Coding Theory: From the Past to the Present

Vitaly Skachek

Institute of Computer Science
University of Tartu

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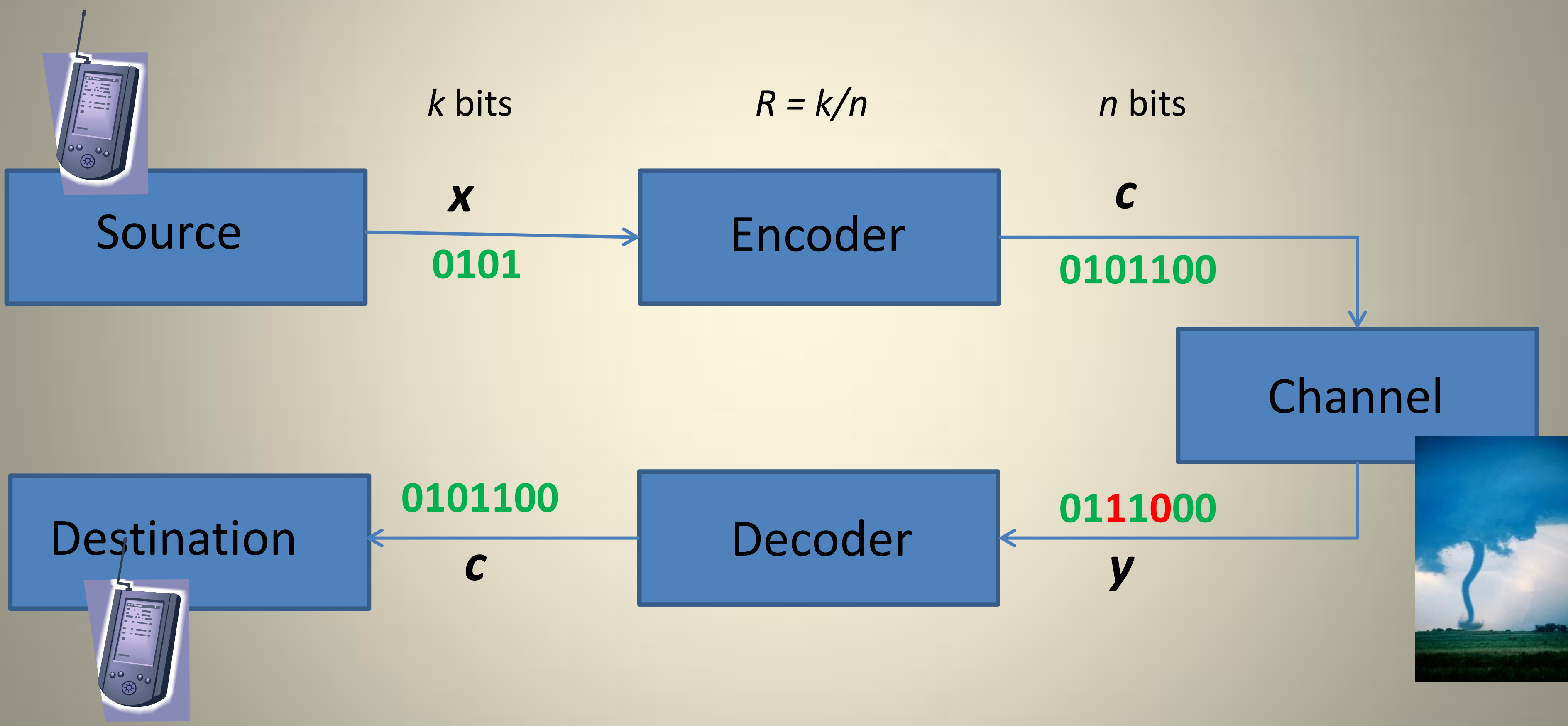
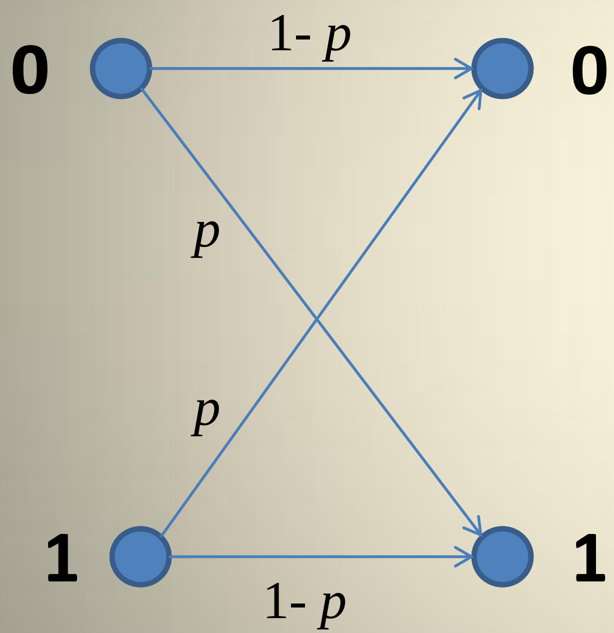
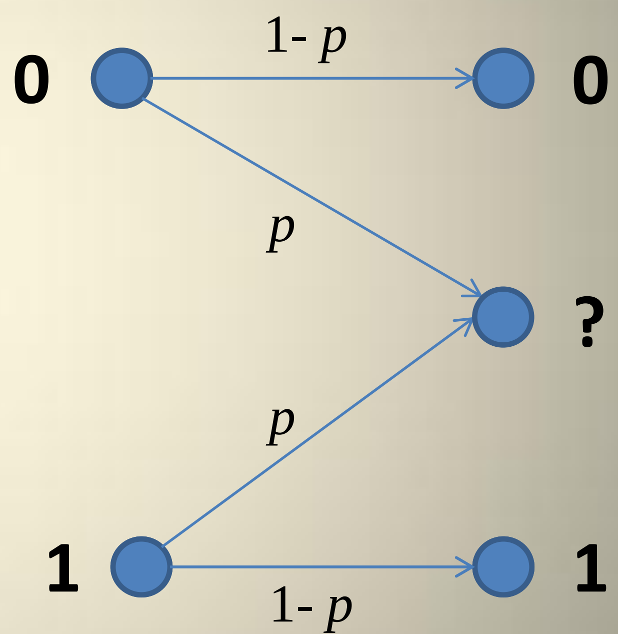
Communications Model



Communications Channels



Binary Symmetric Channel



Binary Erasure Channel

Shannon's Channel Coding Theorems

- A code is a mapping from the set of all vectors of length *k* to a set of vectors of length *n* (over alphabet Σ)

- Given a channel *S*, there is a quantity *C*(*S*) called channel capacity

Claude Shannon
(1916-2001)

Shannon’s Channel Coding Theorems

For any rate $R < C(S)$, there exists an infinite sequence of block codes C_i of growing lengths n_i such that $\frac{k_i}{n_i} \geq R$, and there exists a coding scheme for those codes such that the decoding error probability approaches 0 as $i \rightarrow \infty$.

Shannon's Channel Coding Theorems

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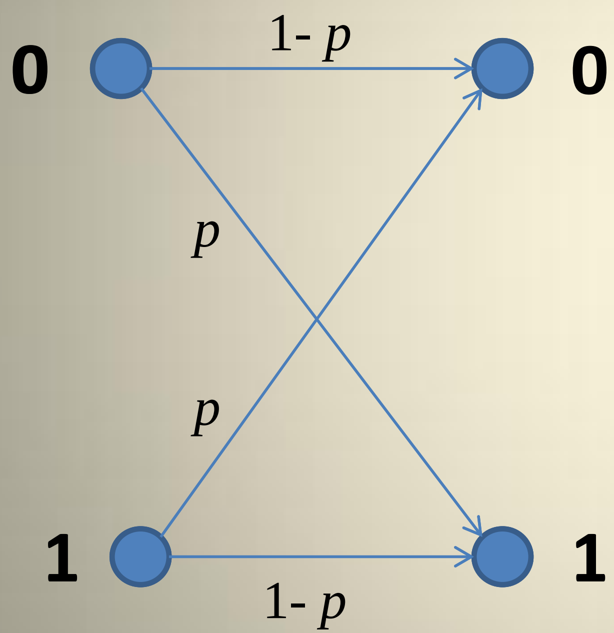
Let $R > C(S)$. For any infinite sequence of block codes C_i of growing lengths n_i such that $\frac{k_i}{n_i} \geq R$, and for any coding scheme for those codes, the decoding error probability is bounded away from 0 as $i \rightarrow \infty$.

Communications Channels



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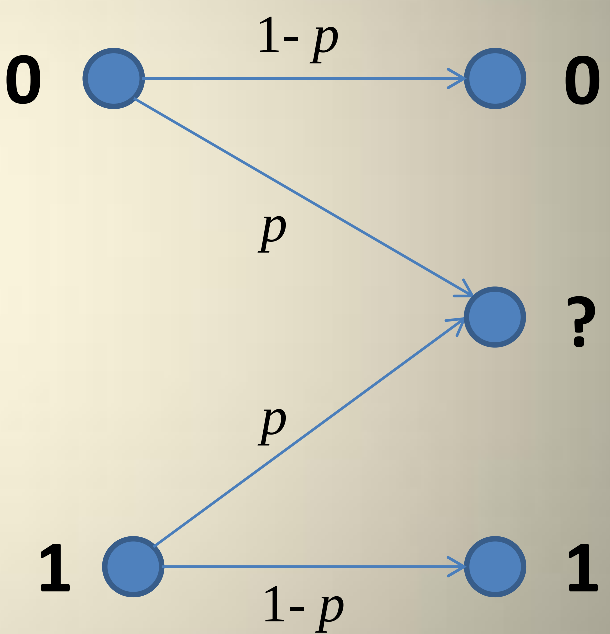
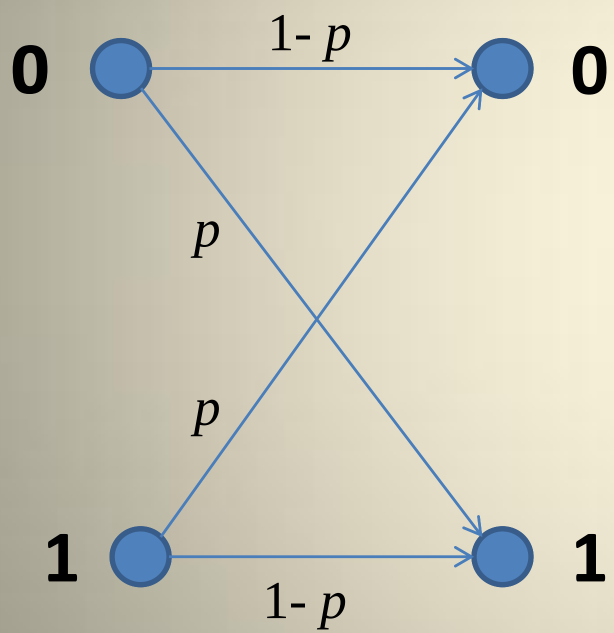


Binary Erasure Channel

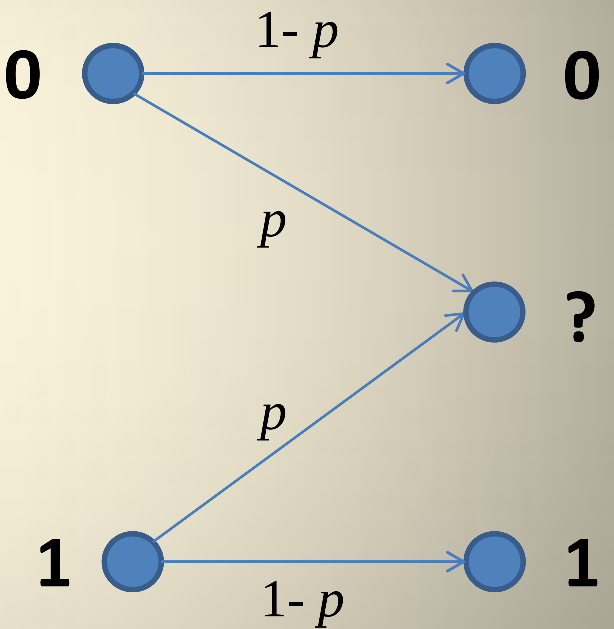
Communications Channels

$C(S)=1-h_2(p)$



Binary Symmetric Channel

$C(S)=1-p$

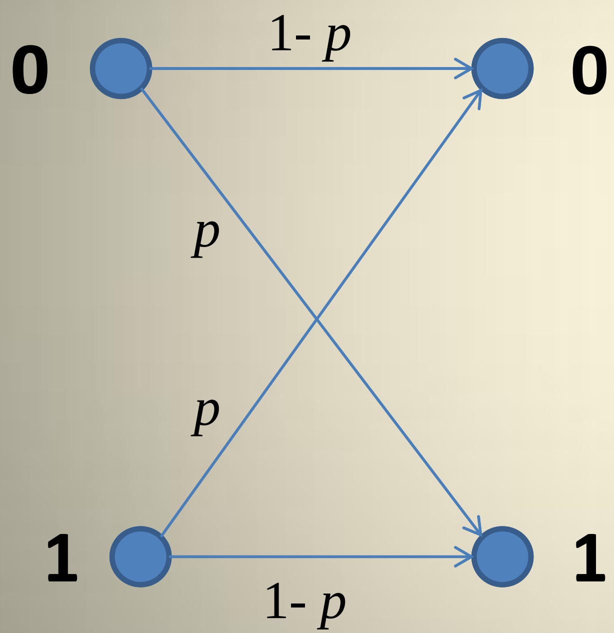


Binary Erasure Channel

Communications Channels

$C(S)$=1-$h_2(p)$

$h_2(x) = -x \log x - (1-x) log(1-x)$



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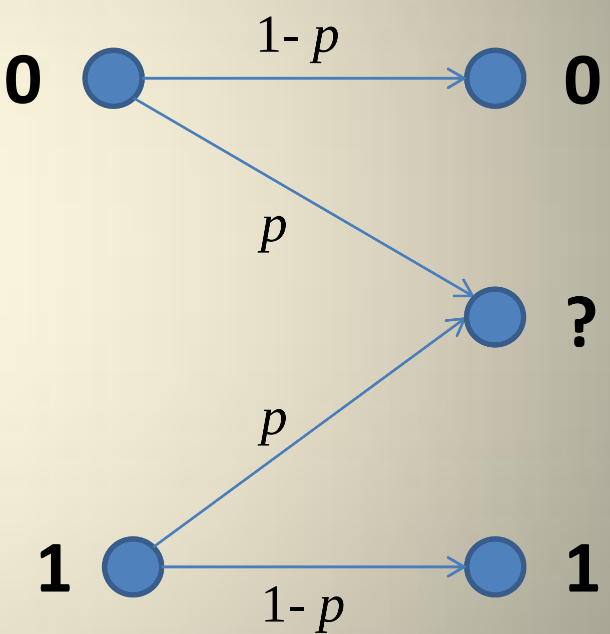

Binary Erasure Channel

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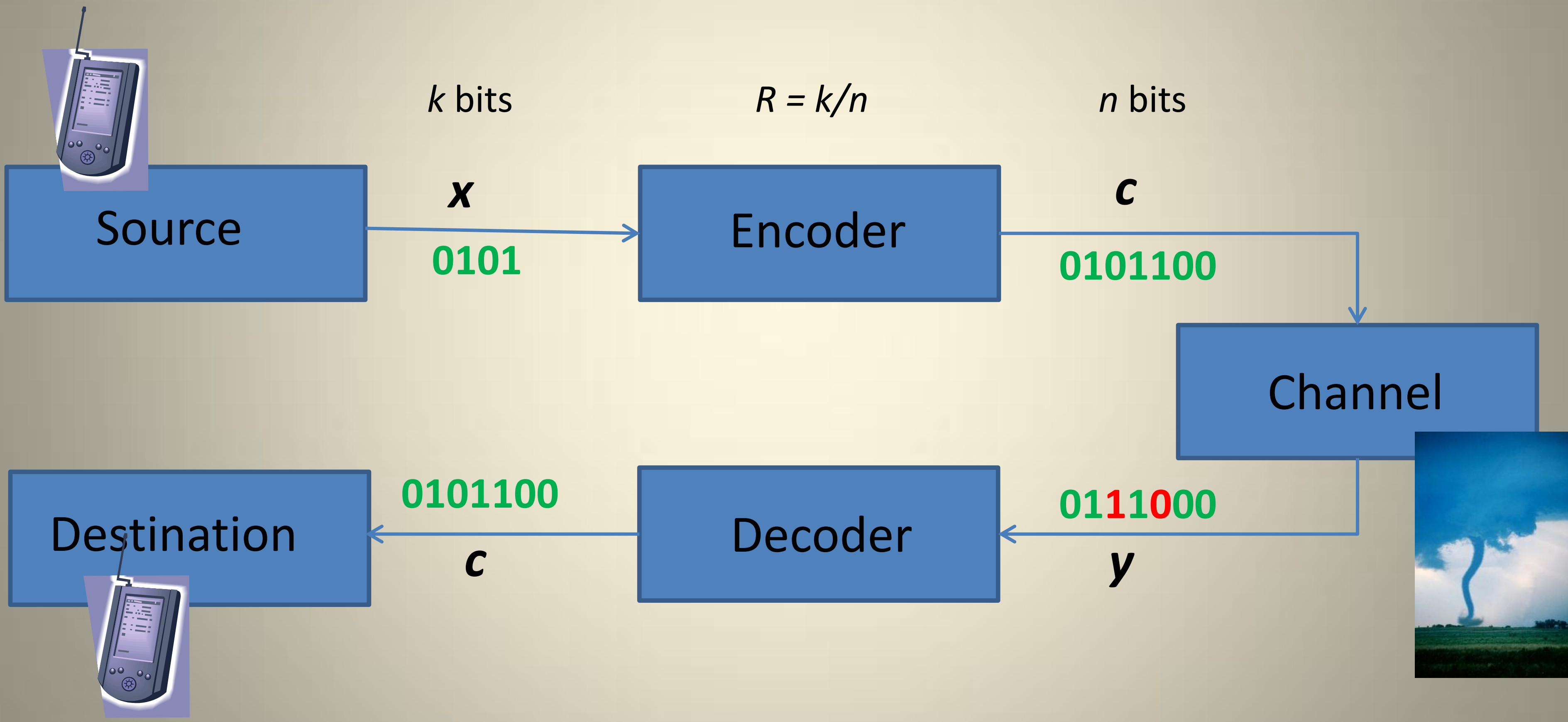


Parameters in Consideration

- Target: optimize the code rate $R = k/n$.

Other parameters in considerations:

- Speed of convergence Pr (err) $\to 0$ as $n \to \infty$. Low error probability for short lengths is needed!
- Time complexity of encoding and decoding algorithms. Structured codes are needed!

Distance

- The Hamming distance between $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$, $d(x, y)$, is the number of pairs of symbols (x_i, y_i), such that $x_i \neq y_i$.

- The minimum distance of a code C is

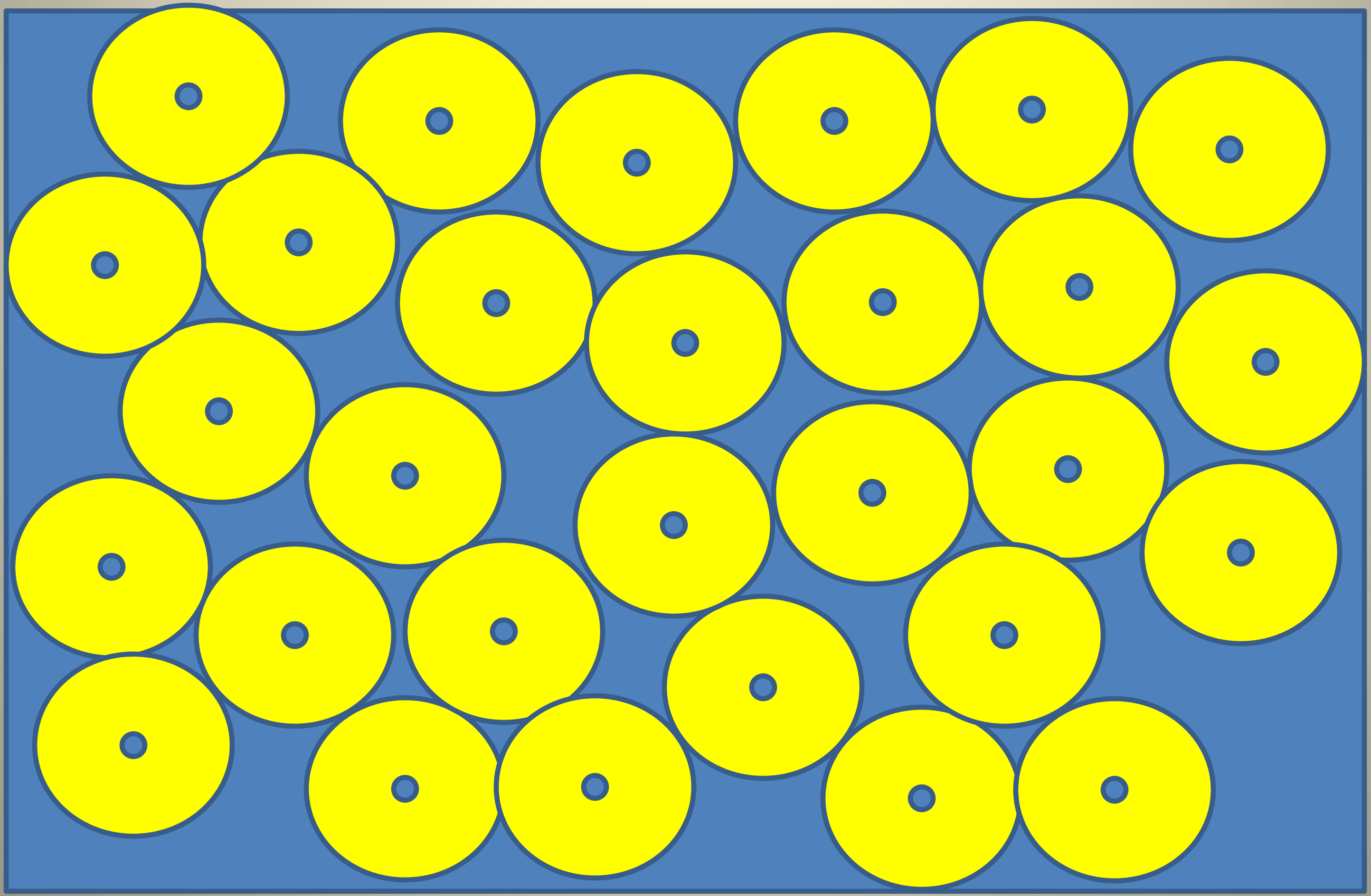
$$d = \min_{\{x, y \in C, x \neq y\}} d(x, y)$$

Linear Codes

- A code C over field *F* is a linear [*n*, *k*, *d*] code if there exists a matrix H with *n* columns and rank n − k such that
$$H \cdot c^T = 0^T \iff c \in C.$$

- The matrix *H* is called a parity-check matrix.
- The value *k* is called the dimension of the code C.
- The ratio *R* = *k*/*n* is called the rate of the code C.
- All words of C are exactly all linear combinations of rows of a generating *k* × *n* matrix *G*.

Sphere-packing idea



Sphere-packing idea

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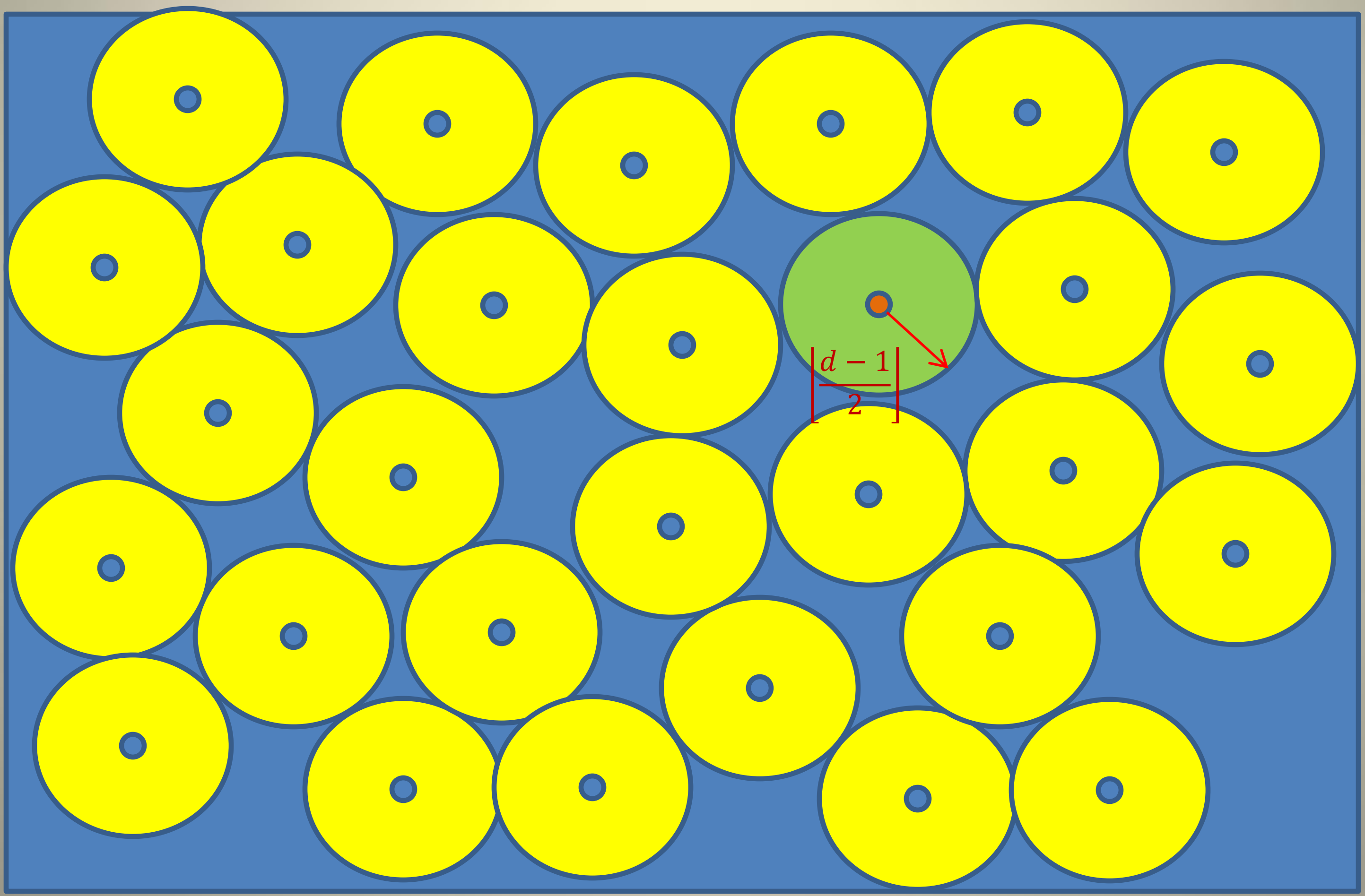

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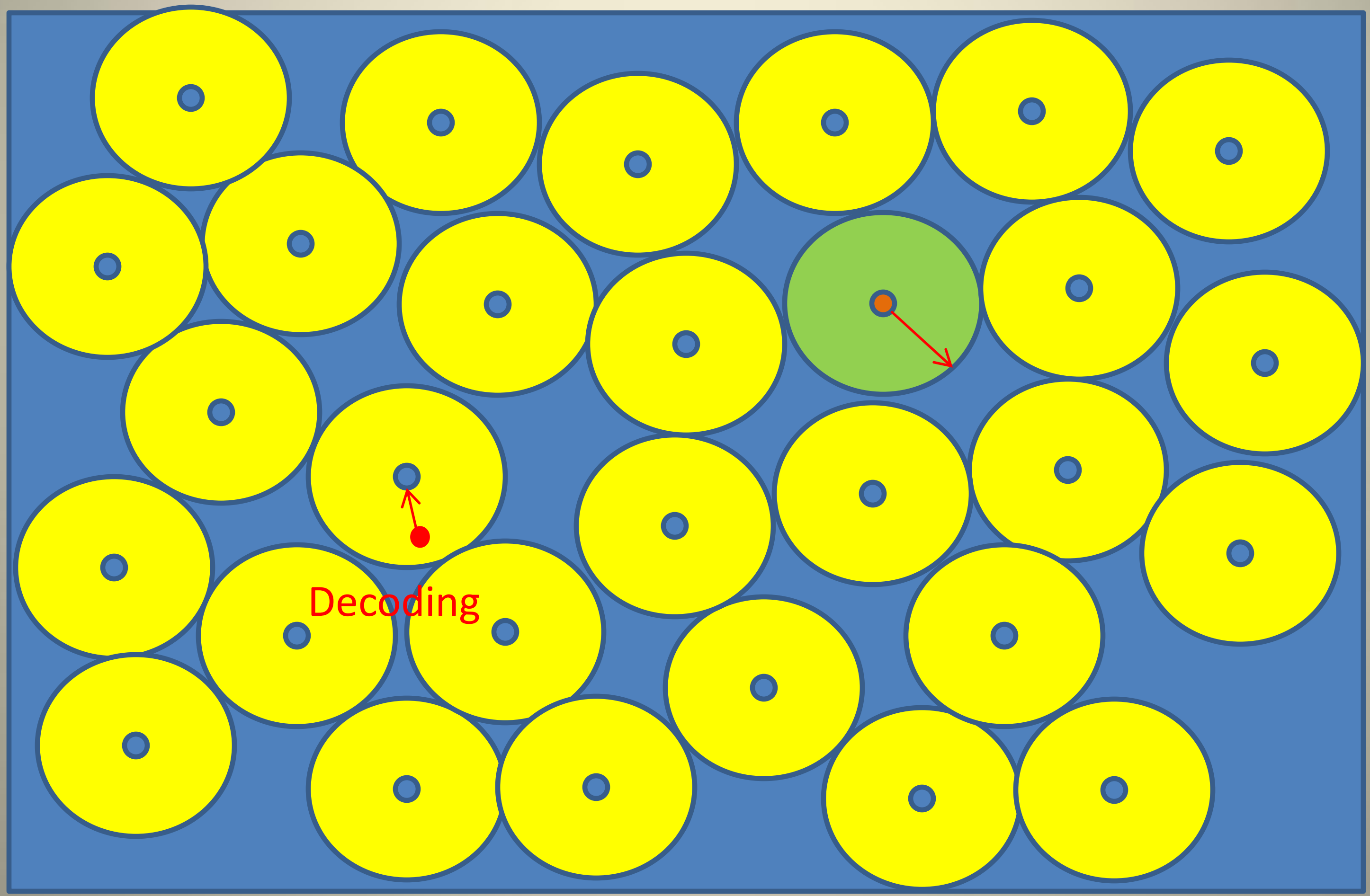


Reed-Solomon Codes

- Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ be *n* distinct elements.
- The generator matrix:

$$G = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ {\alpha_1}^{k-1} & {\alpha_2}^{k-1} & \ldots & {\alpha_n}^{k-1} \end{bmatrix}$$

- Satisfies the Singleton bound: $n = d + k - 1$
  - Optimal trade-off between the parameters

Reed-Solomon Codes (cont.)

- Encoding:

$$[x_0 x_1 \ldots x_{k-1}] \cdot \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ {\alpha_1}^{k-1} & {\alpha_2}^{k-1} & \ldots & {\alpha_n}^{k-1} \end{bmatrix}$$

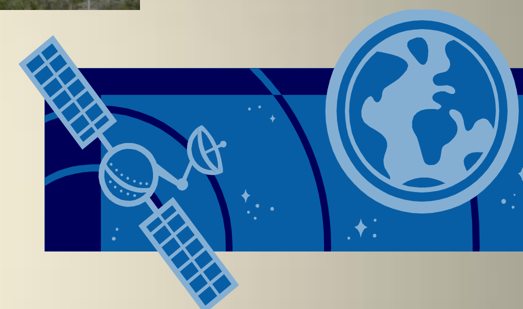
Polynomial Interpolation Viewpoint

- Input vector $[x_0 x_1 \ldots x_{k-1}]$ is associated with polynomial
$$P(z) = x_{k-1}z^{k-1} + x_{k-2}z^{k-2} + x_1 z + x_0$$
- Encoding is a substitution:
$$\left(P(\alpha_1), P(\alpha_2), \ldots, P(\alpha_n)\right)$$
- Decoding is an interpolation by degree $\leq k-1$ polynomial

Reed-Solomon Codes are Used in:

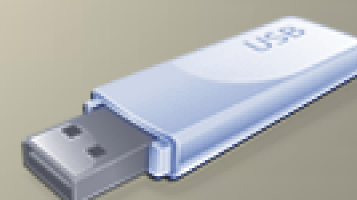
- Wired and wireless communications

- Satellite communications



- Hard drives and compact disks

- Flash memory devices

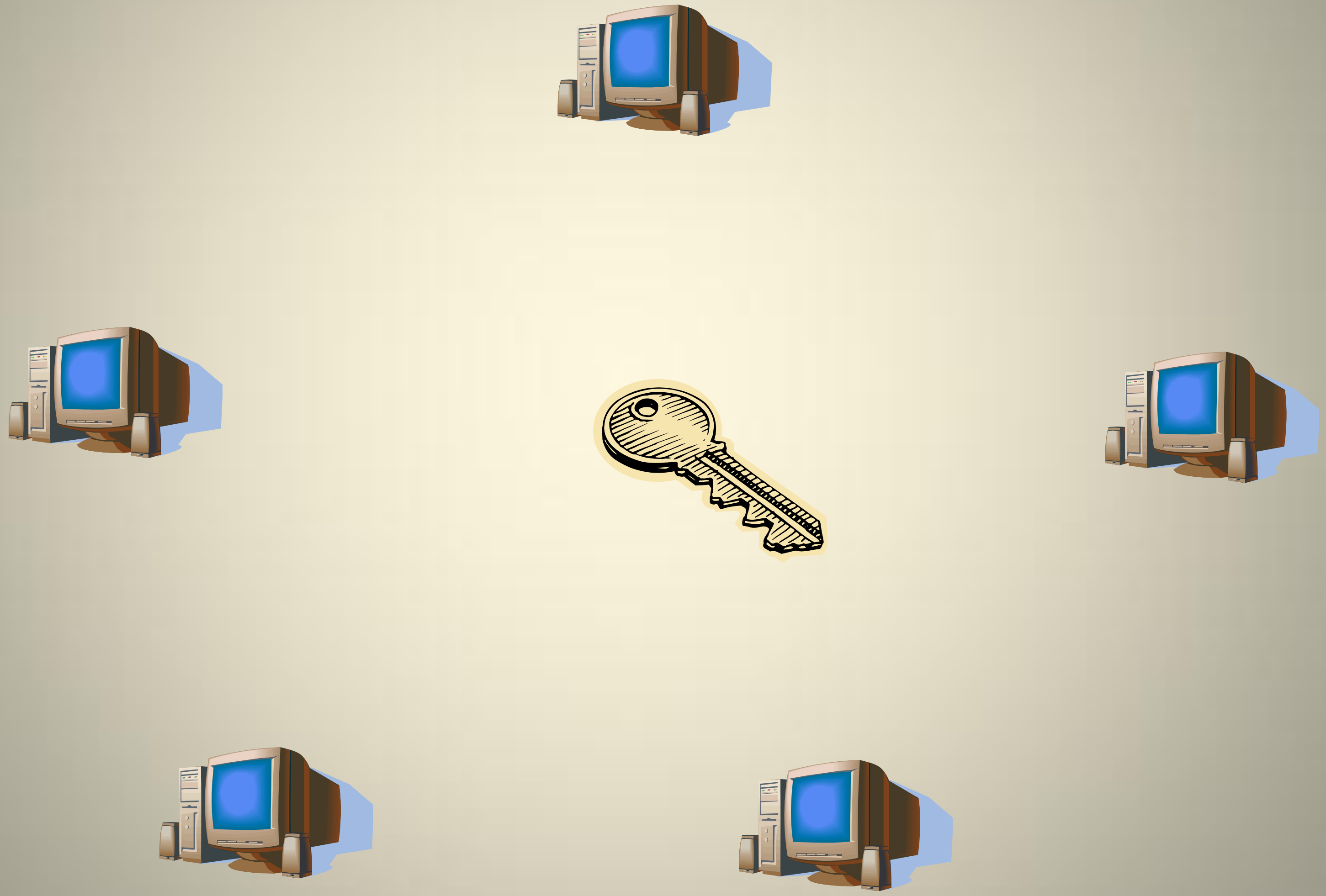


Application of Reed-Solomon Codes

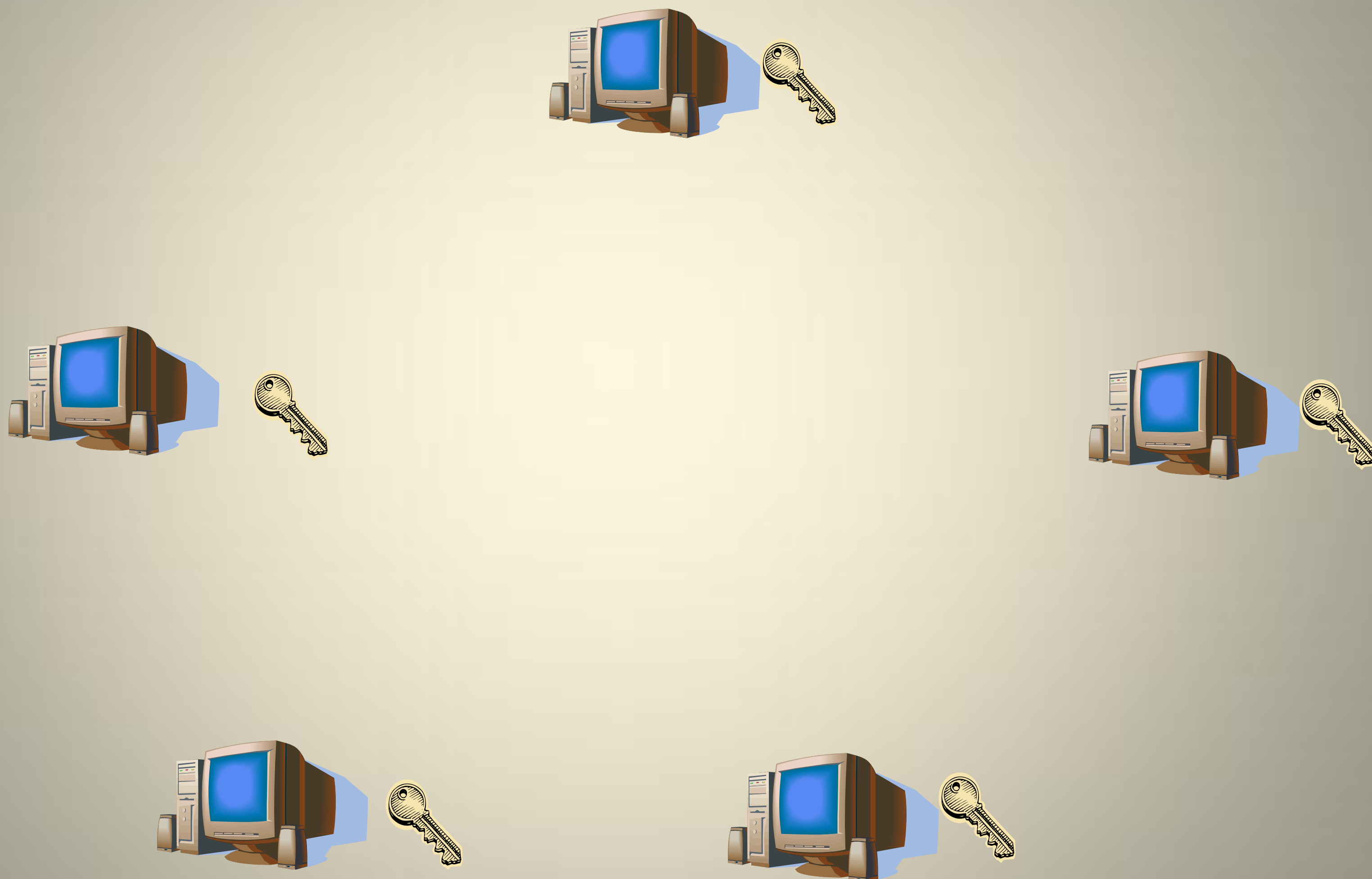
- Shamir's Secret-Sharing Scheme '79
- *n* users
- 1 key (number in *F*)
- Any coalition of $< t$ users does not have any information about the key
- Any coalition of $\geq t$ users can recover the key

Adi Shamir

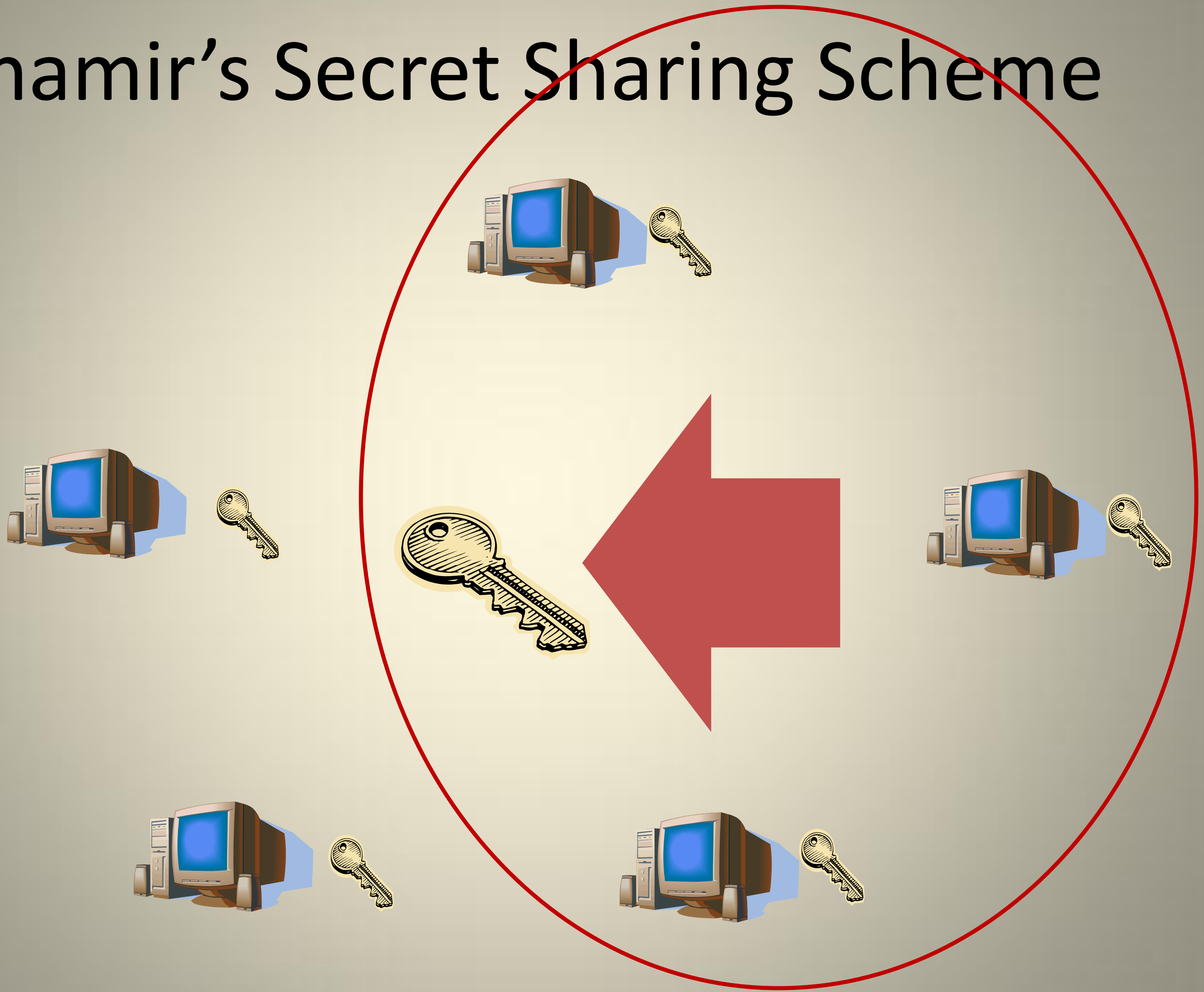
Shamir’s Secret Sharing Scheme



Shamir’s Secret Sharing Scheme



Shamir's Secret Sharing Scheme

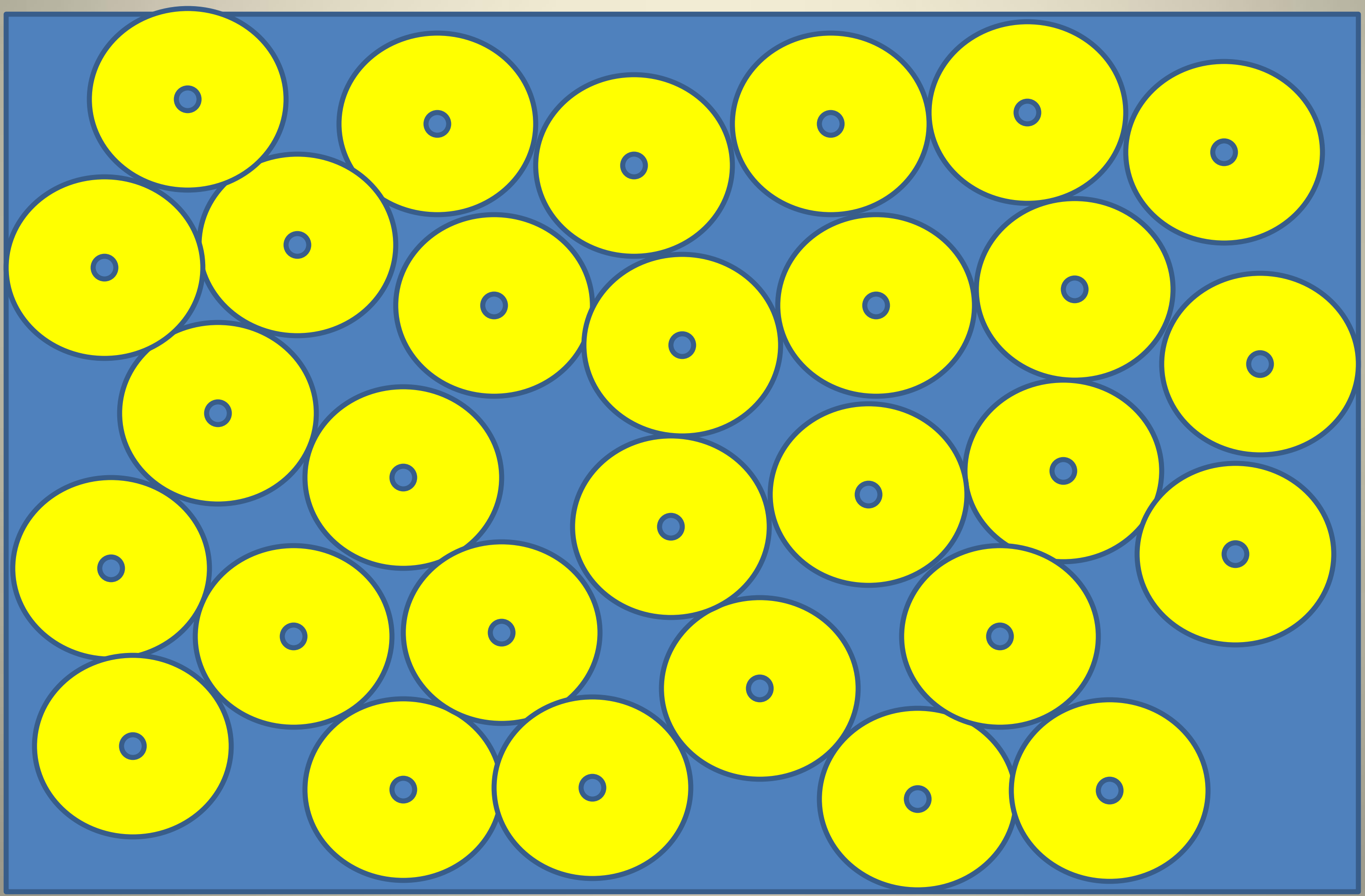


Shamir's Secret Sharing Scheme (cont.)

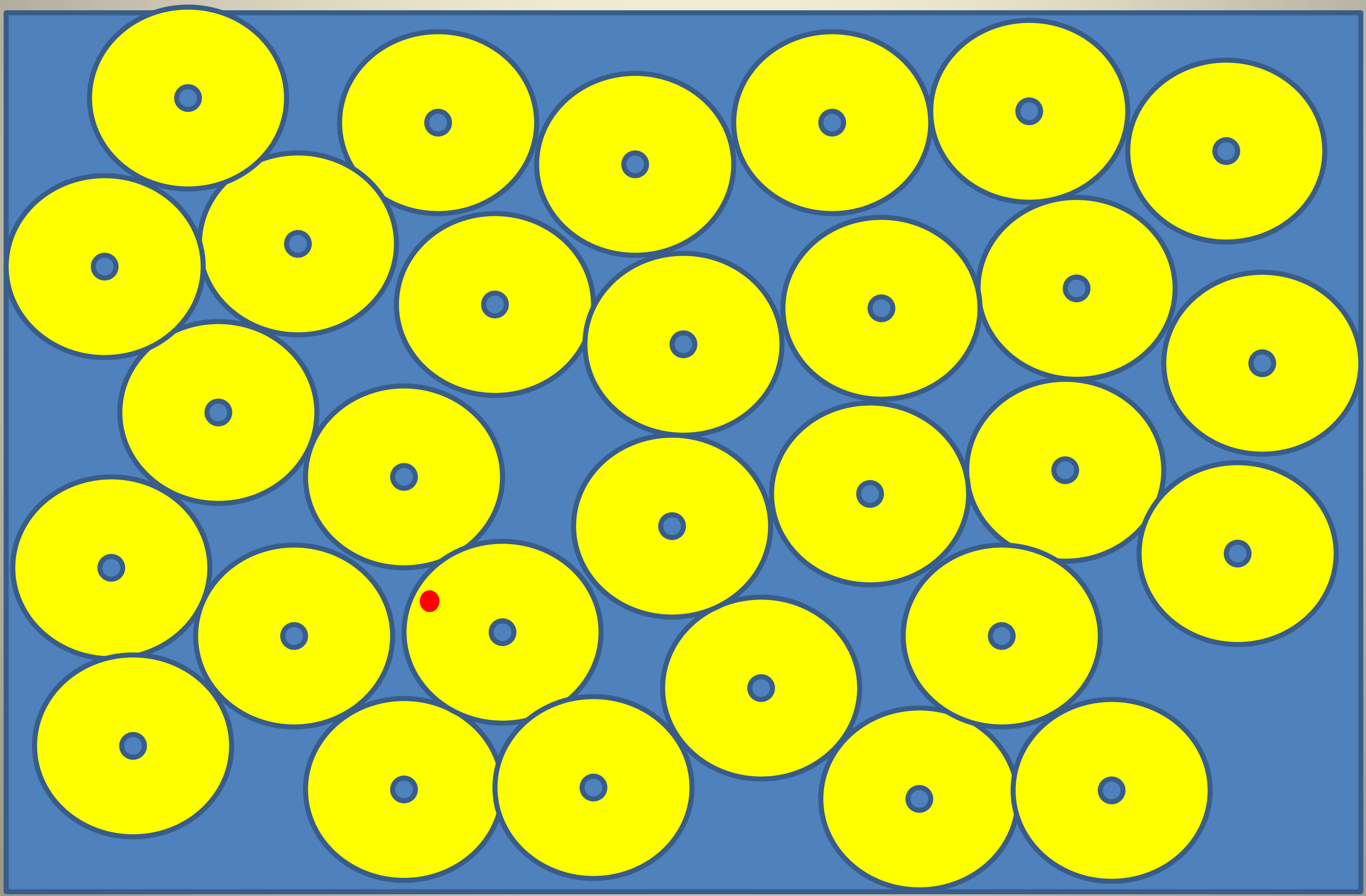
- Select randomly $x_1, x_2, \ldots, x_{k-1}$. Let x_0 be a secret key. Construct polynomial
$$P(z) = x_{k-1}z^{k-1} + x_{k-2}z^{k-2} + x_1 z + x_0$$
- Give $(\alpha_i, P(\alpha_i))$ to user i

- Large coalition has enough points to reconstruct the polynomial
- Small coalition has no information about the polynomial

List-decoding of Reed-Solomon Codes



List-decoding of Reed-Solomon Codes



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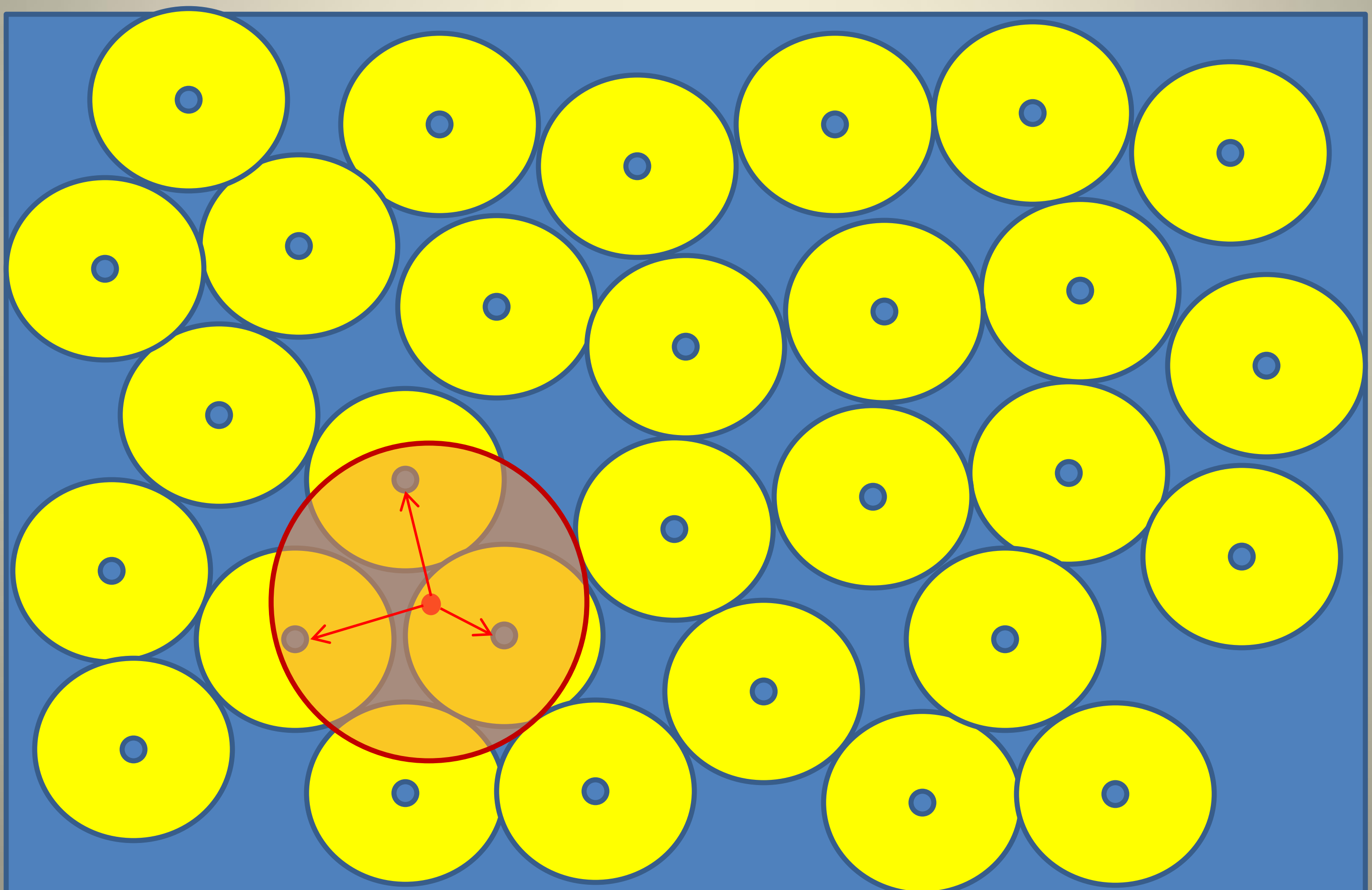
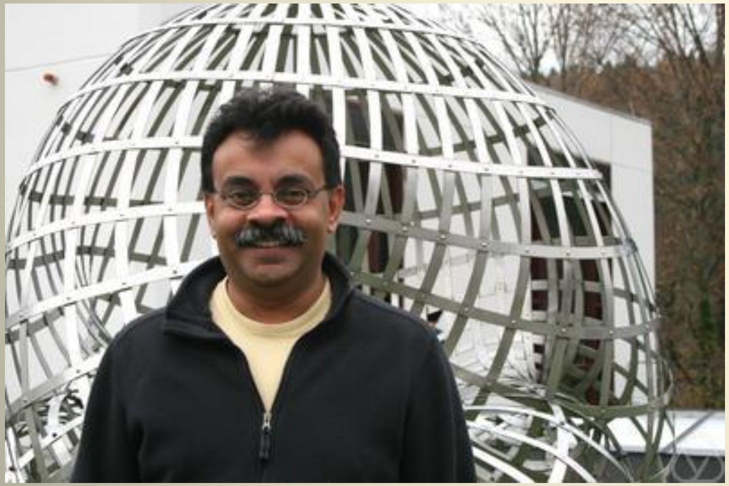


List-decoding of Reed-Solomon Codes

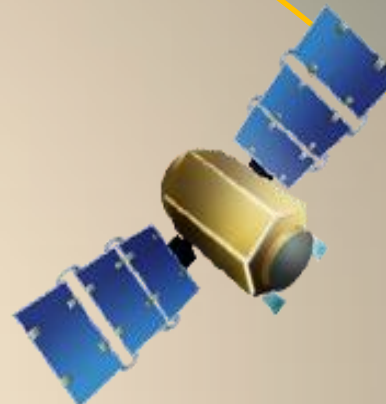
- Sudan '97, Guruswami '99, Vardy-Parvaresh '05, Guruswami-Rudra '06



Madhu Sudan

Venkatesan Guruswami

List Decoding of RS Codes

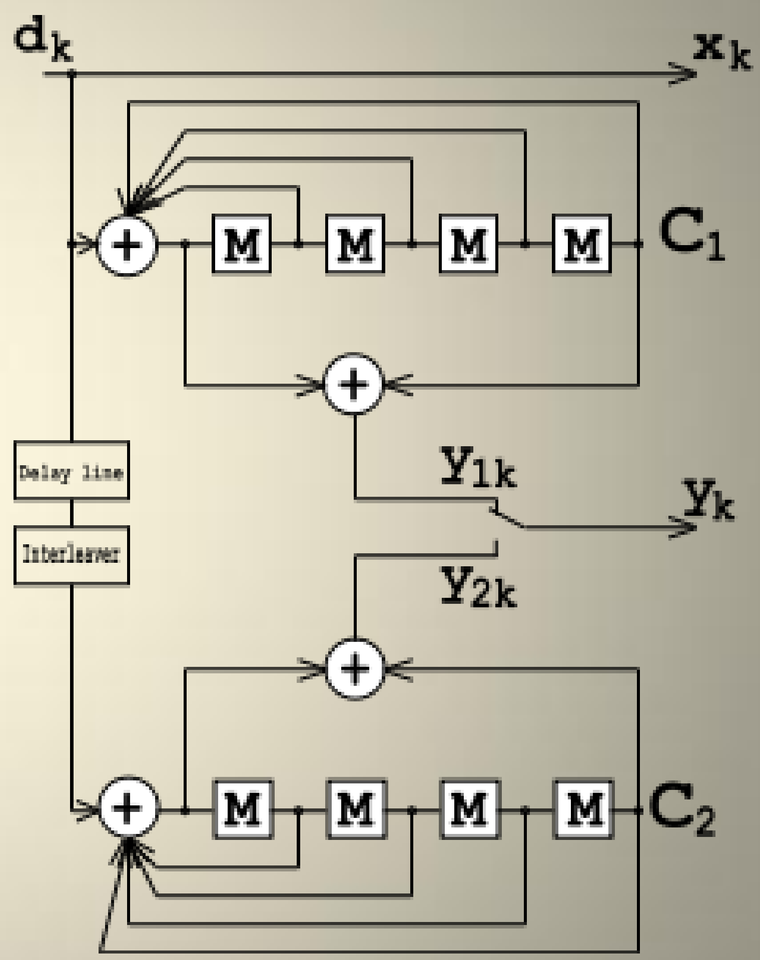


Voyager 1 – the first manmade object to leave the Solar System. Launched in 1977.

Turbo Codes

Berrou, Glavieux and Thitimajshima
(Telecom Bretagne) ’93

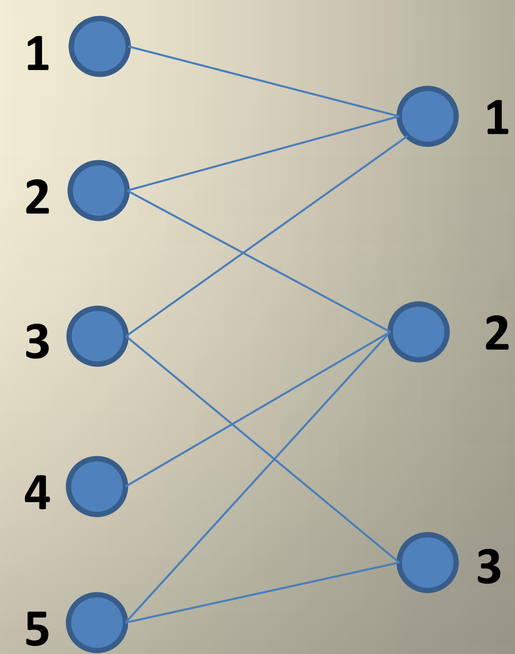
- Non-algebraic codes!
- “Killer” of algebraic coding theory



Low-Density Parity-Check Codes

- Gallager '62
- Urbanke, Richardson and Shokrollahi '01
- Parity-check matrix *H* is sparse
- Performance extremely close to channel capacity
- Decoding complexity linear in *n*

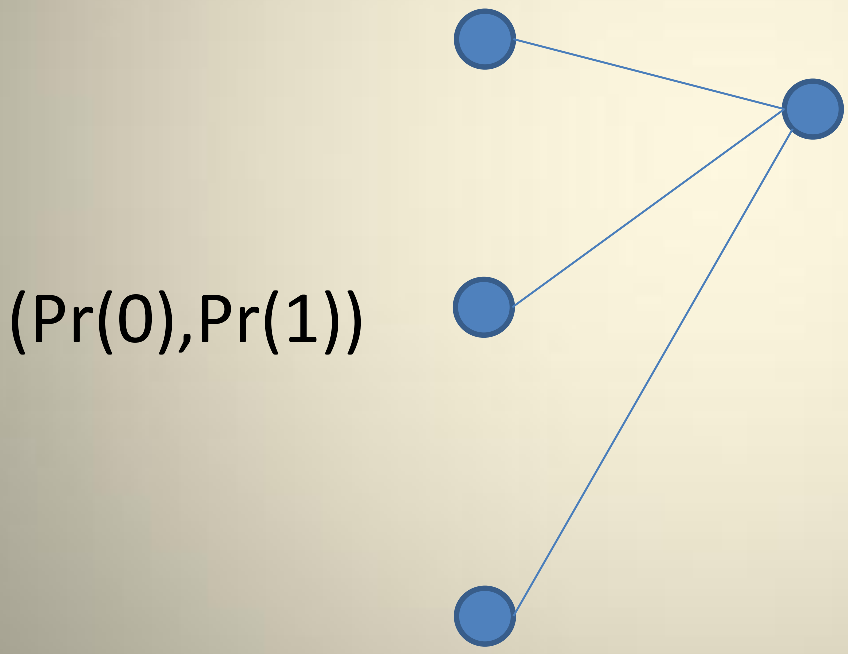
Tanner graph:



$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Low-Density Parity-Check Codes

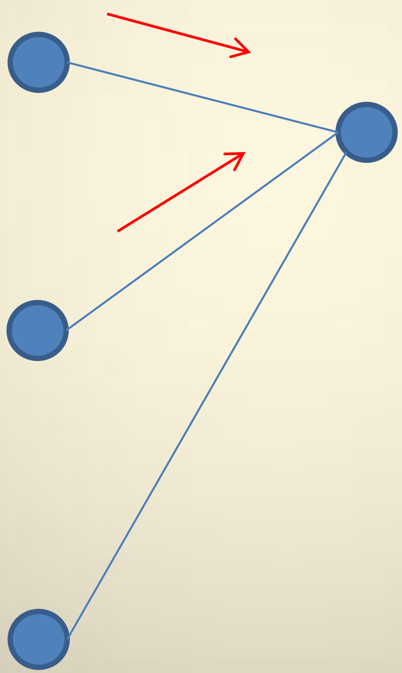
- Belief-propagation decoding algorithm (message-passing algorithm)



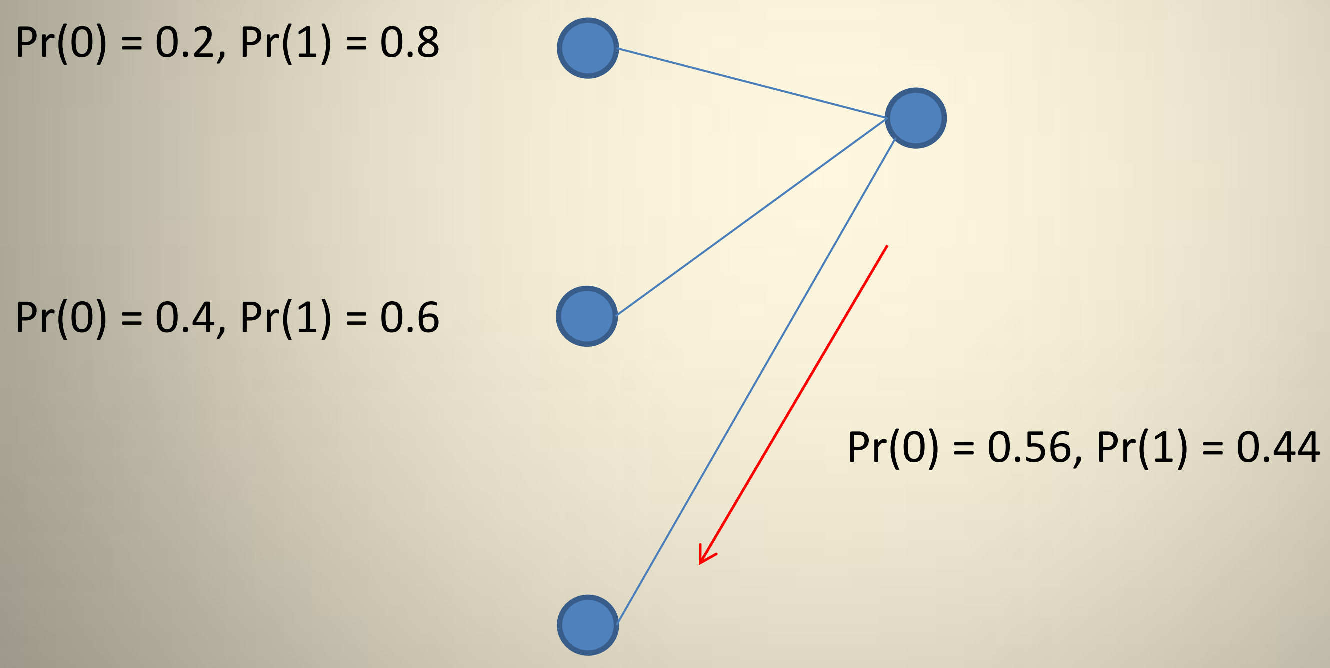
Low-Density Parity-Check Codes

Pr(0) = 0.2, Pr(1) = 0.8

Pr(0) = 0.4, Pr(1) = 0.6



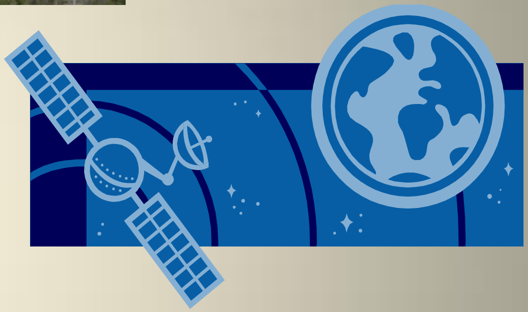
Low-Density Parity-Check Codes



Reed-Solomon Codes are Used in:

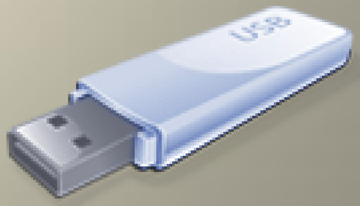
- Wired and wireless communications

- Satellite communications



- Hard drives and compact disks

- Flash memory devices



LDPC Codes are Used in:

- Wired and wireless communications

- Satellite communications

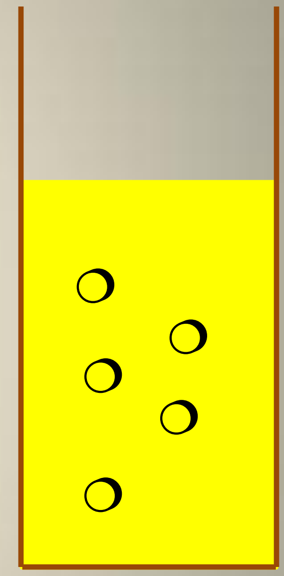
- Hard drives and compact disks

- Flash memory devices

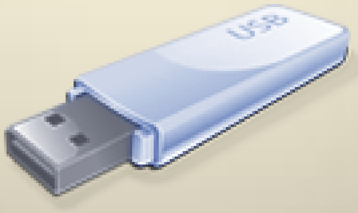
Emerging Applications of Coding Theory

Flash memories

- Easy to add electric charge, hard to remove
- The charge "leaks" with the time
- Neighboring cells influence each other



Flash memory cell

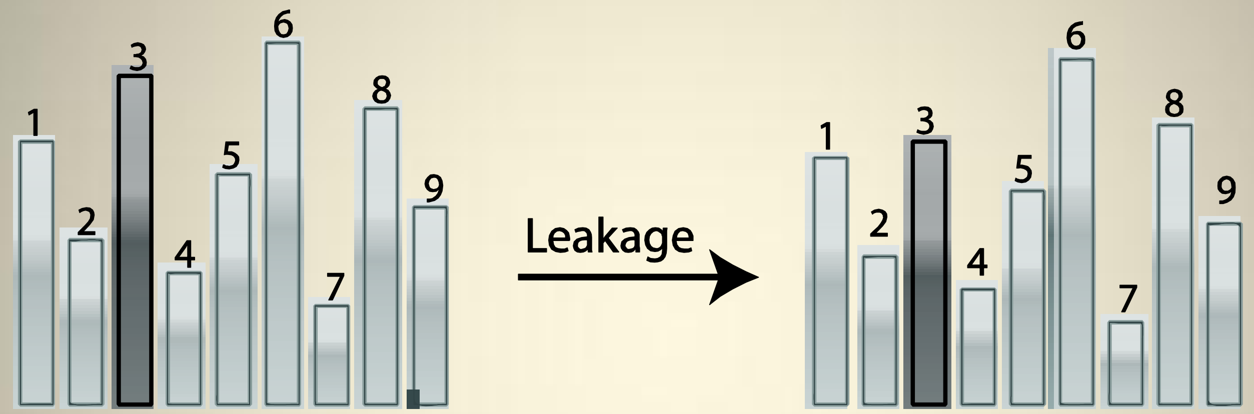


Flash memories

- Rank modulation
- The information is represented using relative levels of charge, invariant to leakage
- Coding over permutations

Jiang, Mateescu, Schwartz, Bruck ‘2006

Flash memories



[6 3 8 1 5 9 2 4 7] → [6 8 3 1 5 9 2 4 7]

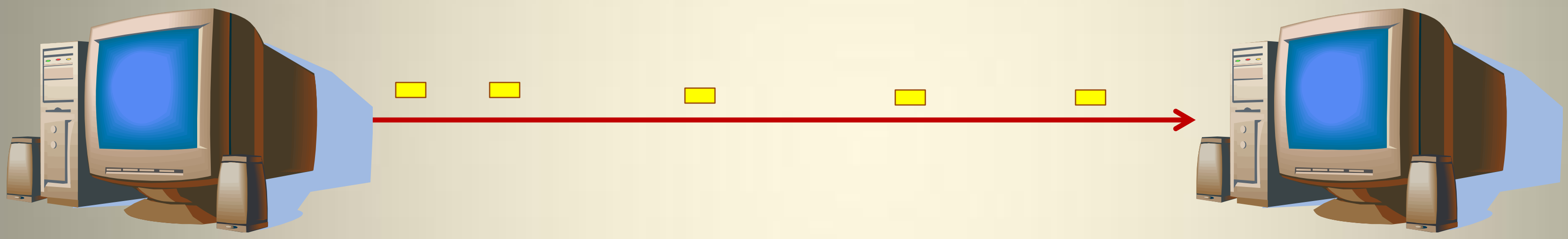
Networking

- Raptor Codes

- A. Shokrollahi ‘2004
- Used in DVB-H standard for IP datacast for handheld devices

Networking

- Raptor Codes



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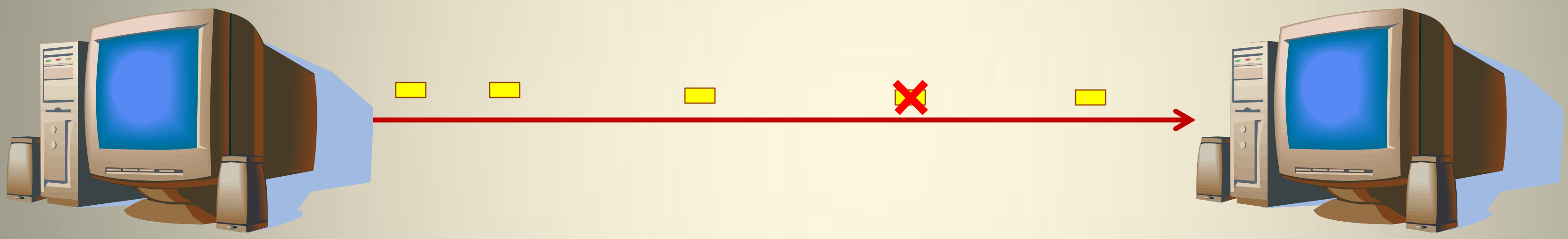

Networking

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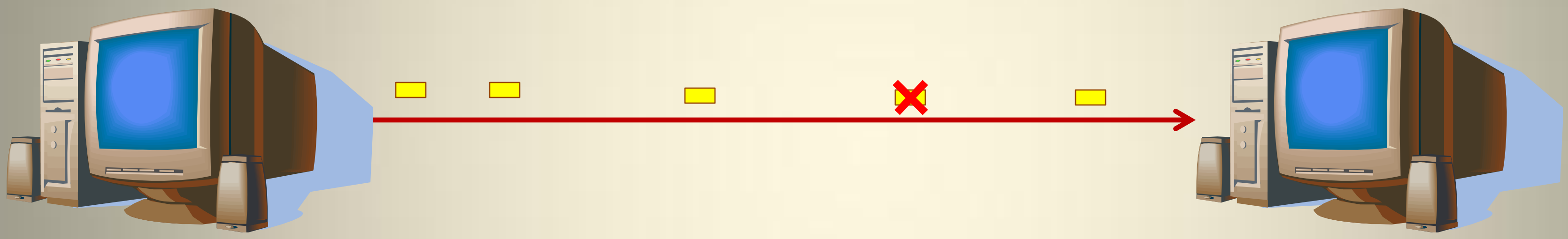


- Possible solution: ARQs (retransmissions) – slow!
- Alternative: large error-correcting code

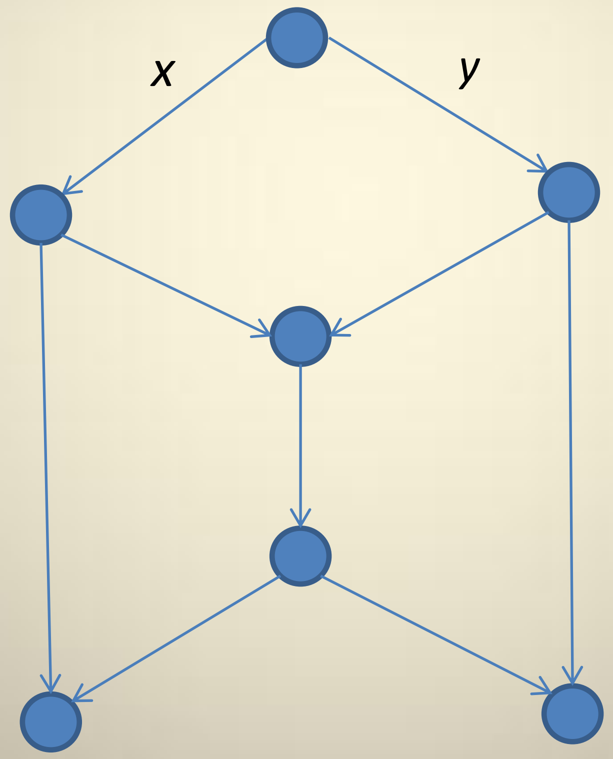
Networking

- Raptor Codes

Network coding

- Butterfly network

Ahlswede, Cai, Li and Yeung, 2000



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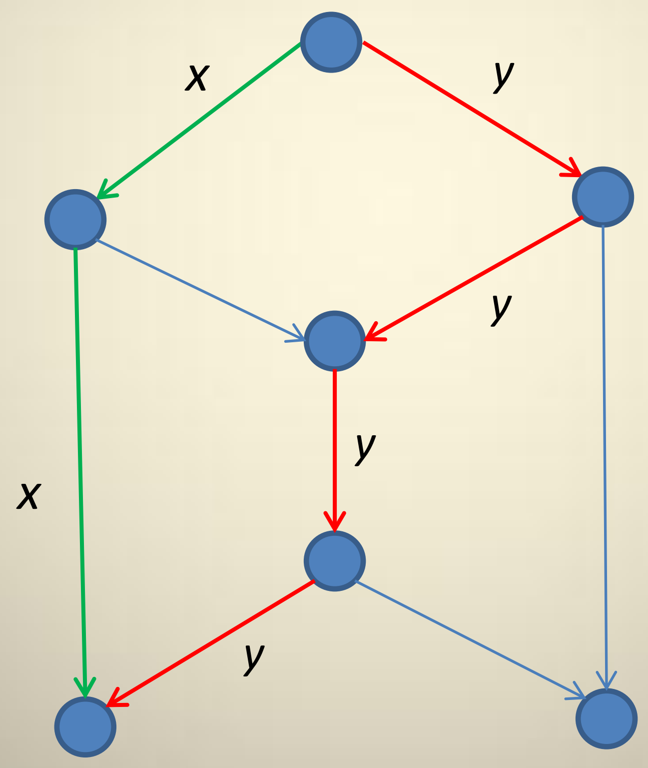

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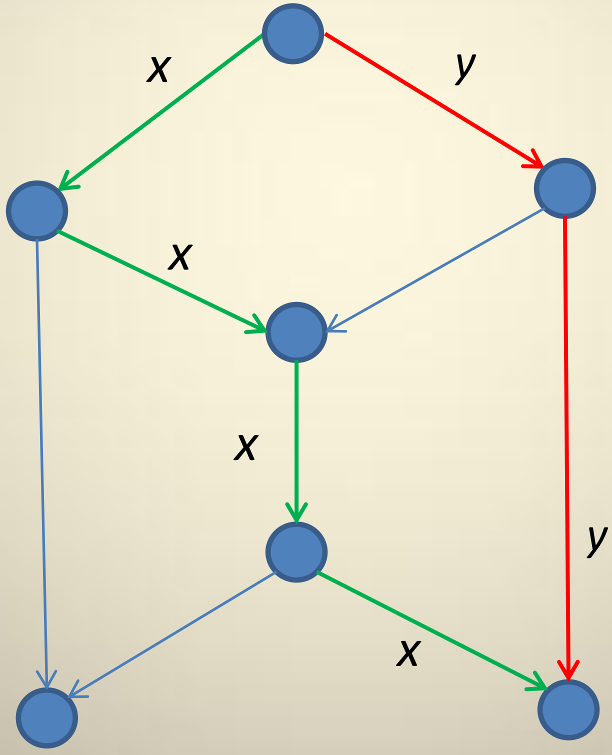

Network coding

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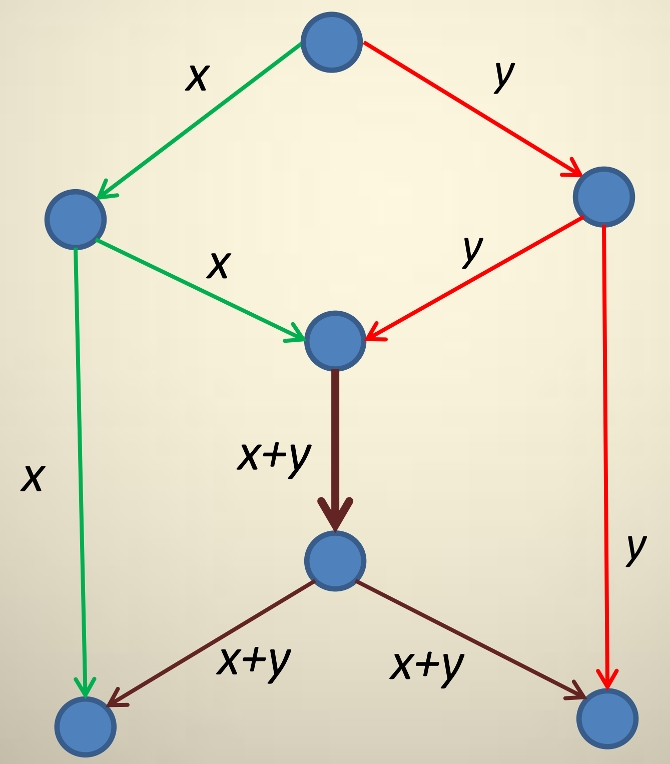
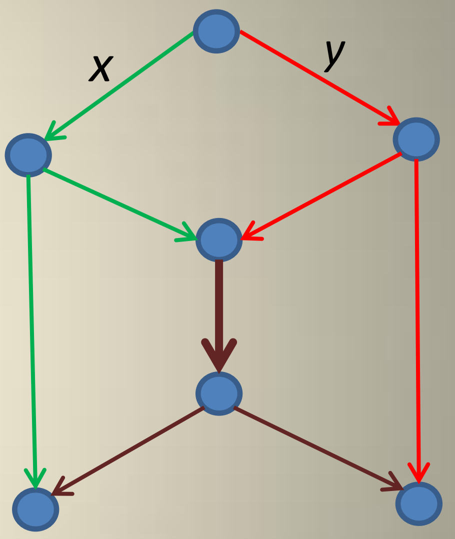


Network coding

- The number of bits deliverable to each destination is equal to min-cut between source and each of destinations
- Avalanche P2P Network (Microsoft, 2005)
- Experiments for use in mobile communications

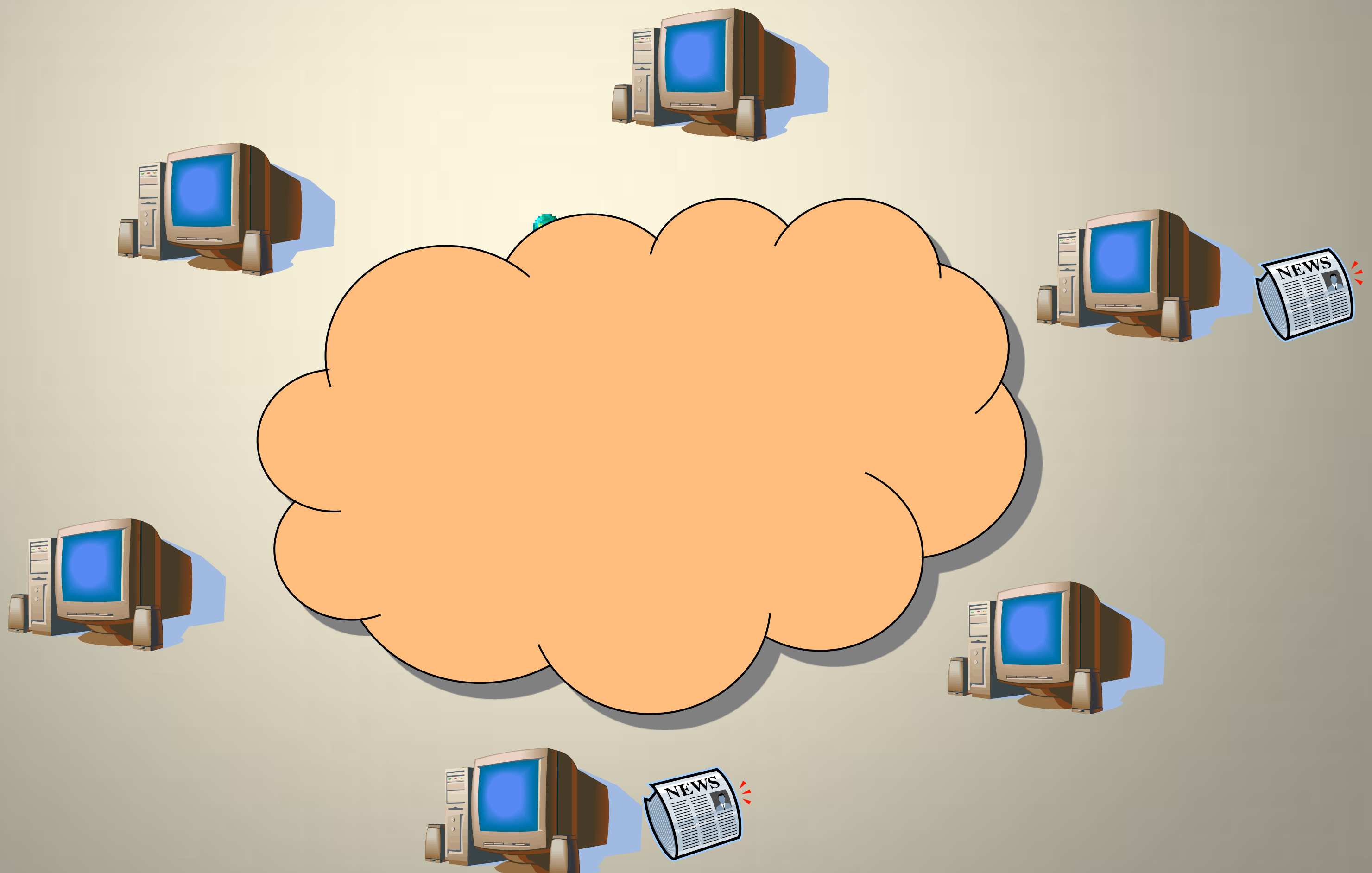


Gossip Algorithms

- *n* users in the network
- *k* of them possess a rumor (packet of data) – each rumor is different
- Each users “calls” another user randomly and sends a rumor to him
- Purpose: to distribute all rumors to all users
- Using coding: send a random linear combination of all rumors in your possession
  - Facilitates convergence of the algorithm
    Deb, Medard and Choute 2006

Gossip Algorithms

- Rumor spreading problem



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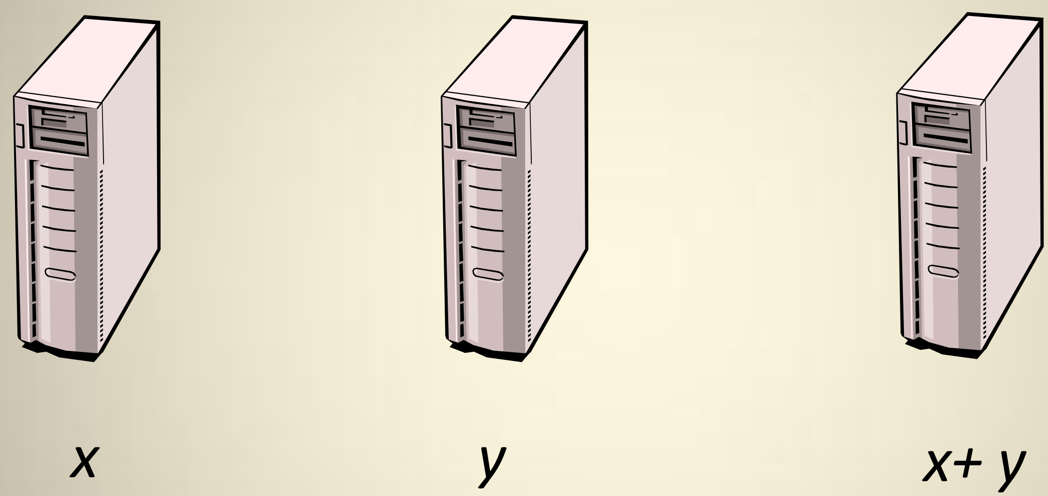
Distributed Storage

- Huge amounts of data stored by big data companies (Google, Amazon, Facebook, Dropbox)

Facebook data center in Oregon

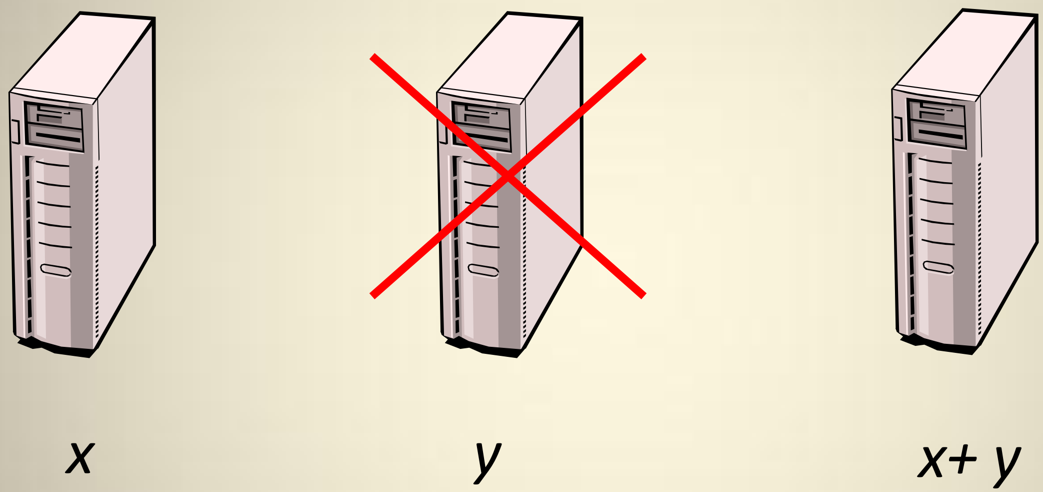
Server room at Wikipedia data center

Distributed data storage

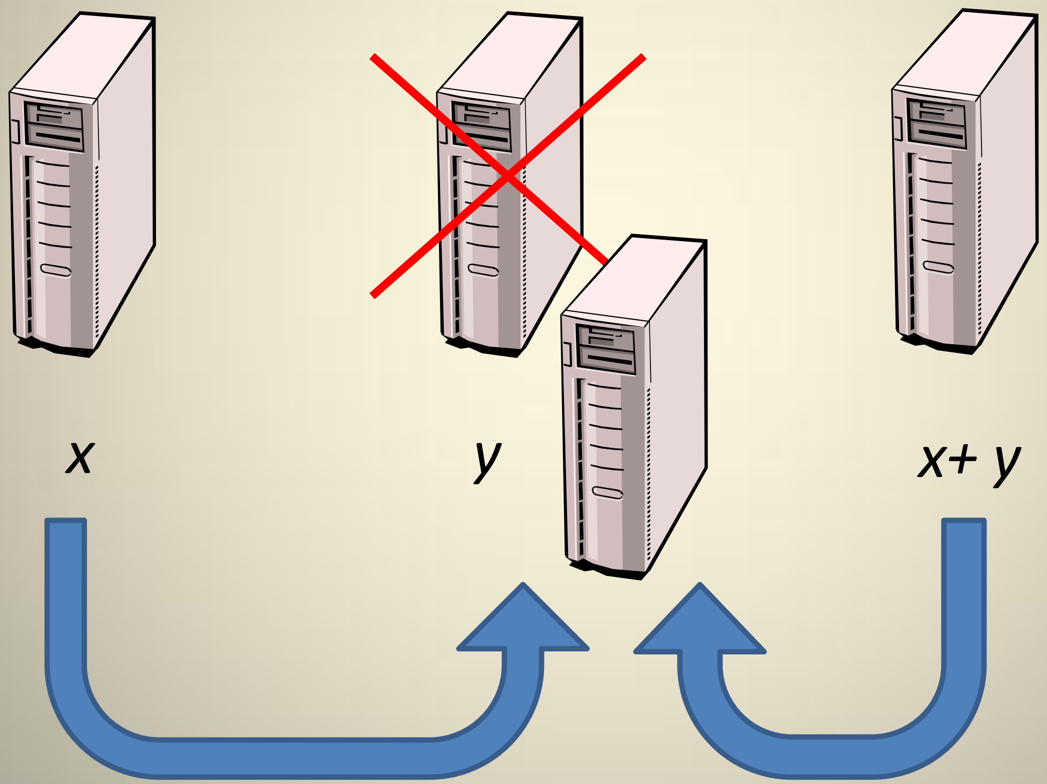


Dimakis, Godfrey, Wu, Wainwright, Ramchandran ‘2008

Distributed data storage



Distributed data storage



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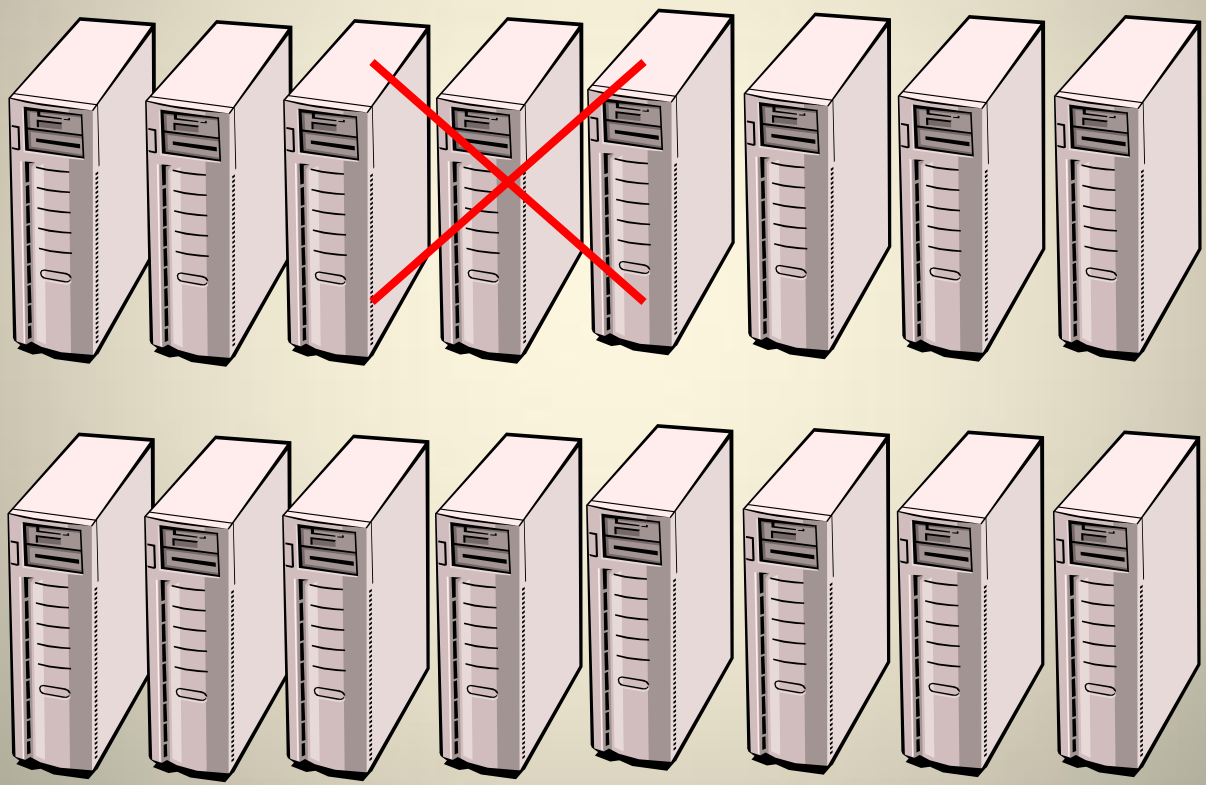

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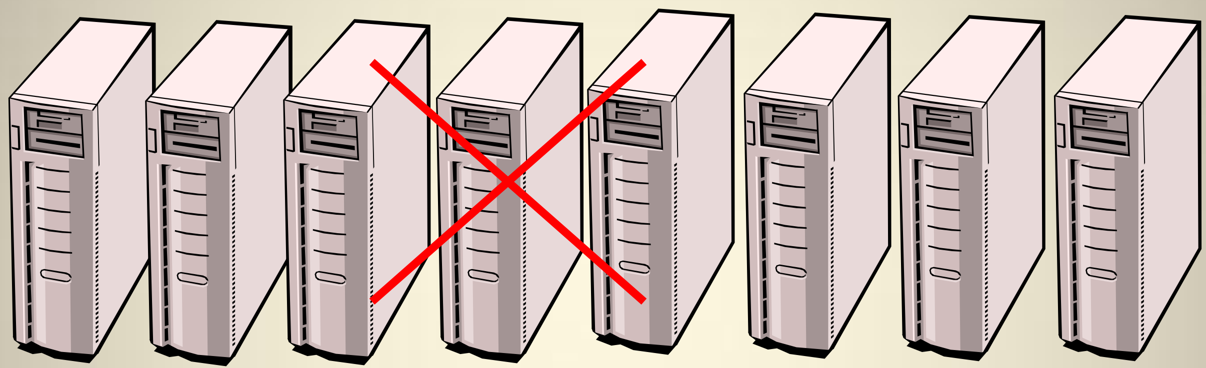
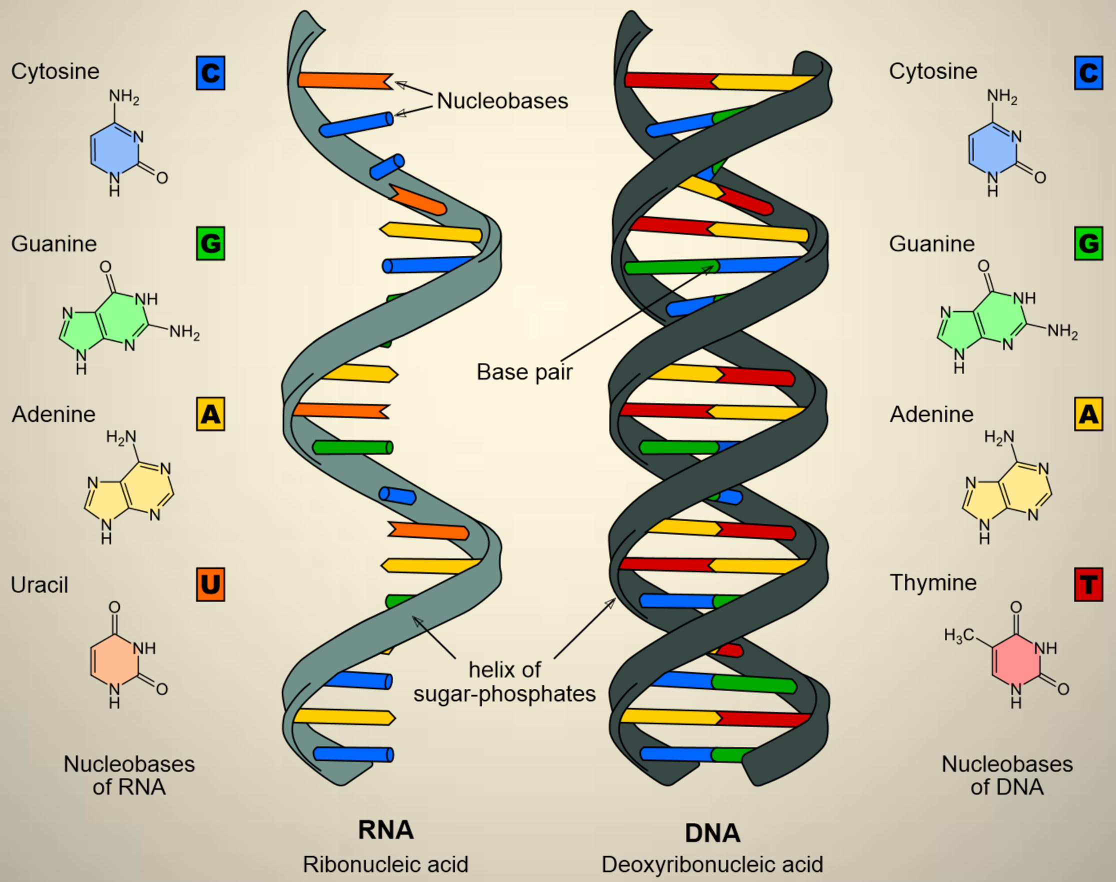


- Classical error-correcting codes can be employed
- Local correction is needed (using few other servers) to facilitate the correction

DNA Analysis



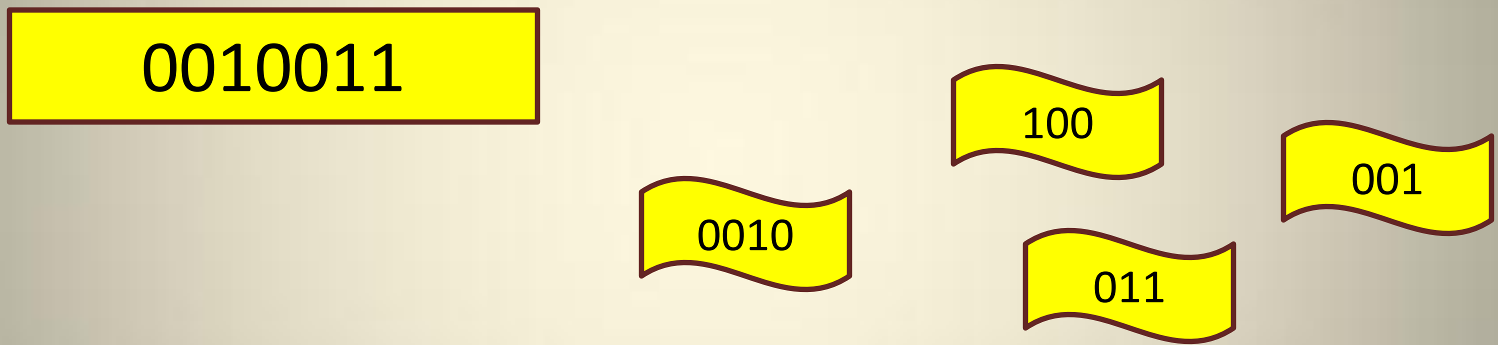
String Reconstruction Problem

- Four amino acids: A, F, G, C
- The composition of each protein can be deduced from its weight
- Each protein-sequence bond is cut independently with the same probability

Acharya, Das, Milenkovic, Orlitsky, and Pan '2011

String Reconstruction Problem

- Binary alphabet {0,1}



Acharya, Das, Milenkovic, Orlitsky, and Pan '2011