A Judgmental Formulation of Modal Logic

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Estonian Theory Days

Jan 30, 2009

Outline

- **Study of logic**
  - **Model theory vs Proof theory**
  - **Classical logic vs Constructive logic**
- Judgmental analysis of propositional logic
- Modal logic
- Summary

Model vs. Proof

Model theory

- Model I
  - Eg. assignment of truth values
- Semantic consequence
  $A_1, \cdots, A_n \models_I C$
  $A_1, \cdots, A_n \models C$

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Proof theory

- Inference rules
  - use premises to obtain the conclusion
- Syntactic entailment
  $A_1, \cdots, A_n \vdash C$

$$\frac{A \quad B}{A \wedge B} \wedge I$$

$$\frac{A \wedge B}{A} \wedge E_L \qquad \frac{A \wedge B}{B} \wedge E_R$$

Classical Logic

- Every proposition is either true or false.
- Concerned with:
  - "whether a given proposition is true or not."
- Tautologies in classical logic

$$A \vee \neg A$$
$$\neg\neg A \supset A$$
$$((A \supset B) \supset A) \supset A$$

Constructive Logic

- We know only what we can prove.
- Concerned with:
  - "how a given proposition becomes true."
- Not provable in constructive logic

$$A \vee \neg A$$
$$\neg\neg A \supset A$$
$$((A \supset B) \supset A) \supset A$$

This talk is about Constructive Proof Theory.

- Per Martin-Löf. On the meaning of the logical constants and the justifications of the logical laws, Nordic Journal of Philosophical Logic, 1(1):11-60, 1996.
- Frank Pfenning and Rowan Davies. A judgmental reconstruction of modal logic, Mathematical Structures in Computer Science, 11(4)-511-540, 2001.

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Judgments and Proofs

- A judgment = an object of knowledge that may or may not be **provable**.
  - If there exists a proof,
    - the judgment becomes evident.
    - we know the judgment.
- Examples
  - *"1 - 1 is equal to 0"* is true.
  - *"1 + 1 is equal to 0"* is false.
  - *"It is snowing"* is true.
  - *"1 - 1 is equal to 0"* is false.

Inference Rules and Axioms

- A proof consists of applications of inference rules.

$$\frac{J_1 \quad J_2 \quad \cdots \quad J_n}{J} R$$

- J_i are premises ($1 \leq i \leq n$).
- J is a conclusion.
  - "If J_1 through J_n (premises) hold, then J (conclusion) holds."

- If $n = 0$ (no premise), the inference rule is an axiom.

Proposition

- A statement such that we know what counts as a verification of it.
  - If A is a proposition, we know how to check the validity of the proof of its truth.

- Example: *"It is raining."*

$$\frac{\textit{``My coat is wet''}\ \text{is true}}{\textit{``It is raining''}\ \text{is true}}\ \mathsf{Rain}$$

- Secondary notion

Proposition

- Without arithmetic rules, what is the meaning of

  "1 - 1 is equal to 0"?

$$\frac{\text{“m } \textit{is equal to} \text{ l” is true} \qquad \text{“l } \textit{is equal to} \text{ m” is true}}{\text{“m } \textit{is equal to} \text{ n” is true}}\ \text{Trans}$$

$$\frac{\text{“m } \textit{is equal to} \text{ n” is true}}{\text{“m-1 } \textit{is equal to} \text{ n-1” is true}}\ \text{Pred}$$

Propositional Logic

- Propositions

$$A \ ::= \ P \mid A \wedge A \mid A \supset A \mid A \vee A \mid \top \mid \bot$$

- Judgments:

$$A \ true \qquad \Longleftrightarrow \qquad A \text{ is true}$$

Natural Deduction System

- Introduced by Gentzen, 1934
- For each connective $\wedge$, $\vee$, $\supset$, ...
  - introduction rule: how to establish a proof
  - elimination rule: how to exploit an existing proof

$$\frac{A \ \ \mathit{true} \qquad B \ \ \mathit{true}}{A \wedge B \ \ \mathit{true}} \wedge\mathrm{I}$$

$$\frac{A \wedge B \ \ \mathit{true}}{A \ \ \mathit{true}} \wedge\mathrm{E_L} \qquad \frac{A \wedge B \ \ \mathit{true}}{B \ \ \mathit{true}} \wedge\mathrm{E_R}$$

Implication

$$\dfrac{\begin{array}{c}\overline{A\ \ true}^{\,x} \\ \vdots \\ B\ \ true\end{array}}{A \supset B\ \ true}\ \supset\!\mathsf{I}^{x}$$

$$\dfrac{A \supset B\ \ true \qquad A\ \ true}{B\ \ true}\ \supset\!\mathsf{E}$$

Disjunction

$$\frac{A \; \mathit{true}}{A \vee B \; \mathit{true}} \vee I_L \qquad \frac{B \; \mathit{true}}{A \vee B \; \mathit{true}} \vee I_R$$

$$\frac{A \vee B \; \mathit{true} \qquad \begin{array}{c} \overline{A \; \mathit{true}}^{\,x} \\ \vdots \\ C \; \mathit{true} \end{array} \qquad \begin{array}{c} \overline{B \; \mathit{true}}^{\,y} \\ \vdots \\ C \; \mathit{true} \end{array}}{C \; \mathit{true}} \vee E^{x,y}$$

Truth and Falsehood

$$\frac{}{\top \;\; true}\;\top I$$

$$\frac{\bot \;\; true}{C \;\; true}\;\bot E$$

What if Elimination Rules were …

- Too strong

$$\frac{A\ \ true \qquad B\ \ true}{A \wedge B\ \ true}\ \wedge I$$

$$\frac{A \wedge B\ \ true}{\bot\ \ true}\ ???$$

- Too weak

$$\frac{A\ \ true \qquad B\ \ true}{A \wedge B\ \ true}\ \wedge I$$

$$\frac{A \wedge B\ \ true}{\top\ \ true}\ ???$$

Elimination Rules are OK

- Local soundness
  - Elimination rules are not too strong.

$$\dfrac{\dfrac{\overset{\mathcal{D}}{A\ \mathit{true}} \quad \overset{\mathcal{E}}{B\ \mathit{true}}}{A \wedge B\ \mathit{true}}\wedge I}{A\ \mathit{true}}\wedge E_L \quad \Longrightarrow_R \quad \overset{\mathcal{D}}{A\ \mathit{true}}$$

- Local completeness
  - Elimination rules are not too weak

$$\overset{\mathcal{D}}{A \wedge B\ \mathit{true}} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\overset{\mathcal{D}}{A \wedge B\ \mathit{true}}}{A\ \mathit{true}}\wedge E_L \quad \dfrac{\overset{\mathcal{D}}{A \wedge B\ \mathit{true}}}{B\ \mathit{true}}\wedge E_R}{A \wedge B\ \mathit{true}}\wedge I$$

Local Soundness and Completeness

$$
\dfrac{\dfrac{\begin{array}{c}\overline{A\ \ true}^{\,x}\\ \vdots\\ B\ \ true\end{array}}{A \supset B\ \ true}\supset\!\mathsf{I}^x \qquad \begin{array}{c}\mathcal{D}\\ A\ \ true\end{array}}{B\ \ true}\supset\!\mathsf{E}
\quad \Longrightarrow_R \quad
\begin{array}{c}\mathcal{D}\\ A\ \ true\\ \vdots\\ B\ \ true\end{array}
$$

$$
\begin{array}{c}\mathcal{D}\\ A \supset B\ \ true\end{array}
\quad \Longrightarrow_E \quad
\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \supset B\ \ true\end{array} \qquad \overline{A\ \ true}^{\,x}}{B\ \ true}\supset\!\mathsf{E}}{A \supset B\ \ true}\supset\!\mathsf{I}^x
$$

Hypothetical Judgments

- Definition

$$A_1 \ \mathit{true}, \cdots, A_n \ \mathit{true} \vdash C \ \mathit{true}$$

$\Longleftrightarrow$ $C \ \mathit{true}$ is provable under assumptions

$A_1 \ \mathit{true}, \cdots, A_n \ \mathit{true}$.

- Substitution principle

If $\Gamma \vdash A \ \mathit{true}$ and $\Gamma, A \ \mathit{true} \vdash J$, then $\Gamma \vdash J$.

Inference Rules

$$\frac{A\ \mathit{true} \in \Gamma}{\Gamma \vdash A\ \mathit{true}}\ \mathsf{Hyp}$$

$$\frac{\Gamma, A\ \mathit{true} \vdash B\ \mathit{true}}{\Gamma \vdash A \supset B\ \mathit{true}} \supset\mathsf{I} \qquad \frac{\Gamma \vdash A \supset B\ \mathit{true} \quad \Gamma \vdash A\ \mathit{true}}{\Gamma \vdash B\ \mathit{true}} \supset\mathsf{E}$$

$$\frac{\Gamma \vdash A\ \mathit{true} \quad \Gamma \vdash B\ \mathit{true}}{\Gamma \vdash A \wedge B\ \mathit{true}} \wedge\mathsf{I} \qquad \frac{\Gamma \vdash A \wedge B\ \mathit{true}}{\Gamma \vdash A\ \mathit{true}} \wedge\mathsf{E_L} \qquad \frac{\Gamma \vdash A \wedge B\ \mathit{true}}{\Gamma \vdash B\ \mathit{true}} \wedge\mathsf{E_R}$$

$$\frac{\Gamma \vdash A\ \mathit{true}}{\Gamma \vdash A \vee B\ \mathit{true}} \vee\mathsf{I_L} \qquad \frac{\Gamma \vdash B\ \mathit{true}}{\Gamma \vdash A \vee B\ \mathit{true}} \vee\mathsf{I_R}$$

$$\frac{\Gamma \vdash A \vee B\ \mathit{true} \quad \Gamma, A\ \mathit{true} \vdash C\ \mathit{true} \quad \Gamma, B\ \mathit{true} \vdash C\ \mathit{true}}{\Gamma \vdash C\ \mathit{true}} \vee\mathsf{E}$$

$$\frac{}{\Gamma \vdash \top\ \mathit{true}} \top\mathsf{I} \qquad \frac{\Gamma \vdash \bot\ \mathit{true}}{\Gamma \vdash C\ \mathit{true}} \bot\mathsf{E}$$

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- Study of logic
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- **Judgmental analysis of modal logic**
  - **Modal necessity □**
  - **Modal possibility ◊**
  - **Lax modality O**
- Summary

POSTECH

Modal Logic

- Modalities $\Box$ and $\Diamond$
  - $\Box A$: necessarily A
  - $\Diamond A$: possibly A
- Spatial interpretation:
  - $\Box A$: everywhere A
  - $\Diamond A$: somewhere A
- Temporal interpretation:
  - $\Box A$: always A
  - $\Diamond A$: sometime A

Modal necessity $\Box$

First Judgments,
Then Propositions.

Validity Judgment

- $A\ valid$

$$A\ \mathit{valid} \quad\Longleftrightarrow\quad \cdot \vdash A\ \mathit{true}$$

$\approx$ A is valid if A is true at a world about which we know nothing, or at any world.

- Modal proposition $\Box A$
  - Introduction rule

$$\frac{A\ \mathit{valid}}{\Box A\ \mathit{true}}\,\Box\mathrm{I}$$

New Forms of Hypothetical Judgments

- Definition

$$B_1 \ valid, \cdots, B_m \ valid; A_1 \ true, \cdots, A_n \ true \vdash C \ true$$

$\Longleftrightarrow C \ true$ is provable under assumptions

$$B_1 \ valid, \cdots, B_m \ valid, A_1 \ true, \cdots, A_n \ true.$$

- Substitution principle

If $\Delta; \Gamma \vdash A \ true$ and $\Delta; \Gamma, A \ true \vdash J$, then $\Delta; \Gamma \vdash J$.

~~If $\Delta; \Gamma \vdash A \ valid$ and $\Delta, A \ valid; \Gamma \vdash J$, then $\Delta; \Gamma \vdash J$.~~

If $\Delta; \cdot \vdash A \ true$ and $\Delta, A \ valid; \Gamma \vdash J$, then $\Delta; \Gamma \vdash J$.

Modal necessity □

$$\frac{\Delta; \cdot \vdash A \ \ true}{\Delta; \Gamma \vdash \Box A \ \ true} \ \Box I$$

$$\frac{\Delta; \Gamma \vdash \Box A \ \ true \qquad \Delta, A \ \ valid; \Gamma \vdash C \ \ true}{\Delta; \Gamma \vdash C \ \ true} \ \Box E$$

Local Soundness and Completeness

$$\dfrac{\dfrac{\Delta; \cdot \overset{\mathcal{D}}{\vdash} A\ \mathit{true}}{\Delta; \Gamma \vdash \Box A\ \mathit{true}}\ \Box I \qquad \Delta, A\ \mathit{valid}; \Gamma \overset{\mathcal{E}}{\vdash} C\ \mathit{true}}{\Delta; \Gamma \vdash C\ \mathit{true}}\ \Box E$$

$$\Longrightarrow_R \qquad \Delta; \Gamma \overset{\mathcal{E}'}{\vdash} C\ \mathit{true}$$

$$\Delta; \Gamma \overset{\mathcal{D}}{\vdash} \Box A\ \mathit{true}$$

$$\Longrightarrow_E \qquad \dfrac{\Delta; \Gamma \overset{\mathcal{D}}{\vdash} \Box A\ \mathit{true} \qquad \dfrac{\overline{\Delta, A\ \mathit{valid}; \cdot \vdash A\ \mathit{true}}\ \mathsf{Hyp}}{\Delta, A\ \mathit{valid}; \Gamma \vdash \Box A\ \mathit{true}}\ \Box I}{\Delta; \Gamma \vdash \Box A\ \mathit{true}}\ \Box E$$

Axiomatic Characterization (S4)

$$\vdash \Box(A \supset B) \supset (\Box A \supset \Box B) \;\; true$$

$$\vdash \Box A \supset A \;\; true$$

$$\vdash \Box A \supset \Box\Box A \;\; true$$

Modal possibility $\Diamond$

Again,
First Judgments,
Then Propositions.

Possibility Judgment

- $A\ poss$

  $\approx A$ is possibly true if A is true at a certain world.

If $\Delta; \Gamma \vdash A\ true$, then $\Delta; \Gamma \vdash A\ poss$.

If $\Delta; \Gamma \vdash A\ poss$ and $\Delta; A\ true \vdash C\ poss$, then $\Delta; \Gamma \vdash C\ poss$.

Modal possibility $\Diamond$

$$\frac{\Delta; \Gamma \vdash A \;\; poss}{\Delta; \Gamma \vdash \Diamond A \;\; true} \; \Diamond I$$

$$\frac{\Delta; \Gamma \vdash \Diamond A \;\; true \qquad \Delta; A \;\; true \vdash C \;\; poss}{\Delta; \Gamma \vdash C \;\; poss} \; \Diamond E$$

Local Soundness and Completeness

$$\dfrac{\dfrac{\overset{\mathcal{D}}{\Delta; \Gamma \vdash A \ poss}}{\Delta; \Gamma \vdash \Diamond A \ true}\,\Diamond\mathrm{I} \qquad \overset{\mathcal{E}}{\Delta; A \ true \vdash C \ poss}}{\Delta; \Gamma \vdash C \ poss}\,\Diamond\mathrm{E}$$

$$\Longrightarrow_R \qquad \overset{\mathcal{E}'}{\Delta; \Gamma \vdash C \ poss}$$

$$\overset{\mathcal{D}}{\Delta; \Gamma \vdash \Diamond A \ true}$$

$$\Longrightarrow_E \qquad \dfrac{\dfrac{\overset{\mathcal{D}}{\Delta; \Gamma \vdash \Diamond A \ true} \qquad \dfrac{}{\Delta; A \ true \vdash A \ true}\,\mathrm{Hyp}}{\Delta; \Gamma \vdash A \ poss}\,\Diamond\mathrm{E}{*}}{\Delta; \Gamma \vdash \Diamond A \ true}\,\Diamond\mathrm{I}$$

Axiomatic Characterization

$$\vdash A \supset \Diamond A \ \ true$$

$$\vdash \Diamond\Diamond A \supset \Diamond A \ \ true$$

$$\vdash \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) \ \ true$$

Lax modality O

Yet again,
First Judgments,
Then Propositions.

Lax Judgment

- $A\ lax$

  $\approx A$ is true under a certain constraint.

If $\Gamma \vdash A\ \ true$, then $\Gamma \vdash A\ \ lax$.

If $\Gamma \vdash A\ \ lax$ and $\Gamma, A\ \ true \vdash C\ \ lax$, then $\Gamma \vdash C\ \ lax$.

Lax Modality O

$$\frac{\Gamma \vdash A \;\; lax}{\Gamma \vdash \bigcirc A \;\; true} \bigcirc\text{I}$$

$$\frac{\Gamma \vdash \bigcirc A \;\; true \qquad \Gamma, A \;\; true \vdash C \;\; lax}{\Gamma \vdash C \;\; lax} \bigcirc\text{E}$$

Local Soundness and Completeness

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash A\ \mathit{lax}\end{array}}{\Gamma \vdash \bigcirc A\ \mathit{true}}\bigcirc\mathsf{I} \qquad \begin{array}{c}\mathcal{E}\\ \Gamma, A\ \mathit{true} \vdash C\ \mathit{lax}\end{array}}{\Gamma \vdash C\ \mathit{lax}}\bigcirc\mathsf{E} \quad \Longrightarrow_R \quad \begin{array}{c}\mathcal{E}'\\ \Gamma \vdash C\ \mathit{lax}\end{array}$$

$$\begin{array}{c}\mathcal{D}\\ \Gamma \vdash \bigcirc A\ \mathit{true}\end{array} \quad \Longrightarrow_E \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \vdash \bigcirc A\ \mathit{true}\end{array} \qquad \dfrac{}{\Gamma, A\ \mathit{true} \vdash A\ \mathit{true}}\mathsf{Hyp}}{\Gamma \vdash A\ \mathit{lax}}\bigcirc\mathsf{E}*}{\Gamma \vdash \bigcirc A\ \mathit{true}}\bigcirc\mathsf{I}$$

Axiomatic Characterization

$$\vdash \; A \supset \bigcirc A \;\; true$$

$$\vdash \; \bigcirc\bigcirc A \supset \bigcirc A \;\; true$$

$$\vdash \; (A \supset B) \supset (\bigcirc A \supset \bigcirc B) \;\; true$$

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Applications

- $\Box$, $\Diamond$
  - Type system for staged computation
  - Type system for distributed computation
- O
  - Type system for effectful computation
  - Monad in functional language Haskell

Internalizing Normal Proofs

- Normal proofs

$$\dfrac{\dfrac{}{A\downarrow}u \quad \vdots \quad B\uparrow}{A \supset B\uparrow} \supset I^u_\uparrow \qquad \dfrac{A \supset B\downarrow \quad A\uparrow}{B\downarrow} \supset E_\downarrow \qquad \dfrac{A\downarrow}{A\uparrow} \downarrow\uparrow \qquad \Bigg\} N^{\uparrow\downarrow}$$

- Internalizing normal proofs using a modality

$$\dfrac{A\uparrow}{\triangle A \ \mathit{true}} \triangle I$$

- Introduction and elimination rules

$$\dfrac{A\uparrow}{\triangle A \ \mathit{true}} \triangle I \qquad \dfrac{\triangle A \ \mathit{true} \quad \begin{array}{c} \overline{A\uparrow}^{v} \\ \vdots \\ B \ \mathit{true} \end{array}}{B \ \mathit{true}} \triangle E^v$$

Sequent Calculus

- Uses two judgments

$$\frac{A \ \ atomic}{\Delta; \Gamma, A \longrightarrow A} Init \qquad \frac{A \ \ atomic}{\Delta; \Gamma, A \Longrightarrow A} Init' \qquad \frac{}{\Delta, A; \Gamma \Longrightarrow A} Sub$$

$$\frac{\Delta, A; \Gamma, \triangle A \longrightarrow C}{\Delta; \Gamma, \triangle A \longrightarrow C} \triangle L \qquad \frac{\Delta; \cdot \Longrightarrow A}{\Delta; \Gamma \longrightarrow \triangle A} \triangle R$$

$$\frac{\Delta, A; \Gamma, \triangle A \Longrightarrow C}{\Delta; \Gamma, \triangle A \Longrightarrow C} \triangle L' \qquad \frac{\Delta; \cdot \Longrightarrow A}{\Delta; \Gamma \Longrightarrow \triangle A} \triangle R'$$

$$\frac{\Delta; \Gamma, A \supset B \longrightarrow A \qquad \Delta; \Gamma, A \supset B, B \longrightarrow C}{\Delta; \Gamma, A \supset B \longrightarrow C} \supset L \qquad \frac{\Delta; \Gamma, A \longrightarrow B}{\Delta; \Gamma \longrightarrow A \supset B} \supset R$$

$$\frac{\Delta; \Gamma, A \supset B \longrightarrow A \qquad \Delta; \Gamma, A \supset B, B \Longrightarrow C}{\Delta; \Gamma, A \supset B \Longrightarrow C} \supset L' \qquad \frac{\Delta; \Gamma, A \Longrightarrow B}{\Delta; \Gamma \Longrightarrow A \supset B} \supset R'$$

- Satisfies cut-elimination

Thank you.

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