A Colorful Introduction to Cellular Automata

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Overview

- Cellular automata (CA) are local presentations of global dynamics
- They are powerful tools for qualitative analysis
- They display several interesting theoretical features
- We will set some of them in action

History of cellular automata

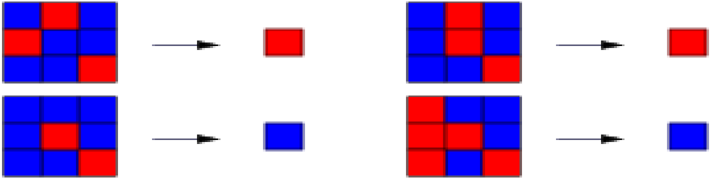
- von Neumann, 1950s:
  mechanical model of self-reproduction
- Moore and Myhill, 1962:
  the Garden of Eden problem
- Hedlund, 1969:
  shift dynamical systems
- Hardy, de Pazzis, Pomeau 1976:
  lattice gas automata
- Amoroso and Patt, 1972; Kari, 1990:
  the invertibility problem
- Machì and Mignosi, 1993:
  cellular automata on Cayley graphs

Life is a Game

Ideated by John Horton Conway (1960s) popularized by Martin Gardner.
The checkboard is an infinite square grid.
Each case (cell) of the checkboard is "surrounded" by those within a chess' king's move, and can be "living" or "dead".

1. A dead cell surrounded by exactly three living cells, becomes living.
2. A living cell surrounded by two or three living cells, survives.
3. A living cell surrounded by less than two living cells, dies of isolation.
4. A living cell surrounded by more than three living cells, dies of overpopulation.

Game of Life situations



Cellular automata

Conway's Game of Life is an example of cellular automaton.

Definition

A cellular automaton (CA) on a regular lattice $\mathcal{L}$ is a triple $\langle S, \mathcal{N}, f \rangle$ where

1. S is a finite set of states
2. $\mathcal{N} = \{\nu_1, \ldots, \nu_N\}$ is a finite neighborhood index on $\mathcal{L}$
3. $f : S^N \to S$ is the local function

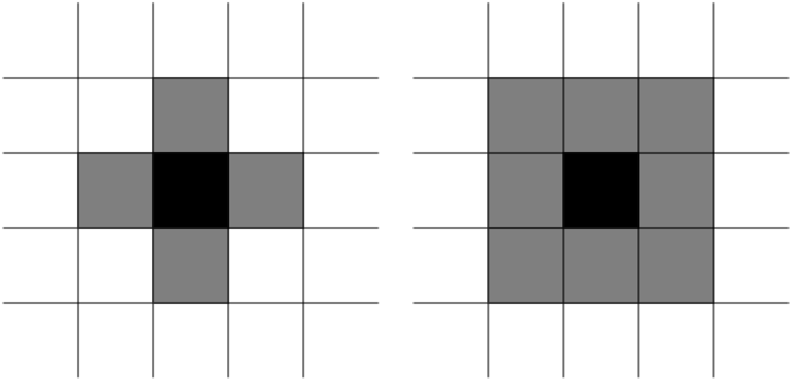
The local function induces a global function on $S^{\mathcal{L}}$

$$G(c)(z) = f(c(z + \nu_1), \ldots, c(z + \nu_N))$$

The evolution from configuration c is thus

$$c_z^{t+1} = f\left(c_{z+\nu_1}^t, \ldots, c_{z+\nu_N}^t\right)$$

von Neumann and Moore neighborhoods on the square grid



Wolfram's enumeration of 1D CA rules

Given a 1-dimensional, 2-state rule with neighborhood $\mathrm{vN}(1)$,

1. identify the sequence $(x, y, z) \in \{0, 1\}^{\mathrm{vN}(1)}$ with the the binary number xyz, and
2. associate to the rule f the number $\sum_{j=0}^{7} 2^j f(j)$.

Applications of cellular automata

- Population dynamics
- Economics
- Fluid dynamics
- Simulations of geological phenomena
- Symbolic dynamics
- Approximation of differential equations
- Screen savers
- And many more...

Implementations

CA are straightforward to implement on a computer.

- Define the space.
- Implement the local rule
- Run an update.

More difficult is to provide a general framework for CA.

- Hardware
  - CAM6 (Toffoli and Margolus; PC-XT expansion card)
  - CAM8 (Toffoli and Margolus; SPARCStation-driven device)
- Software
  - JCASim (Weimar; in Java)
  - SIMP/STEP (Bach and Toffoli; in Python)

- Developed by Edward (Ted) Bach as his PhD project under the supervision of Tommaso Toffoli.
- Currently in its 0.7 release.
- Written as a Python module.
- Employs the NumPy and PyGame modules.
- Allows implementation of several kinds of lattices.

And now for something totally different...

Reversible cellular automata

A reversible cellular automaton (briefly, RCA) is a cellular automaton $\mathcal{A}$ such that:

- The global function F is bijective.
- There exists a CA $\mathcal{A}'$ whose global function is F^{-1}.

It is well-known that

if the global function is bijective
then the CA is a RCA.

Reversible CA are ubiquitous

Toffoli embedding theorem (1979)

Every d-dimensional CA can be simulated by a $(d+1)$-dimensional RCA.

Reason why

- History can be stored by a second layer and the additional dimension.
- The additional layer is shifted—reversible.
- The original function on first layer is XOR'ed with second—reversible.

… however, reversibility is problematic

Theorem (Amoroso and Patt, 1972)

Reversibility of 1D CA is decidable.

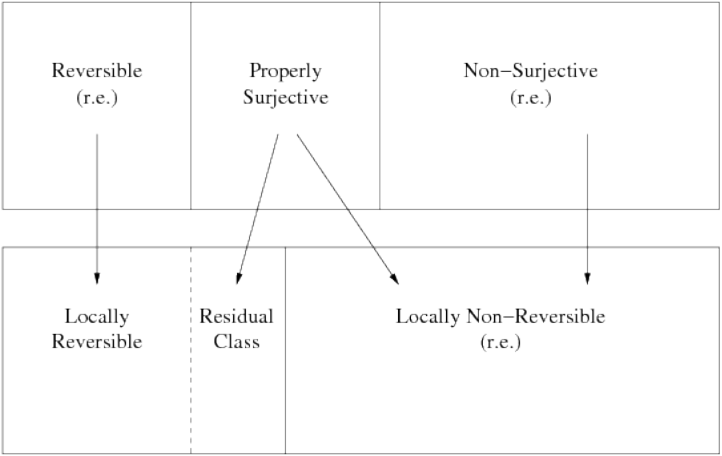
Reason why: tool provided by de Bruijn graphs.

Theorem (Kari, 1990)

Reversibility of 2D CA is undecidable.

Reason why: obstacle from undecidability of tiling problem.

CA from infinite to finite lattices



Block automata

They are a model of “watertight compartments” computation.

- Space is partitioned into equally-shaped blocks
- Each block updates at the same time
- Each block updates independently of the others

Block automata may be thought of as zero-range, coarse-grained CA.

Block automata are ubiquitous!

Theorem (Kari, 1996)

Every reversible 1D and 2D CA can be rewritten as a composition of block automata and partial shifts.

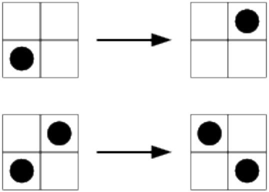
Theorem (Durand-Lôse, 2001)

Every reversible CA can be simulated by a composition of block automata and partial shifts.

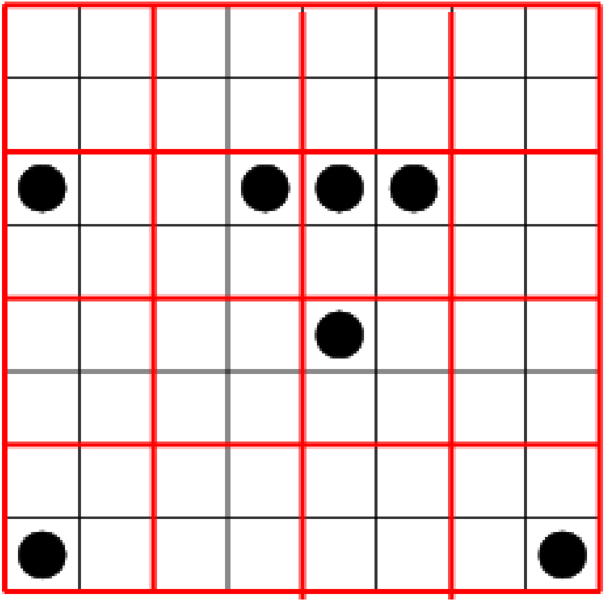
The Margolus neighborhood

Key ideas:

- Split plane into 2×2 blocks.
- Change center of splitting at each step.
- Make symmetric, bijective rule.



Update with the Margolus neighborhood



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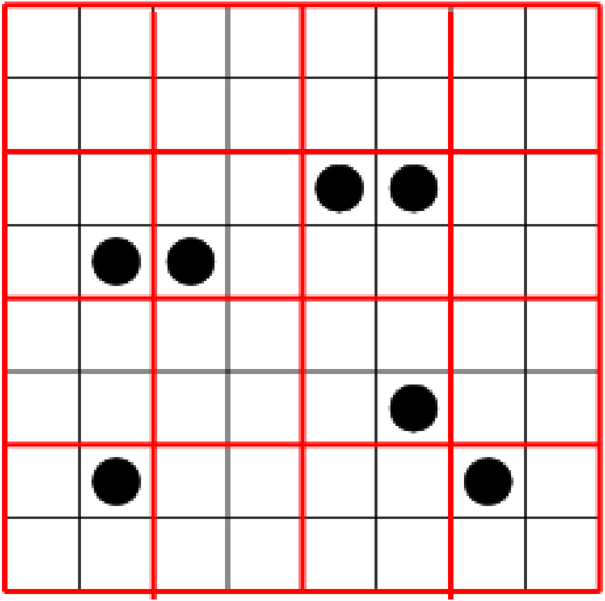
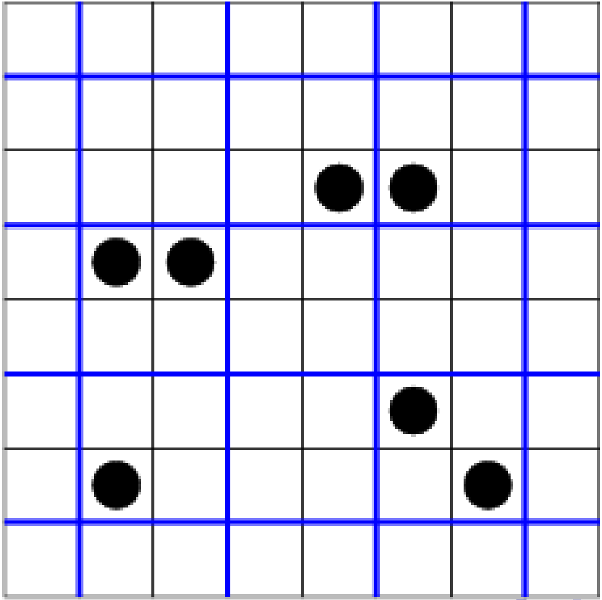
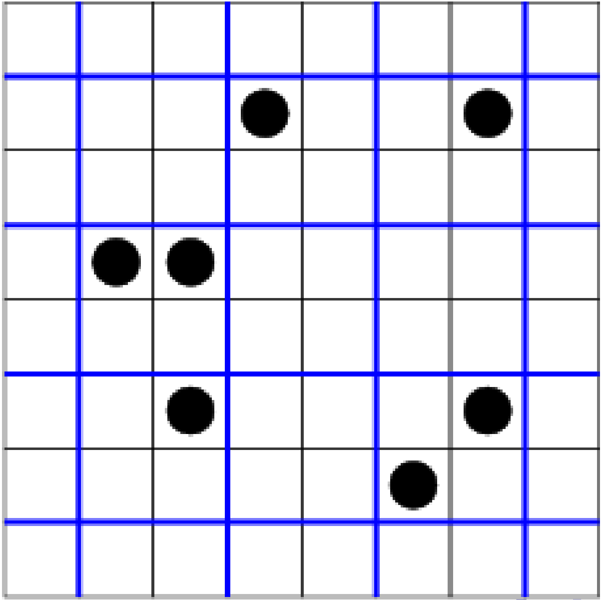


Update with the Margolus neighborhood



Update with the Margolus neighborhood



Example: Fredkin's billiard ball model

Implementation by Toffoli and Margolus, 1986

- Square grid with Margolus neighborhood.
- Walls are represented by paired lines of particles.
- Balls are represented by pairs of particles on a diagonal with an empty space between them.
- Block rule:
  - If one: proceed.
  - If two from opposite directions: bounce 90°.
  - Otherwise: nothing.

Lattice-gas automata: A two-steps discipline

Collision

- Strictly pointwise process
- Same number for inputs and outputs
- Same types for inputs and outputs

Propagation

- Each signal to one neighbour
- No replication
- No reuse

Characterization of reversible lattice-gas automata

Theorem

Let $\mathcal{A}$ be a lattice-gas automaton with collision function $f : S^N \to S^N$. TFAE.

1. $\mathcal{A}$ is reversible.
2. f is a permutation.

Reason why

- Propagation is reversible by construction.
- Collision is a collection of processes on isolated points.
- But any such collection is globally reversible iff it is a collection of local reversible processes.

Example: HPP

- Square grid on the plane.
- Up to four particles per node, in the four directions.
- Collision rule:
  - If from opposite directions: bounce 90^{o}.
  - Otherwise: proceed.

Example: FHP

- Triangular grid on the plane.
- Up to six particles per node, in the six directions.
- Collision rule:
  - If two from opposite directions: bounce 60^o in random direction.
  - If three 120^o apart: bounce 60^o.
  - Otherwise: proceed.

Second-order dynamics

We call second-order a dynamics of the form

$$x^{t+1} = F(x^t, x^{t-1}) \tag{1}$$

- In "first-order" dynamics, the converse of $x^{t+1} = F(x^t)$ is $x^t = G(x^{t+1})$ with $G = F^{-1}$.
- In second-order dynamics, the converse of (1) should have the form

$$x^{t-1} = G(x^t, x^{t+1})$$

  for some G.
- What should the shape of G be?

Characterization of second-order reversibility

The following are equivalent.

1. The following second-order system is reversible:

$$x^{t+1} = F(x^t, x^{t-1})$$

2. The following second-order system is reversible:

$$(x^{t+1}, y^{t+1}) = (F(x^t, y^t), x^t)$$

3. For every $p \in X$, the following map is a bijection:

$$F_p(x) = F(p, x)$$

Thus, a second-order dynamical system is reversible iff

the future is a permutation of the past
parameterized by the present.

Second-order cellular automata

In a second-order CA the local function maps S^{N+1} into S.
The dynamics has the form

$$c_x^{t+1} = f\left(c^t_{x+\mathcal{N}(1)}, \ldots, c^t_{x+\mathcal{N}(N)}; c_x^{t-1}\right) \tag{2}$$

We have the following trick, due to Fredkin:

- Consider the first-order CA:

$$c_x^{t+1} = f\left(c^t_{x+\mathcal{N}(1)}, \ldots, c^t_{x+\mathcal{N}(N)}\right)$$

  where the states are integers modulo m.

- Then

$$c^{t+1} = f\left(c^t_{x+\mathcal{N}(1)}, \ldots, c^t_{x+\mathcal{N}(N)}\right) - c_x^{t-1}$$

  is a reversible second-order CA!

Reversibility in second-order CA

Let $\mathcal{A} = \langle d, S, \mathcal{N}, f \rangle$ be a second-order CA. TFAE.

1. $\mathcal{A}$ is reversible.
2. f is a permutation of its last argument, parameterized by its first $|\mathcal{N}|$ arguments.

Moreover, any second-order CA can be rewritten isomorphically as a lattice-gas automaton. (Toffoli, C., and Mentrasti, 2004)

Two-step second-order

Divide the grid into two sub-grids, even and odd, so that:

1. Even cells only have odd neighbors.
2. Odd cells only have even neighbors.

Separate the updates so that:

1. Even cells only update at even times.
2. Odd cells only update at odd times.

Example: Ising model on the plane

- Square grid, von Neumann neighborhood.
- Nodes contain up/down dipoles.
- Edges represent links.
- A link is excited if orientation of dipoles is opposite.
- A link is relaxed if orientation of dipoles is same.
- Update rule:
  - If as many excited as relaxed: flip node.
  - Otherwise: nothing.

On the Web

- Cellular automata FAQ
  www.cafaq.com
- Jarkko Kari's tutorial
  users.utu.fi/jkari/ca/CAintro.pdf
- Ted Bach's SIMP/STEP
  sourceforge.net/projects/simpstep/
  www.ioc.ee/~silvio/simp.html
- Guillaume Theyssier's ACML
  www.lama.univ-savoie.fr/~theyssier/acml/
- Jörg R. Weimar's JCASim
  www.jweimar.de/jcasim/
- Golly (Game of Life simulator)
  golly.sourceforge.net/
- Stephen Wolfram's articles
  www.stephenwolfram.com/publications/articles/ca/

On paper

- J. Kari. Theory of Cellular Automata: a survey. *Theor. Comp. Sci.* **334** (2005) 3–33.
- T. Toffoli, N. Margolus. Invertible cellular automata: A review. *Physica D* **45** (1990) 229–253.
- T. Toffoli, S. Capobianco, P. Mentrasti. How to turn a second-order cellular automaton into a lattice gas: a new inversion scheme. *Theor. Comp. Sci.* **325** (2004) 329–344.
- T. Toffoli, S. Capobianco, P. Mentrasti. When—and how—can a cellular automaton be rewritten as a lattice gas? *Theor. Comp. Sci.* **403** (2008) 71–88.

Thank you for attention!

Any questions?