Walking through infinite trees with mixed induction and coinduction

Keiko Nakata and Tarmo Uustalu with Marc Bezem
Institute of Cybernetics, Tallinn
(many thanks to T. Altenkirch and H. Herbelin)

Theory Days, Nelijärve, 4 February 2011

Constructive reasoning

Sensitive to the use of axioms, in particular to the principle of the Excluded Middle

$$\forall P.\, P \vee \neg P$$

giving us a universal method of obtaining, for any P, either a proof of P or a proof of $\neg P$.

E.g., Either my program terminates or diverges.

We shall look at the word through refined eyes with the glasses of constructive type theory.

Modal properties about computations

A program will print “hello” infinitely often.

A program will eventually terminate.

A server is responsive, ie. any interactive query is eventually replied.

A server is almost always busy.

Infinite binary trees

$$\frac{}{R : color} \quad \frac{}{B : color}$$

$$\frac{c : color \quad t_0 : tree \quad t_1 : tree}{t_0\ c\ t_1 : tree}$$

Extensional equality, aka bisimilarity, on trees defined by coinduction:

$$\frac{t_0 \approx t_0' \quad t_1 \approx t_1'}{t_0\ c\ t_1 \approx t_0'\ c\ t_1'}$$

$\approx$ is an equivalence.

(NB: double horizontal line – coinduction, single – induction)

Paths and sequences

A *path* $p : path$ is an infinite sequence of *directions*, l (for left) and r (for right).

$$\frac{}{l : dir} \quad \frac{}{r : dir} \qquad \frac{d : dir \quad p : path}{d\ p : path}$$

with bisimilarity defined coinductively

$$\frac{p \approx p'}{d\ p \approx d\ p'}$$

We define infinite sequences $s : seq$ of colors and bisimilarity on them.

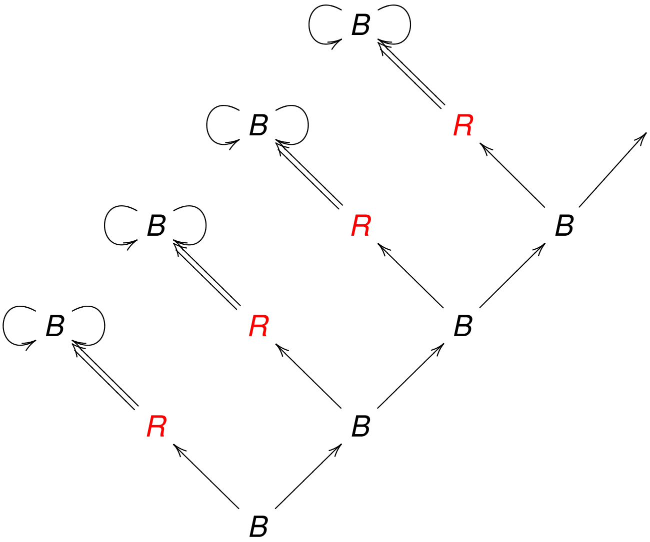
$$\frac{c : color \quad s : seq}{c\ s : seq} \qquad \frac{s \approx s'}{c\ s \approx c\ s'}$$

Walking into a tree along a path

The function $[|t|]_p$ walks into the tree t : *tree* along the path p : *path*, building a sequence. It is defined by corecursion

$$
\begin{aligned}
[|(t_0\ c\ t_1)|]_{(l\ p)} &= c\ [|t_0|]_p \\
[|(t_0\ c\ t_1)|]_{(r\ p)} &= c\ [|t_1|]_p
\end{aligned}
$$

Almost always black tree



Properties on trees

1. A tree is always black.
   Every path (of a tree) is always black
2. A tree is eventually red.
   Every path is eventually red.
3. A tree is infinitely often red.
   Every path is infinitely often red.
4. A tree is eventually always black.
   Every path is eventually always black.
5. A tree is almost always black.
   Every path of a tree is almost always black.

Every path/tree is always black

A sequence $s : seq$ is always black, $\mathbf{G}_{black}\ s$, where

$$\frac{\mathbf{G}_{black}\ s}{\mathbf{G}_{black}\ (B\ s)}$$

Every path of a tree $t : tree$ is always black, i.e.,

$$\forall p : path.\ \mathbf{G}_{black}\ [|t|]_p$$

A tree $t : tree$ is always black, $\mathbf{G}_{black}\ t$

$$\frac{\mathbf{G}_{black}\ t_0 \quad \mathbf{G}_{black}\ t_1}{\mathbf{G}_{black}\ (t_0\ B\ t_1)}$$

Always black

A tree $t : tree$ is always black, $\mathbf{G}_{black}\ t$,
if and only if
every path of a tree $t : tree$ is always black,
$\forall p : path.\, \mathbf{G}_{black}\ [|t|]_p$.

Every path/tree is eventually red

A sequence $s : seq$ is eventually red, $\mathbf{F}_{red}\ s$, where

$$\frac{}{\mathbf{F}_{red}\ (R\ s)} \qquad \frac{\mathbf{F}_{red}\ s}{\mathbf{F}_{red}\ (c\ s)}$$

Every path of a tree $t : tree$ is eventually red, i.e.,

$$\forall p : path.\, \mathbf{F}_{red}\ [|t|]_p$$

A tree $t : tree$ is eventually red, $\mathbf{F}_{red}\ t$

$$\frac{}{\mathbf{F}_{red}\ (t_0\ R\ t_1)} \qquad \frac{\mathbf{F}_{red}\ t_0 \quad \mathbf{F}_{red}\ t_1}{\mathbf{F}_{red}\ (t_0\ c\ t_1)}$$

Eventually red

A tree t : *tree* is eventually red
implies
every path of t is eventually red constructively.

Every path of t is eventually red
implies with FAN
t is eventually red.

FAN says that a tree with only finite paths is finite.

Every path/tree is eventually always black

A sequence $s : seq$ is eventually always black, $\mathbf{FG}_{black}\ s$, where

$$\frac{\mathbf{G}_{black}\ s}{\mathbf{FG}_{black}\ s} \qquad \frac{\mathbf{FG}_{black}\ s}{\mathbf{FG}_{black}\ (c\ s)}$$

Every path of a tree $t : tree$ is eventually always black,

$$\forall p : path.\, \mathbf{FG}_{black}\ [|t|]_p$$

A tree $t : tree$ is eventually always black, $\mathbf{FG}_{black}\ t$, where

$$\frac{\mathbf{G}_{black}\ t}{\mathbf{FG}_{black}\ t} \qquad \frac{\mathbf{FG}_{black}\ t_0 \quad \mathbf{FG}_{black}\ t_1}{\mathbf{FG}_{black}\ (t_0\ c\ t_1)}$$

Eventually always black

A tree t : *tree* is eventually always black
implies
every path of t is eventually always black.

Every path of t : *tree* is eventually always black
implies with weak continuity and a variation of FAN
t is eventually always black.

Weak continuity says functions can only depend on finite information.

Every path/tree is almost always black

A sequence $s : seq$ is almost always black, $pre\ s$, where

$$\frac{on_X\ s}{\overline{on_X\ (B\ s)}} \quad \frac{X\ s}{\overline{on_X\ (R\ s)}} \qquad\qquad \frac{pre\ s}{pre\ (R\ s)} \quad \frac{on_{pre}\ s}{pre\ (B\ s)}$$

Every path of a tree $t : tree$ is almost always black,

$$\forall p : path.\ pre\ [|t|]_p$$

A tree $t : tree$ is almost always black, $\mathbf{FG}_{black}\ t$, where

$$\frac{on_X\ t_0 \quad on_X\ t_1}{\overline{on_X\ (t_0\ B\ t_1)}} \quad \frac{X\ t_0 \quad X\ t_1}{\overline{on_X\ (t_0\ R\ t_1)}}$$

$$\frac{on_{pre}\ t_0 \quad on_{pre}\ t_1}{pre\ (t_0\ B\ t_1)} \quad \frac{pre\ t_0 \quad pre\ t_1}{pre\ (t_0\ R\ t_1)}$$

Almost always black

A tree t : *tree* is almost always black
implies
every path of t is almost always black.

Every path of t is almost always black
implies with Bar induction
t is almost always black.

Almost always black tree