eq-Comp

c-Eq-Comp

W1-Eq-Comp
Some open problems

1. Description of classes $AEq$-Comp, $Ac-Eq-Com$ and $AW-Eq-Comp$.

2. Description of monoids over which two subclasses of absolutely injective monoids coincide.

3. Find conditions for an equationally compact act to be congruence compact or *vice versa*.

4. Description of (commutative) monoids over which all congruence compact acts are equationally compact.

5. Description of self-equationally compact monoids.