Performance of ML Decoding for Ensembles of Binary and Nonbinary Regular LDPC Codes of Finite Lengths

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***Abstract*—The Gallager ensembles of binary regular LDPC codes and binary images of nonbinary regular LDPC codes are studied. Recurrent procedure for computing average spectra for these two ensembles is presented. By using the existing bounding techniques, estimates on the error probability of the maximum-likelihood (ML) decoding over an AWGN channel with BPSK signaling for short codes from different ensembles of LDPC codes are obtained. The numerical results show performance of the ML decoding for different code ensembles. Conclusions drawn based on the average code spectra are then verified by near-ML decoding simulations for both randomly selected and the best known short codes. The asymptotic ML decoding thresholds for AWGN and BSC channels are calculated.**

As expected, codes with the ML decoding performance superior to that of the average code in the ensemble, are easy to find. However, comparison of the the presented results with simulation results for belief propagation (BP) decoding shows that the ML decoding performance should not be used as a target for searching for good iteratively decodable codes.

I. Introduction

Nowadays, LDPC codes are widely used in communication systems and they are among the most probable candidates for the future communication standards. In some scenarios, e.g. wireless communications, relatively short codes (hundreds of bits or a few thousands of bits long) are used. It is important to understand which code ensembles are the most promising in terms of their performance, and how far the currently used codes are from the theoretical limits.

In this paper, we study the maximum-likelihood (ML) decoding performance of the binary LDPC codes and of the binary images of nonbinary (NB) LDPC codes in the finite length regime. The additive white Gaussian noise (AWGN) channel model with BPSK signaling is assumed throughout this paper.

For general linear codes, the detailed overviews of the finite length lower and upper bounds on the error probability of the ML decoding over AWGN channel are presented by Sason and Shamai in [1] and by Polyansky *et al.* in [2]. As it follows from these references, the lower Shannon bound [3] and the Poltyrev tangential sphere (TS) bound [4] are still the best benchmarks for the linear code ML decoding performance. Note that the bound in [4] requires knowledge of the weight spectrum of the code.

A straightforward way to evaluate the code ensemble performance under the ML decoding is to substitute the estimates on the average code weight spectra for these ensembles into the TS bound [5]. For the ensembles of binary regular LDPC codes and LDPC codes over arbitrary nonbinary fields, the average weight spectra are derived in [6]. Estimates on the average spectrum of the binary regular LDPC codes are used in [7] in order to demonstrate very good performance of this ensemble. For a detailed analysis of the asymptotic weight spectrum of the ensemble of nonbinary (NB) protograph-based LDPC codes, as well as of binary images of NB protograph-based LDPC codes, see [8]. Binary images of irregular NB LDPC codes were analyzed in [9]. Estimates on the thresholds of the ML decoding over an AWGN channel for the binary images of NB LDPC codes are also presented in [8].

In this work, we derive new estimates on the ML decoding error probability for the AWGN channel by using precise average weight enumerators for both binary random LDPC codes and for binary images of random nonbinary LDPC codes. In [10], this technique was used for analysis of distance properties of LDPC codes including stopping distance and stopping redundancy. In this work, by substituting the new exact average spectra into the TS bound [4], new bounds for binary codes, which are tighter than their counterparts in [7] are obtained. Moreover, new bounds for nonbinary codes are found.

The new asymptotic analysis of the Gallager ensemble of nonbinary LDPC codes leads us to conclusions, which are different from the conclusions drawn in [8], [9] for distinct ensembles of nonbinary LDPC codes. Unlike BP decoding thresholds for the ensembles studied in [8], [9], the ML decoding thresholds for the Gallager ensemble are monotonically improving with the increasing code alphabet size q.

This paper is organized as follows. In Section II, we present our technique for computing the code spectra. In Section III, the formulas for computing the decoding error probability bounds are given. In Section IV, we present numerical results for moderately long (length about 1000 bits) codes and compare the new bounds with the BP decoding simulation results for known LDPC codes. Then, we compare the theoretical

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bounds with the ML decoding performance of the newly found short codes. As expected, codes with performance superior to that of the average code in the ensemble are easy to find. However, short LDPC codes with near-optimum minimum distance and low ML decoding frame error rate (FER) do not necessarily demonstrate the best performance under the BP decoding. In Section V, we analyze the asymptotic behavior of the average weight distribution for the average binary image of the ensemble of regular NB LDPC codes. The obtained results are used to compute the AWGN and BSC ML decoding thresholds for this ensemble.

II. Computing exact coefficients of the average weight distribution for ensembles of regular LDPC codes

In this section, we consider the Gallager ensemble of binary regular (J,K)-LDPC codes and the average binary image of the Gallager ensemble of nonbinary regular (J,K)-LDPC codes over a finite field of characteristic two [6]. In the sequel, we refer to the binary images of nonbinary LDPC codes as nonbinary LDPC codes. Let n and k denote the code length and dimension, respectively, and let H be an $r \times n$ parity-check matrix, $r = n-k$. We compute the average spectrum coefficients $\mathrm{E}\{A_{n,w}\}$, where $A_{n,w}$ is the random variable representing the number of binary codewords of weight w and length n and $\mathrm{E}\{\cdot\}$ is the expected value over the code ensemble.

In order to compute the coefficients of the average weight distribution for binary and nonbinary random LDPC codes, we use their generating functions derived in [6] and [10], respectively. In general, even if an analytical expression of the weight generating function is known, computing the coefficients $A_{n,w}$ is a rather difficult task. However, if a weight generating function can be expressed as a degree of another weight generating function, then the spectrum coefficients can be computed by using a recurrent procedure with complexity linear in n.

For the Gallager ensemble of (J,K)-regular codes, the parity-check matrix consists of J strips $H^{\mathrm{T}} = \left(H_1^{\mathrm{T}} \,|\, H_2^{\mathrm{T}} \,|\, \ldots \,|\, H_J^{\mathrm{T}}\right)^{\mathrm{T}}$, where each strip H_i of width $M = r/J$ is a random permutation of the first strip.

The generating function of the number of binary sequences $\boldsymbol{x}$ of weight w and length n satisfying the equality $\boldsymbol{x}H_i^{\mathrm{T}} = \mathbf{0}$, $i \in \{1, 2, \ldots, J\}$, is given by

$$G(s) = \sum_{w=0}^{n} G_{n,w} s^w = \big(g(s)\big)^M , \tag{1}$$

where $g(s) = \sum_{i=0}^{K} g_i s^i = \left((1+s)^K + (1-s)^K\right)/2$, $g_i = \binom{K}{i}$ if i is even, and is equal to 0 otherwise.

From (1), we obtain the recurrent relation

$$G_{1,i} = g_i, \quad i = 0, 1, ..., K , \tag{2}$$

$$G_{j,w} = \sum_{i=0}^{K} g_i G_{j-1,w-i}, \; j = 2, ..., M, \; w = 0, ..., jK . \tag{3}$$

For the ensemble of NB LDPC codes over GF(q), $q = 2^m$, $m \geq 2$, the generating function $G(s)$ of the binary image weight distribution is expressed via the generating function $F(\rho)$ of the nonbinary weight distribution as

$$G(s) = F(\rho)\,\big|_{\rho=\phi(s)} , \tag{4}$$

$$F(\rho) = \sum_{w=0}^{KM} F_w \rho^w = (f(\rho))^M , \tag{5}$$

where $f(\rho) = \left((1+(q-1)\rho)^K + (q-1)(1-\rho)^K\right)/q$ (see [6, Chapter 5]) and

$$\phi(s) = \sum_{i=1}^{m} \phi_i s^i = \frac{(1+s)^m - 1}{q-1} , \tag{6}$$

$$\phi_i = \frac{1}{q-1}\binom{m}{i} . \tag{7}$$

In order to simplify the computation of the coefficients f_i in the series expansion $f(\rho) = \sum_{i=0}^{K} f_i \rho^i$, we use the following recursion

$$\alpha_0 = 1, \; \alpha_i = (q-1)^{i-1} - \alpha_{i-1} , \tag{8}$$

$$f_i = \binom{K}{i}\alpha_i . \tag{9}$$

The procedure for computing the average spectrum consists of the following steps:

1) For $w = 0, 1, ..., Km$, compute the coefficients of the average binary weight distribution for the blocks of j consecutive nonzero symbols from GF(q) using the recurrent relation

$$\Phi_{1,w} = \phi_w, \; w = 0, \ldots, m , \tag{10}$$

$$\Phi_{j,w} = \sum_{i=0}^{m} \phi_i \Phi_{j-1,w-i}, \quad j = 2, ..., K , \tag{11}$$

where ϕ_w is determined by (7).

2) Apply equations (8)–(9) to compute the weight distribution of the nonbinary symbols satisfying one (nonbinary) parity check.

3) Find the average weight distribution of the average binary image satisfying one nonbinary parity check as

$$F_w = \sum_{j=0}^{K} f_j \Phi_{j,w}, \; w = 0, ..., Km , \tag{12}$$

where f_j and $\Phi_{j,w}$ are determined by (9) and (11), respectively.

4) Compute one-strip binary weight distribution by using the recursion

$$G_{1,0} = 1, \; G_{1,w} = F_w, w = 1, \ldots, Km , \tag{13}$$

$$G_{j,w} = \sum_{i=0}^{Km} F_i G_{j-1,w-i}, \quad j = 2, ..., M . \tag{14}$$

The average spectrum coefficients are obtained as

$$\mathrm{E}\{A_{n,w}\} = \binom{n}{w} p(w)^J = \binom{n}{w}^{1-J} G_{n,w}^J , \tag{15}$$

where $p(w) = \binom{n}{w}^{-1} G_{n,w}, n = mKM$, and for $G_{n,w}$ we can either use (13)–(14) in the general case, or a simpler recursion (2)–(3) for the binary codes. The computational complexity is determined by the main step in the recursion (14), and it is proportional to $M \cdot Km = n$.

III. Bounds on error probability of ML decoding

1) Lower bound: Let n, R, and σ denote the code length, code rate and standard noise deviation for an AWGN channel, respectively. We use the notations and formulas in [3] for the cone half-angle $\theta \in [0, \pi]$, which corresponds to the solid angle of an n-dimensional circular cone, and for the solid angle of the whole space

$$\Omega_n(\theta) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_0^{\theta} (\sin\phi)^{n-2} d\phi, \quad \Omega_n(\pi) = \frac{2\pi^{n/2}}{\Gamma(n/2)} .$$

respectively. For a given code of length n and cardinality 2^{nR} the parameter θ_0 is selected as a solution of the equation

$$\frac{\Omega_n(\theta_0)}{\Omega_n(\pi)} = 2^{-nR}.$$

Then, for the FER $P_{\text{sh}}(n, R, \sigma)$, we use approximation [11] for the Shannon lower bound [3]

$$\begin{aligned} P_{\text{sh}}(n, R, \sigma) &\approx \frac{1}{\sqrt{n\pi}} \cdot \frac{1}{\sqrt{1 + G^2} \sin\theta_0} \\ &\times \frac{\left[G \sin\theta_0 \exp\left(-\frac{1}{2\sigma^2} + \frac{G}{2\sigma}\cos\theta_0\right)\right]^n}{\frac{G}{\sigma}\sin^2\theta_0 - \cos\theta_0}, \end{aligned} \quad (16)$$

where $G = \frac{1}{2\sigma}\left(\cos\theta + \sqrt{\cos^2\theta + 4\sigma^2}\ \right)$.

2) Upper bound: We present here the Poltyrev bound [4] for completeness,

$$\begin{aligned} P_e \ \leq\ & \int_{-\infty}^{\sqrt{n}} f\left(\frac{x}{\sigma}\right) \left\{ \sum_{w \leq w_0} S_w \Theta_w(x) + \right. \\ + & \left. 1 - \chi^2_{n-1}\left(\frac{r_x^2}{\sigma^2}\right) \right\} dx + Q\left(\frac{\sqrt{n}}{\sigma}\right). \end{aligned} \quad (17)$$

Here $f(x) = (1/\sqrt{2\pi}) \exp -x^2/2$ is the Gaussian probability density function, $Q(x) = \int_x^{\infty} f(x)dx$,

$$\begin{aligned} \Theta_w(x) &= \int_{\beta_w(x)}^{r_x} f\left(\frac{y}{\sigma}\right) \chi^2_{n-2}\left(\frac{r_x^2 - y^2}{\sigma^2}\right) dy \ , \\ w_0 &= \left\lfloor \frac{r_0^2 n}{r_0^2 + n} \right\rfloor, r_x = r_0\left(1 - \frac{x}{\sqrt{n}}\right) \ , \\ \mu_w(r) &= \frac{1}{r}\sqrt{\frac{w}{1 - w/n}}, \beta_w(x) = \left(1 - \frac{x}{\sqrt{n}}\right)\sqrt{\frac{w}{1 - w/n}} \ , \end{aligned}$$

S_w is the w-th spectrum coefficient, and χ^2_n denotes the probability density function of chi-squared distribution with n degrees of freedom.

Parameter r_0 is a solution with respect to r of the equation

$$\sum_{w:\mu_w(r)<1} S_w \int_0^{\arccos \mu_w(r)} \sin^{n-3}\phi \quad d\phi = \sqrt{\pi} \frac{\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} . \quad (18)$$

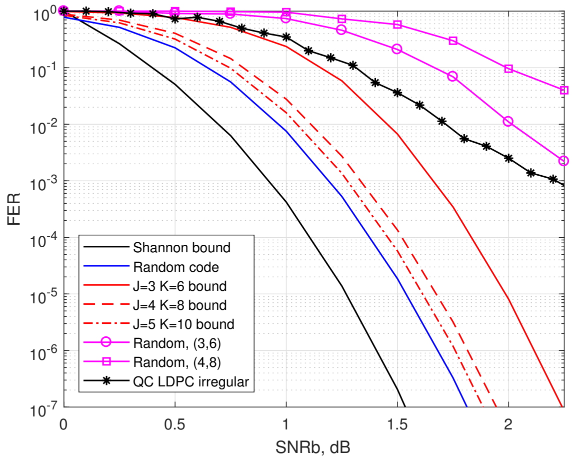

Fig. 1. Bounds and BP decoding simulation results for binary LDPC codes $n = 1000$, $R = 1/2$

IV. Numerical results and simulations

We assume that the BPSK signaling is used for transmitting over the AWGN channel with signal-to-noise ratio per bit $\text{SNRb} = -10 \log_{10} 2R\sigma^2$, where $R = k/n$.

We start this section with comparison of the Shannon lower bound (16) and upper bound (17) computed for random linear codes and random binary and nonbinary regular LDPC codes of lengths about 1000 and 100 bits. As it is shown in Figures 1–5, for $J > 3$ the upper bound for the random binary (J, K)-regular LDPC codes of length 1000 is rather close to the upper bound for the random linear codes. However, for binary codes of length 100 even for $J > 3$, there is a rather large gap between the upper bound for the random linear codes and the upper bound for the binary LDPC code ensemble. Moreover, it is easy to see that $J = 4$ is the optimum value in this case. As for the random nonbinary regular LDPC codes of length 1000, already for $J = 2$ and $m > 4$, the upper bound is close to the bound for the random linear codes. If $J = 3$, then it suffices to have $m > 2$ to be close to the upper bound for the random linear codes. For nonbinary regular LDPC codes of length 100 the gap between the bounds is much more narrow than in the binary case already for $m = 2$.

A. Moderate length codes

Next, we compare the aforementioned bounds with the simulated FER performance of the BP decoding for the rate $R = 1/2$ LDPC codes of length $n = 1000$. In Figure 1, we compare randomly generated binary (J, K)-regular LDPC codes selected according to their FER performance of the BP decoding and a nonrandom binary irregular quasi-cyclic (QC) LDPC code constructed by using the optimization technique in [12]. In Figure 2, the comparison is performed for randomly generated and selected nonbinary $(2, 4)$-regular LDPC codes over $\text{GF}(2^m)$, $m = 6, 8$, and a nonrandom nonbinary $(2, 4)$-regular code over $\text{GF}(2^8)$ constructed by using the optimization technique in [12]. In Figure 3, the FER performance of the BP decoding for randomly generated and selected according to

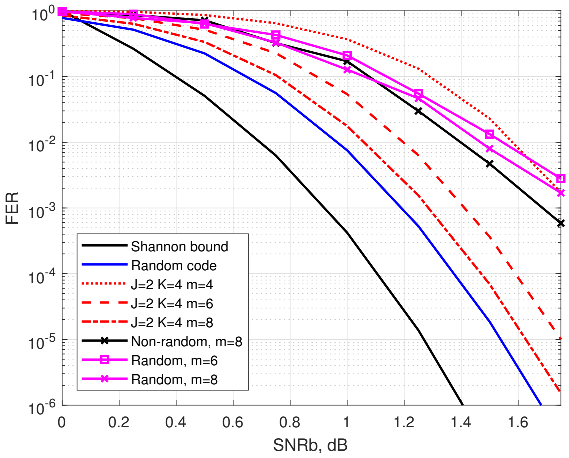

Fig. 2. Bounds and BP decoding simulation results for non-binary LDPC codes $n = 1000$, $R = 1/2$, $J = 2$

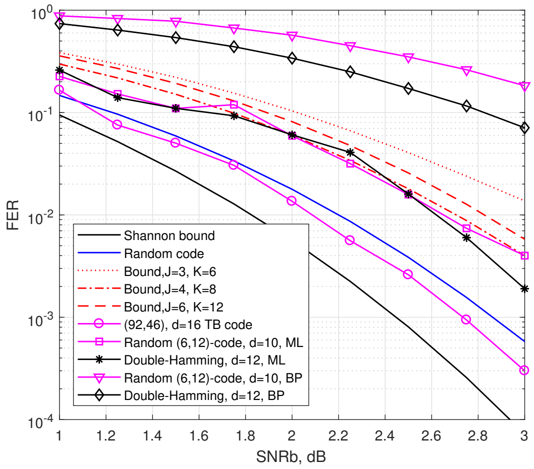

Fig. 4. Bounds and BP and near-ML decoding [13] simulation results for binary LDPC codes $n = 96$, $R = 1/2$

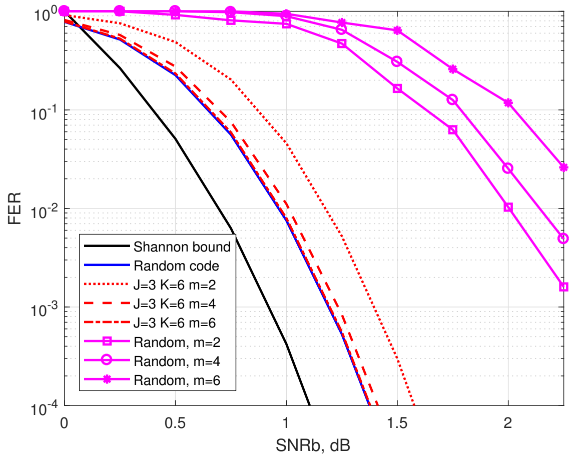

Fig. 3. Bounds and BP decoding simulation results for non-binary LDPC codes $n = 1000$, $R = 1/2$, $J = 3$

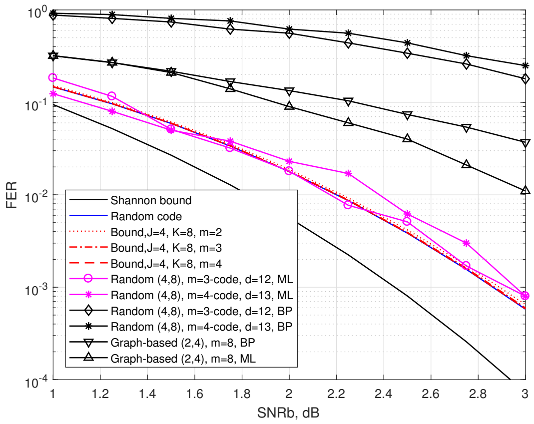

Fig. 5. Bounds and BP and near-ML [13] decoding simulation results for non-binary LDPC codes $n = 96$, $R = 1/2$

their FER performance nonbinary $(3, 6)$-regular LDPC codes over GF(2^m), $m = 2, 4, 6$, are presented.

We observe that unlike the corresponding bounds on the ML decoding performance, the FER performance of BP decoding does not improve with increasing J. The optimized irregular LDPC code demonstrates the best FER performance among all the binary LDPC codes under consideration.

Among nonbinary LDPC codes with $J = 2$, the nonrandom LDPC code over GF(2^8) optimized according to [12] has the best FER performance. The gap between the FER performance of the BP decoding for nonbinary LDPC codes with $J = 3$ and the corresponding bounds is larger than for $J = 2$. The best FER performance was obtained for $m = 8$.

B. Short codes

In this section, we compare the bounds (16) and (17) computed for random linear codes and for both binary and nonbinary (J, K)-regular random LDPC codes of length $n = 96$ with the actual FER performance of both BP and near-ML decoding [13] of short randomly generated codes.

In Figure 4, we present the FER performance of the ML-decoding of the binary tail-biting (TB) $(92, 46)$ code with $d_{\min} = 16$, which appears in [14] and the FER performance of both BP and ML-decoding for randomly generated and selected according to their minimum distance binary $(6, 12)$-regular LDPC code with $d_{\min} = 10$, and with the double-Hamming $(4, 8)$-regular LDPC code with $d_{\min} = 12$ constructed in [15].

In Figure 5, we compare the FER performance of both BP and ML decoding for randomly generated and selected according to their minimum distance nonbinary $(4, 8)$-regular LDPC codes with $d_{\min} = 12$ and $d_{\min} = 13$, $m = 3$ and $m = 4$, respectively. The FER performance of BP and ML decoding for the nonbinary graph-based $(2, 4)$-regular LDPC code with $d_{\min} = 12$ and $m = 8$ constructed by technique in [12] is presented in the same figure.

The TB linear code simulation shows that the FER of the ML decoding is lower than the upper bound (17) for the random linear codes. We conclude that in the binary case, randomly selected codes with the optimum value $J = 6$ have the FER performance close to that of the upper bound (17)

TABLE I
ASYMPTOTIC ML DECODING THRESHOLDS FOR BINARY AND NONBINARY LDPC RATE $R = 1/2$ CODES ON AWGN CHANNEL

m	J				
	2	3	4	5	10
1	3.418	0.794	0.426	0.341	0.308
2	2.138	0.546	0.353	0.318	0.308
3	1.397	0.421	0.324	0.311	0.308
4	0.975	0.360	0.313	0.309	0.308
8	0.421	0.310	0.308	0.308	0.308
16	0.313	0.308	0.308	0.308	0.308

TABLE II
ASYMPTOTIC DECODING THRESHOLDS FOR BINARY AND NONBINARY LDPC RATE $R = 1/2$ CODES ON BSC

m	J				
	2	3	4	5	10
1	0.0396	0.0915	0.1045	0.1082	0.1100
2	0.0566	0.0998	0.1076	0.1094	0.1100
3.	0.0725	0.1045	0.1090	0.1098	0.1100
4.	0.0848	0.1071	0.1096	0.1100	0.1100
8	0.1037	0.1098	0.1100	0.1100	0.1100
16	0.1093	0.1100	0.1100	0.1100	0.1100

for the binary $(6, 12)$-LDPC codes, which is inferior to the corresponding upper bound for the binary linear codes. The same holds true for the constructed $(4, 8)$-regular LDPC code. However, the FER performance of the BP decoding for more dense $(6, 12)$-regular LDPC code is inferior to that of the double-Hamming LDPC code.

In the nonbinary case, we observe that LDPC codes with $J > 2$ and with larger values of $d_{\min}$ have the FER performance of the ML decoding close to that of the upper bound (17) for $(4, 8)$-LDPC codes, which is, in turn, close to the upper bound for the random linear codes. However, their ML decoding performance is inferior to that of the LDPC code with $J = 2$ under the BP decoding. The FER performance of the LDPC code with $J = 2$ under the ML and BP decoding are very close to each other.

V. THRESHOLDS

From [16], we have the following formula for the upper bound on the minimum SNR per bit (the decoding threshold for the ML decoding over the binary input AWGN channel)

$$\mathrm{SNRb} \le \max_{0<\delta<1} \frac{1-\delta}{2\delta R}\left(1 - e^{-2a(\delta)}\right) , \qquad (19)$$

where $a(\delta)$ is the asymptotic exponent of the average spectrum as a function of normalized codeword weight $\delta = w/n$,

$$a(\delta) = \lim_{n\to\infty} \frac{A_{n,\delta n}}{n} . \qquad (20)$$

The asymptotic exponent of the average binary spectrum of the NB LDPC code [10] is

$$a_{\mathrm{NB}}(\delta) = \min_{\rho}\left\{(1-J)h(\delta) + \frac{J}{Km}\ln\left(g(\phi(e^{\rho}))\right) - \rho\delta J\right\},$$

where $h(\delta) = -\delta\ln\delta - (1-\delta)\ln(1-\delta)$ is the binary entropy function in nats.

For the BSC, the threshold crossover probability p_{thr} is equal to the solution of the following equation with respect to p [17, Theorem 4.1])

$$\max_{\delta\in(0,2p)} a(\delta) + \delta\ln 2 + (1-\delta)h\left(\frac{p-\delta/2}{1-\delta}\right) = h(p) . \quad (21)$$

For $m = 1$ (binary codes) thresholds in Table I coincide with thresholds found in [18]. For $m = 2, ..., 8$ and $J = 2$ the thresholds in Table I are close to those in [8, Fig.25].

VI. CONCLUSIONS

The new upper bounds on the ML decoding performance of the ensembles of regular LDPC codes when communicating over AWGN channels are obtained. They are based on the precise coefficients of the average weight spectra computed by the low-complexity recurrent procedure. The new values of the ML decoding thresholds for nonbinary random LDPC codes are computed.

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