Wreath product of set-valued functors and tensor multiplication

Valdis Laan

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Wreath product of monoids and acts

Def. 1 (Act) Let $\mathbf{A}$ be a monoid. A nonempty set M is called a *left* $\mathbf{A}$-*act* (notation ${}_{\mathbf{A}}M$), if there is a mapping $\mathbf{A} \times M \to M$, $(k, m) \mapsto km$, such that

1. $k_1(k_2m) = (k_1k_2)m$ for every $k_1, k_2 \in \mathbf{A}$ and $m \in M$;

2. $1m = m$ for every $m \in M$.

Def. 2 (WP of monoids) Let $\mathbf{A}, \mathbf{B}$ be monoids and ${}_{\mathbf{B}}N$ a left $\mathbf{B}$-act. On the set $\mathbf{A}^N \times \mathbf{B}$ we define a multiplication by

$$(\Psi, g)(\Phi, f) = ({}^f\Psi * \Phi, gf),$$

$\Phi, \Psi : N \to \mathbf{A}$, $f, g \in \mathbf{B}$, where

$$({}^f\Psi * \Phi)(n) = \Psi(fn)\Phi(n)$$

for every $n \in N$. With this multiplication $\mathbf{A}^N \times \mathbf{B}$ becomes a monoid, which is called the *wreath product* of $\mathbf{A}$ and $\mathbf{B}$ through ${}_{\mathbf{B}}N$ and denoted $\mathbf{A}\,\mathrm{wr}^N\,\mathbf{B}$.

Def. 3 (WP of acts) Let $\mathbf{A}, \mathbf{B}$ be monoids and ${}_{\mathbf{A}}M, {}_{\mathbf{B}}N$ left acts. Then $M \times N$ becomes a left $(\mathbf{A}\,\mathrm{wr}^N\,\mathbf{B})$-act if we define

$$(\Phi, f)(m, n) = (\Phi(n)m, fn),$$

$\Phi : N \to \mathbf{A}$, $f \in \mathbf{B}$, $m \in M, n \in N$. This act is called the *wreath product* of acts ${}_{\mathbf{A}}M$ and ${}_{\mathbf{B}}N$ and is denoted by $M\,\mathrm{wr}\,N$.

Theorem 4 (Normak) *$M\,\mathrm{wr}\,N$ is pullback flat iff ${}_{\mathbf{A}}M$ and ${}_{\mathbf{B}}N$ are pullback flat and*

- *$\mathbf{A}$ is right collapsible, or*
- *$\mathbf{A}$ is left reversible and for all $f_1, f_2 \in \mathbf{B}$, $n \in N$ with $f_1 n = f_2 n$ there exists $g \in \mathbf{B}$ such that $f_1 g = f_2 g$ and $gN = \{n\}$, or*
- *for all $f_1, f_2 \in \mathbf{B}$, $n_1, n_2 \in N$ with $f_1 n_1 = f_2 n_2$ there exist $g_1, g_2 \in \mathbf{B}$ such that $f_1 g_1 = f_2 g_2$ and $g_1 N = \{n_1\}$, $g_2 N = \{n_2\}$.*

Wreath product of categories

Def. 5 (WP of categories) Given small categories $\mathbf{A}$ and $\mathbf{B}$ and a functor $B : \mathbf{B} \to \mathbf{Set}$, the *(discrete) wreath product* $\mathbf{A} \operatorname{wr}^B \mathbf{B}$ is a category defined as follows:

WP1 The objects of $\mathbf{A} \operatorname{wr}^B \mathbf{B}$ are pairs (α, b), where b is an object of $\mathbf{B}$ and $\alpha : B(b) \to \mathrm{Ob}(\mathbf{A})$ is a mapping.

WP2 A morphism $(\Phi, f) : (\alpha, b) \to (\alpha', b')$ of $\mathbf{A} \operatorname{wr}^B \mathbf{B}$ has $f : b \to b'$ a morphism of $\mathbf{B}$ and $\Phi = (\Phi_n)_{n \in B(b)}$ where $\Phi_n : \alpha(n) \to (\alpha' \circ B(f))(n)$ in $\mathbf{A}$.

WP3 If $(\Phi, f) : (\alpha, b) \to (\alpha', b')$ and $(\Psi, g) : (\alpha', b') \to (\alpha'', b'')$ are morphisms of $\mathbf{A} \operatorname{wr}^B \mathbf{B}$, then

$$(\Psi, g) \circ (\Phi, f) = ({}^f\Psi * \Phi, g \circ f) : (\alpha, b) \to (\alpha'', b''),$$

where

$$({}^f\Psi * \Phi)_n = \Psi_{B(f)(n)} \circ \Phi_n.$$

for every $n \in B(b)$.

Wreath product of Set-valued functors

Def. 6 (WP of functors) Given small categories $\mathbf{A}$ and $\mathbf{B}$ and functors $A : \mathbf{A} \to \textbf{Set}$ and $B : \mathbf{B} \to \textbf{Set}$, the *wreath product* $A \,\mathsf{wr}\, B$ is a functor $\mathbf{A} \,\mathsf{wr}^B\, \mathbf{B} \to \textbf{Set}$, defined as follows:

WF1 For an object (α, b) of $\mathbf{A} \,\mathsf{wr}^B\, \mathbf{B}$,
$(A \,\mathsf{wr}\, B)(\alpha, b) = \{(l, n) \mid n \in B(b), l \in A(\alpha(n))\}$.

WF2 If $(\Phi, f) : (\alpha, b) \to (\alpha', b')$ is a morphism of $\mathbf{A} \,\mathsf{wr}^B\, \mathbf{B}$ and $(l, n) \in (A \,\mathsf{wr}\, B)(\alpha, b)$ then

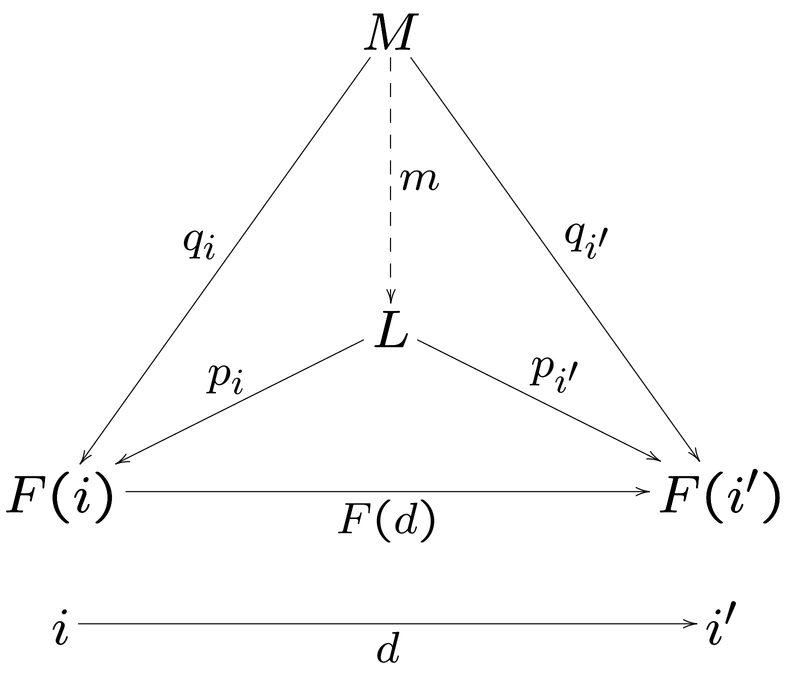
$$(A \,\mathsf{wr}\, B)(\Phi, f)(l, n) = (A(\Phi_n)(l), B(f)(n)).$$

$$\begin{array}{ccccc}
(\alpha, b) & \longmapsto & (A \,\mathsf{wr}\, B)(\alpha, b) & \ni & (l, n) \\
\Big\downarrow{\scriptstyle (\Phi, f)} & & \Big\downarrow{\scriptstyle (A \,\mathsf{wr}\, B)(\Phi, f)} & & \Big\downarrow \\
(\alpha', b') & \longmapsto & (A \,\mathsf{wr}\, B)(\alpha', b') & \ni & (A(\Phi_n)(l), B(f)(n))
\end{array}$$

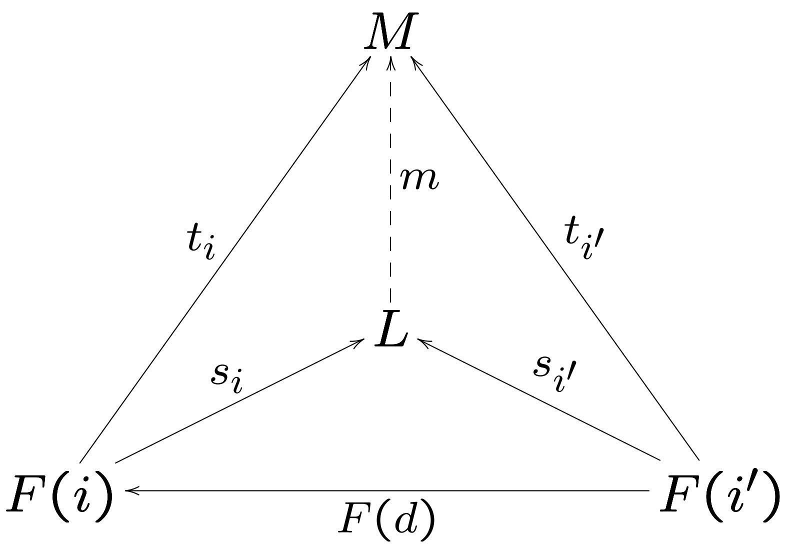
Limits and colimits

Let $F : \mathbf{D} \to \mathbf{A}$ be a functor and denote $I = \mathrm{Ob}(\mathbf{D})$.

$(L, (p_i)_{i \in I}) = \lim F$:



$(L, (s_i)_{i \in I}) = \mathrm{colim}\, F$:



Lemma 7 *If* $\mathbf{D}$ *is a small category,* $I = \mathrm{Ob}(\mathbf{D})$ *and* $F : \mathbf{D} \to \mathbf{Set}$ *is a functor then*

$\lim F = \{(x_i)_{i \in I} \mid x_i \in F(i), \forall d : j \to i$ *in* $\mathbf{D}$ $F(d)(x_j) = x_i\}$,

with the obvious projections.

A *zig-zag* connecting objects c and c' in a category $\mathbf{C}$:

$$c \xrightarrow{f_1} b_1 \xleftarrow{g_1} a_1 \xrightarrow{f_2} b_2 \xleftarrow{g_2} \ldots \xrightarrow{f_n} b_n \xleftarrow{g_n} c'.$$

If there is a zig-zag connecting two objects, we say that these objects are *connected*. Connectedness is an equivalence relation on the set of objects of a small category $\mathbf{C}$, we denote it by $\sim$ and the equivalence class of an object c by $[c]$.

Lemma 8 *If* $\mathbf{C}$ *is a small category and* $F : \mathbf{C} \to \mathbf{Set}$ *is a functor then*

$$\mathrm{colim}\, F = \mathrm{Ob}(\mathrm{el}(F))/\sim,$$

where the injections $s_c : F(c) \to \mathrm{colim}\, F$, $c \in \mathrm{Ob}(\mathbf{C})$, *are defined by*

$$s_c(x) = [(c, x)],$$

where $x \in F(c)$ *and* $[(c, x)]$ *is the equivalence class of* $(c, x) \in \mathrm{Ob}(\mathrm{el}(F))$ *by* $\sim$.

Preservation of limits

Let $(L, (p_i)_{i \in I})$ be the limit of a functor $F : \mathbf{D} \to \mathbf{A}$, where $I = \mathsf{Ob}(\mathbf{D})$. A functor $G : \mathbf{A} \to \mathsf{Set}$ *preserves* it if $(G(L), (G(p_i))_{i \in I})$ is the limit of GF.

Category of elements of a functor

Consider a functor $J : \mathbf{C} \to \mathsf{Set}$. The *category of elements* of J (denoted by $\mathsf{el}(J)$) has:

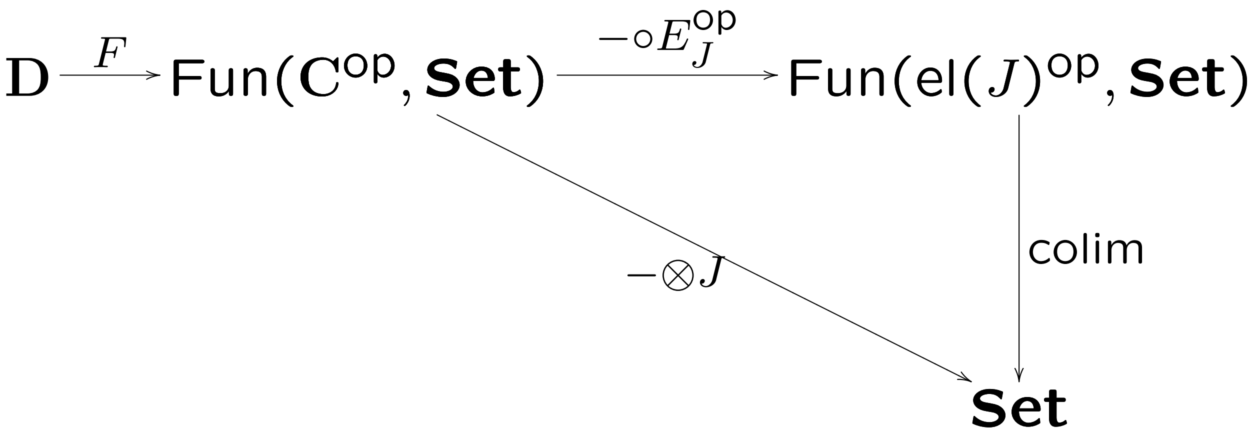
- objects: pairs (c, x), $c \in \mathsf{Ob}(\mathbf{C})$, $x \in J(c)$,
- morphisms $(c, x) \longrightarrow (c', x')$ are $\mathbf{C}$-morphisms $f : c \to c'$ such that $J(f)(x) = x'$.

There is a forgetful functor $E_J : \mathsf{el}(J) \to \mathbf{C}$,

$$E_J(c, x) = c, \;\; E_J(f) = f,$$

and a functor $E_J^{\mathsf{op}} : \mathsf{el}(J)^{\mathsf{op}} \to \mathbf{C}^{\mathsf{op}}$.

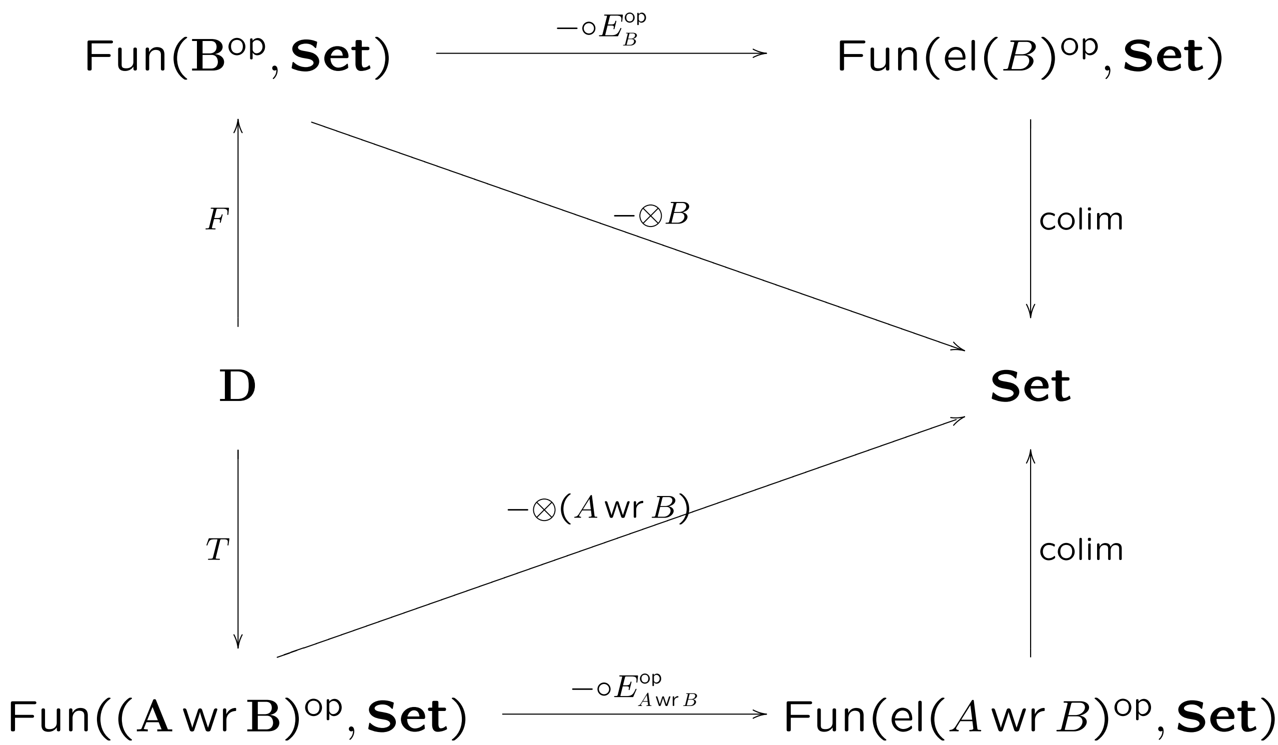
Tensor products



We are interested in the situation where $\mathbf{C}$ is small, $\mathbf{A} = \mathsf{Fun}(\mathbf{C}^{\mathsf{op}}, \mathsf{Set})$, $J : \mathbf{C} \to \mathbf{Set}$, and

$$G = - \otimes J = \mathsf{colim} \circ (- \circ E_J) : \mathsf{Fun}(\mathbf{C}^{\mathsf{op}}, \mathsf{Set}) \longrightarrow \mathsf{Set}$$

is the functor of tensor multiplication by J.



Results

Let $\mathbf{D}$ be a small category.

Theorem 9 *If the functor* $- \otimes (A \,\mathsf{wr}\, B)$ *preserves* $\mathbf{D}$*-limits, then the functor* $- \otimes B$ *preserves* $\mathbf{D}$*-limits.*

Theorem 10

1. *If the functor* $- \otimes (A \,\mathsf{wr}\, B)$ *preserves* $\mathbf{D}$*-limits of representables, then the functor* $- \otimes A$ *preserves* $\mathbf{D}$*-limits of representables.*

2. *If the functor* $- \otimes (A \,\mathsf{wr}\, B)$ *preserves* $\mathbf{D}$*-limits of representables, then the functor* $- \otimes B$ *preserves* $\mathbf{D}$*-limits of representables.*

If $a \in \mathsf{Ob}(\mathbf{A})$ and $b \in \mathsf{Ob}(\mathbf{B})$, then $\delta_a^b : B(b) \to \mathsf{Ob}(\mathbf{A})$ denotes the constant mapping on a.
If $b \in \mathsf{Ob}(\mathbf{B})$ and $k : a \to a'$ is a morphism in $\mathbf{A}$, then denoting

$$\Gamma^k = (k)_{n \in B(b)}$$

we have $(\Gamma^k, 1_b) : (\delta_a^b, b) \to (\delta_{a'}^b, b)$ in $\mathbf{A} \,\mathsf{wr}^B\, \mathbf{B}$.

(*) For every functor $T : \mathbf{D} \to \mathsf{Fun}((\mathbf{A} \,\mathsf{wr}^B\, \mathbf{B})^{\mathsf{op}}, \mathbf{Set})$, every morphism $(\Lambda, f) : (\delta_a^b, b) \to (\delta_a^{b'}, b')$ in $\mathbf{A} \,\mathsf{wr}^B\, \mathbf{B}$ (that is, $f : b \to b'$ in $\mathbf{B}$ and $\Lambda = (\Lambda_n)_{n \in B(b)}$ where $\Lambda_n : a \to a$ for every $n \in B(b)$)
and every $i \in I = \mathsf{Ob}(\mathbf{D})$

$$T_i\left((\Lambda, f)^{\mathsf{op}}\right) = T_i\left((\Gamma^{1_a}, f)^{\mathsf{op}}\right).$$

(**) For every functor $T : \mathbf{D} \to \mathsf{Fun}((\mathbf{A} \,\mathsf{wr}^B\, \mathbf{B})^{\mathsf{op}}, \mathbf{Set})$, every morphism $k : a \to a$ in $\mathbf{A}$
and every object $b \in \mathsf{Ob}(\mathbf{B})$

$$T_i\left((\Gamma^k, 1_b)^{\mathsf{op}}\right) = 1_{T_i(\delta_a^b, b)}.$$

Theorem 11 *Suppose that* $\mathsf{Ob}(\mathbf{A}) = \{a\}$ *and* $\mathbf{A} \,\mathsf{wr}^B\, \mathbf{B}$ *satisfies (*) and (**). If* $- \otimes A$ *and* $- \otimes B$ *preserve* $\mathbf{D}$*-limits, then* $- \otimes (A \,\mathsf{wr}\, B)$ *preserves* $\mathbf{D}$*-limits.*

References

1 G. Kelly, *On clubs and doctrines*, in Proceedings of Sydney Category Theory Seminar 1972/1973, G. Kelly, editor, volume 420 of Lecture Notes in Mathematics, Springer-Verlag.

2 M. Kilp, U. Knauer, A. V. Mikhalev, *Monoids, Acts and Categories*, Walter de Gruyter, Berlin New York, 2000.

3 P. Normak, *Strong flatness and projectivity of the wreath product of acts,* in Abelevy Gruppy i Moduli, Tomsk, 1982.

4 M. Barr, Ch. Wells, *Electronic supplement to Category Theory for Computing Science*, http://www.cwru.edu/artsci/math/wells/pub /papers.html