Compactness properties of acts over semigroups

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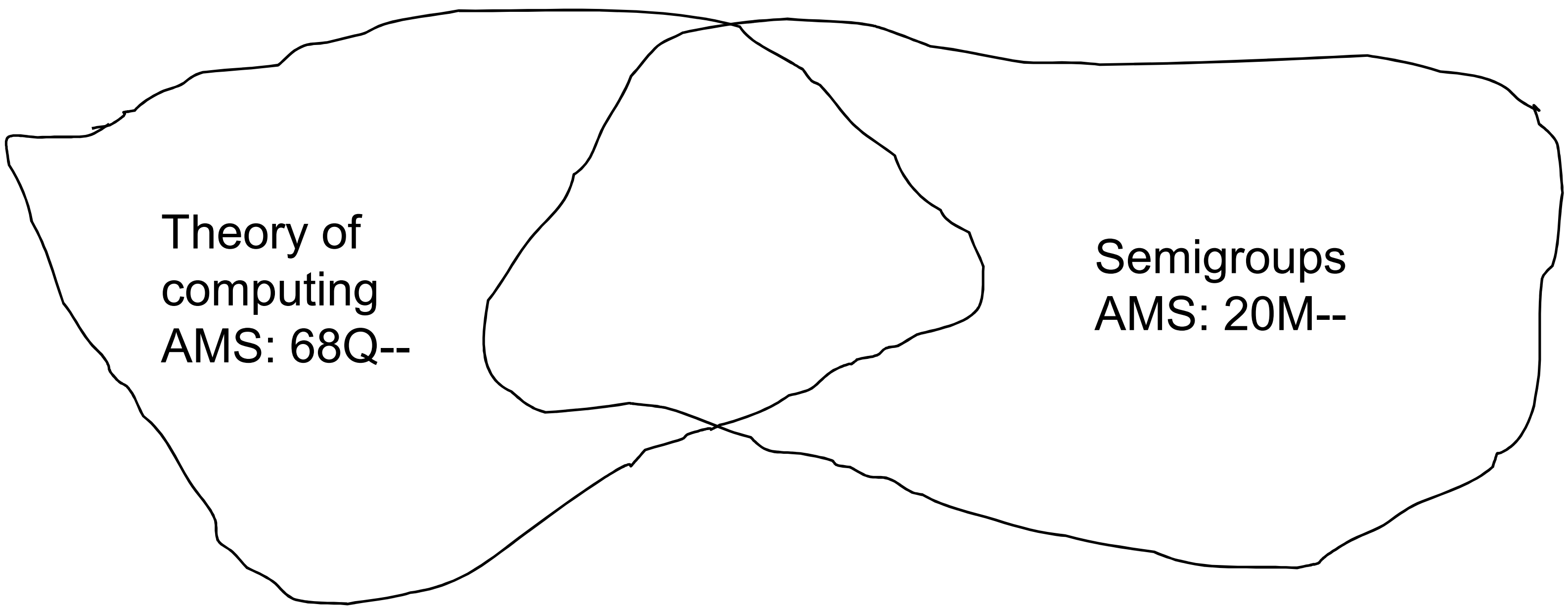
Two partly overlapping communities

Theoretical computer science Discrete mathematics/algebra



Although the objects of study - automata and acts over semigrous - are similar, communities studying them are quite different.

The plan

1. Represent the finite deterministic (FD) automata in algebraic form.
2. Formulate a classification problem of FD automata/input monoids.
3. Generalize the representation to acts over arbitrary monoids.
4. Demonstrate the classification of monoids by example of equational compactness.
5. List some open problems.

Automata

A finite (deterministic) automaton:

$$M=M(Q,\Sigma,\delta,q_0,F),\ \delta:Q\times\Sigma\rightarrow Q,\ q_0\in Q,\ F\subseteq Q,\ |Q|,|\Sigma|<\infty.$$

Mapping δ can be iteratively extended to Σ^*, the set of all words over the alphabet Σ:

$$\delta(q,\varepsilon)=q,\ \delta(q,wa)=\delta(\delta(q,w),a),\ q\in Q,\ w\in\Sigma^*,\ a\in\Sigma,\ \varepsilon \text{ - empty word.}$$

Language accepted by M:

$$L = L(M) = \{w\in \Sigma^* \mid \delta(q_0,w)\in F\}.$$

Action congruences on Σ^*

Define a binary operation $\bullet$ on Σ^*: $w_1 \bullet w_2 = w_1 w_2$, $\forall w_1, w_2 \in \Sigma^*$.

$(\Sigma^*, \bullet)$ is a free semigroup with the generating set Σ and empty word ε as the identity element.

Define binary relations ρ and σ on Σ^*:

$$w_1 \rho w_2 \Leftrightarrow \delta(q, w_1) = \delta(q, w_2) \text{ for all } q \in Q,$$

$$w_1 \sigma\, w_2 \Leftrightarrow \delta(q_0, w_1) = \delta(q_0, w_2).$$

ρ is a congruence and σ is a right congruence on Σ^* with $\rho \subseteq \sigma$.

Therefore Σ^*/ρ is a semigroup S with identity (a monoid).

Algebraic notation

Assuming that all elements of Q can be reached from q_0 we get that Q and the factor set Σ^*/σ can be identified: $Q = \Sigma^*/\sigma$.

Instead of input alphabet Σ, we can consider the input monoid $S=\Sigma^*/\rho$.

The automaton $M(Q,\Sigma,\delta,q_0,F)$ can be written in the form

$$M=M(\Sigma^*/\sigma,S,\delta',\underline{\varepsilon},\underline{L}).$$

In the language of mathematics/algebra:

M is a cyclic (generated by a single element $\underline{\varepsilon}$) act over monoid S together with a distinguished subset $N(=\underline{L})$.

Here both M and S are finite.

Back to automata: classification

Any automaton $M(Q,\Sigma,\delta,q_0,F)$ determines uniquely a congruence ρ on Σ^* by $w_1 \rho w_2 \Leftrightarrow \delta(q,w_1) = \delta(q,w_2)$ for all $q \in Q$:

$$M \mapsto \rho = \rho(M).$$

Consider the class **Aut-S,** $S=\Sigma^*/\rho$, of all automata that have S as its input monoid; assign $M \mapsto$ **Aut-S**.

$$\text{In fact } \mathbf{Aut\text{-}S} = \{X \mid \rho \subseteq \rho(X)\}.$$

A classification problem: having fixed a monoid S, describe the elements of **Aut-S** (find characterizing properties of automata from **Aut-S**).

The opposite problem: having fixed a property of automata in **Aut-S**, find out what properties should have the input monoid S of these automata.

Generalization: an act over a monoid

Definition. Let S be an arbitrary monoid and let M be a set. S is called a *right action on* M (equivalently, M is a right S-act) if there exists a mapping $M \times S \to M$ such that $m\varepsilon = m$ and $m(st) = (ms)t$ for all $m \in M$ and $s,t \in S$.

Differences from classical automata case:

1. S is not necessarily finite nor finitely generated: the input alphabet of corresponding automaton can be infinite.
2. M is not necessarily finite nor finitely generated: the corresponding automaton can have more than one initial state - generators of M.

Any subset of M of elements having certain property can be considered as a set of final states.

Some typical problems

Description of the structure of S-acts: find conditions for an S-act for having a certain property. Example: equationally compact act.

Homological characterization of monoids: having fixed a certain property P of S-acts, find conditions for monoid S so that all S-acts would have the property P. Example: equational compactness.

Decomposition of S-acts: find conditions under which an S-act that has certain property P, can be presented as a composition of “smaller” S-acts each of which has property P.

Finite *versus* infinite

An attempt in 1980-ies in Estonia to marry theory and practice.

A possible solution: compactness properties.

A property is called *a compactness property* if from the fact that certain conditions are fulfilled for arbitrary finite subsets of certain set, it follows that these conditions are fulfilled for the whole set.

Examples: equational compactness and its generalizations, congruence compactness.

Equational compactness

(Polynomial) equations on a right S-act M:

$xs = yt$, $xs = xt$, $xs = a$, where $s,t \in S$, $a \in M$, x,y are variables.

Definition. An S-act M is called *equationally compact* if every system of polynomial equations (with constants from M) has a solution in M provided that every finite subsystem has a solution in M.

Example: All finite acts are equationally compact.

Negative example: M - a set of natural numbers; $S = (N,\bullet)$ - monoid of natural numbers; $\{x_1=x_2\bullet 2,\ x_2=x_3\bullet 2,\ x_3=x_4\bullet 2,\ \ldots\}$ is not solvable.

Generalizations of equational compactness

Type of system of equations	**Term**
Finite number of variables	*f-equational compactness*
Only one variable	*1-equational compactness*
No constants	*Weak equational compactness*
Every maximal connected subsystem contains constants	*c-equational compactness*

Example: M - a set of natural numbers; S = (N,•) - monoid of natural numbers; M is both f-equationally compact and 1-equationally compact.

Injectivity

An S-act M is called injective if for any homomorphism $\varphi: X \to M$ and any monomorphism $\iota: X \to Y$ there exists a unique homomorphism $\psi: Y \to M$ such that the following diagram

$$\begin{tikzcd} & M \\ Y \arrow[ur, "\psi"] & X \arrow[l, "\iota"'] \arrow[u, "\varphi"'] \end{tikzcd}$$

is commutative.

Non-formal description of injectivity: if M simulates an act X then it simulates any bigger act containing X as well.

Theorem. An act is equationally compact $\Leftrightarrow$ it is injective in relation to pure embeddings.

Homological description of monoids

Definition. A monoid S is called right *absolutely f-equationally compact* if all right S-acts are f-equationally compact.

Theorem. The following conditions on a monoid S are equivalent:

1) S is right absolutely f-equationally compact;

2) S is right absolutely 1-equationally compact;

3) All right ideals of S are finitely generated and for every right congruence ρ of S and every finite set $\{s_1,\dots, s_k\}\subseteq S$ there exist an element $u\in S$ and a finitely generated subcongruence $\psi\subseteq\rho$ such that $(s_i,s_iu)\in\rho$ for every i and $\rho\subseteq\psi^u\cup\rho^u|_J$, where J is the right ideal of S generated by the set $\{s_1,\dots,s_k\}$.

Here $s\psi^u t \Leftrightarrow su\psi tu$.

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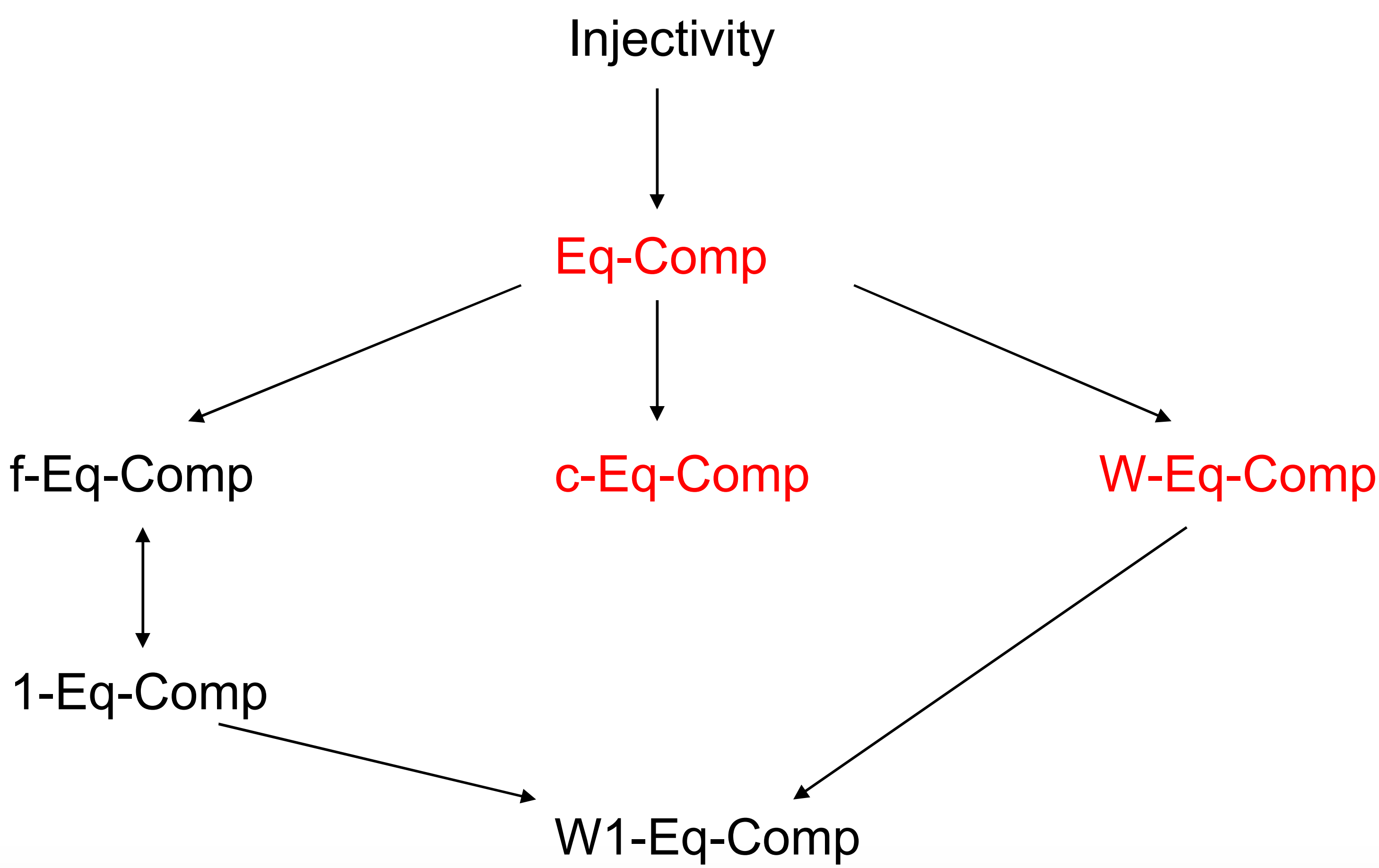
Dependences between properties



Some open problems

1. Description of classes *AEq-Comp*, *Ac-Eq-Com* and *AW-Eq-Comp*.

2. Description of monoids over which two subclasses of absolutely injective monoids coincide.

3. Find conditions for an equationally compact act to be congruence compact or *vice versa*.

4. Description of (commutative) monoids over which all congruence compact acts are equationally compact.

5. Description of self-equationally compact monoids.