Linear Batch Codes

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Distributed storage systems

- Enormous amounts of data are stored in a huge number of servers.
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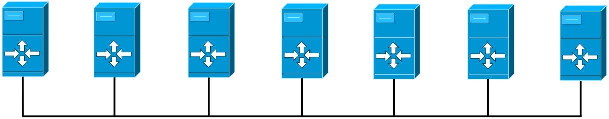

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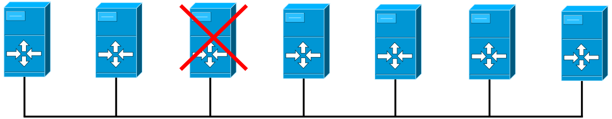

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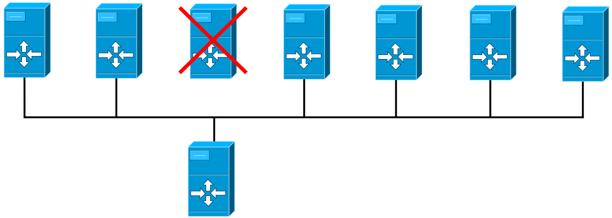

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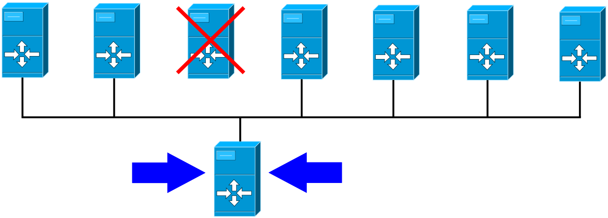

Locally repairable codes

- Consideration: minimize amount of transferred data.
- Proposed in [Dimakis, Godfrey, Wu, Wainwright, Ramchandran 2008].
- Error-correcting codes.
- Additional property: symbols can be corrected by using a small number of other symbols (locality).

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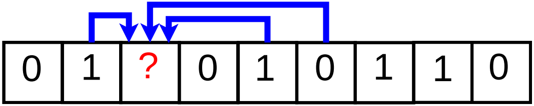

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Constructions:

- [Ishai *et al.* 2004]: algebraic, expander graphs, subsets, RM codes, locally-decodable codes

Prior art

Design-based constructions and bounds:

- [Stinson, Wei, Paterson 2009]
- [Brualdi, Kiernan, Meyer, Schroeder 2010]
- [Bujtas, Tuza 2011]
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Application to distributed storage:

- [Rawat, Papailiopoulos, Dimakis, Vishwanath 2014]
- [Silberstein 2014]

Batch codes

Definition [Ishai *et al.* 2004]

$\mathcal{C}$ is an $(n, N, m, M, t)_\Sigma$ batch code over Σ if it encodes any string $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \Sigma^n$ into M strings (buckets) of total length N over Σ, namely $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_M$, such that for each m-tuple (batch) of (not neccessarily distinct) indices $i_1, i_2, \cdots, i_m \in [n]$, the symbols $x_{i_1}, x_{i_2}, \cdots, x_{i_m}$ can be retrieved by m users, respectively, by reading $\leq t$ symbols from each bucket, such that x_{i_ℓ} is recovered from the symbols read by the ℓ-th user alone.

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Definition

An $(n, N, m, M, t)_q$ batch code is *linear*, if every symbol in every bucket is a linear combination of original symbols.

Small buckets

In what follows, consider *linear codes* with $t = 1$ and $N = M$: each encoded bucket contains just one symbol in $\mathbb{F}_q$.

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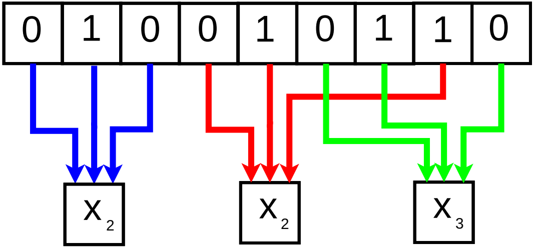

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- Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ be an information string.
- Let $\mathbf{y} = (y_1, y_2, \cdots, y_M)$ be an encoding of $\mathbf{x}$.
- Each encoded symbol y_i, $i \in [M]$, is written as $y_i = \sum_{j=1}^{n} g_{j,i} x_j$.
- Form the matrix $\mathbf{G}$:

$$\mathbf{G} = \left(g_{j,i} \right)_{j \in [n], i \in [M]} ;$$

the encoding is $\mathbf{y} = \mathbf{x}\mathbf{G}$.

Retrieval

Theorem

Let $\mathcal{C}$ be an $[M, n, m]_q$ batch code. It is possible to retrieve $x_{i_1}, x_{i_2}, \cdots, x_{i_m}$ simultaneously if and only if there exist m non-intersecting sets $T_1, T_2, \cdots, T_m$ of indices of columns in $\mathbf{G}$, and for T_r there exists a linear combination of columns of $\mathbf{G}$ indexed by that set, which equals to the column vector $\mathbf{e}_{i_r}^T$, for all $r \in [m]$.

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Example

[Ishai *et al.* 2004] Consider the following linear binary batch code $\mathcal{C}$ whose 4×9 generator matrix is given by

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} .$$

Retrieval (cont.)

Example

Let $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $\mathbf{y} = \mathbf{x}\mathbf{G}$.
Assume that we want to retrieve the values of (x_1, x_1, x_2, x_2). We can retrieve (x_1, x_1, x_2, x_2) from the following set of equations:

$$\begin{cases} x_1 &= y_1 \\ x_1 &= y_2 + y_3 \\ x_2 &= y_5 + y_8 \\ x_2 &= y_4 + y_6 + y_7 + y_9 \end{cases} .$$

It is straightforward to verify that any 4-tuple $(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})$, where $i_1, i_2, i_3, i_4 \in [4]$, can be retrieved by using columns indexed by some four non-intersecting sets of indices in [9]. Therefore, the code $\mathcal{C}$ is a $[9, 4, 4]_2$ batch code.

Properties of linear batch codes

Lemma

Let $\mathcal{C}$ be an $[M, n, m]_q$ batch code. Then, the matrix $\mathbf{G}$ is full rank.

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Theorem

Let $\mathcal{C}$ be an $[M, n, m]_2$ batch code $\mathcal{C}$ over $\mathbb{F}_2$. Then, $\mathbf{G}$ is a generator matrix of the classical error-correcting $[M, n, \geq m]_2$ code.

Example

The converse is not true. For example, take $\mathbf{G}$ to be a generator matrix of the classical $[4,3,2]_2$ ECC as follows:

$$\mathbf{G} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} .$$

Let $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3, y_4) = \mathbf{xG}$.
It is impossible to retrieve (x_2, x_3). This can be verified by the fact that

$$x_2 = y_1 + y_2 = y_3 + y_4 \quad \text{and} \quad x_3 = y_1 + y_3 = y_2 + y_4 ,$$

and so one of the y_i's is always needed to compute each of x_2 and x_3.

Bounds on the parameters

- Various well-studied properties of linear ECCs, such as MacWilliams identities, apply also to linear batch codes (for $t = 1$, $M = N$ and $q = 2$).

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- A variety of bounds on the parameters of ECCs, such as sphere-packing bound, Plotkin bound, Griesmer bound, Elias-Bassalygo bound, McEliece-Rodemich-Rumsey-Welch bound apply to the parameters of $[M, n, m]_2$ batch codes.

Construction 1

Theorem

Let $\mathcal{C}_1$ be an $[M_1, n, m_1]_q$ batch code and $\mathcal{C}_2$ be an $[M_2, n, m_2]_q$ batch code. Then, there exists an $[M_1 + M_2, n, m_1 + m_2]_q$ batch code.

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Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be $n \times M_1$ and $n \times M_2$ generator matrices of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Take $n \times (M_1 + M_2)$ matrix

$$\hat{\mathbf{G}} = [\ \mathbf{G}_1 \mid \mathbf{G}_2\ ]\ .$$

Construction 2

Theorem

Let $\mathcal{C}_1$ be an $[M_1, n_1, m_1]_q$ batch code and $\mathcal{C}_2$ be an $[M_2, n_2, m_2]_q$ batch code. Then, there exists an $[M_1 + M_2, n_1 + n_2, \min\{m_1, m_2\}]_q$ batch code.

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Denote by $\mathbf{G}_1$ and $\mathbf{G}_2$ the $n_1 \times M_1$ and $n_2 \times M_2$ generator matrices corresponding to $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Take the following $(n_1 + n_2) \times (M_1 + M_2)$ matrix

$$\hat{\mathbf{G}} = \left[\begin{array}{c|c} \mathbf{G}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{G}_2 \end{array} \right] .$$

Theorem

Let $\mathcal{C}$ be an $[M, n, m]_q$ batch code, and let $\mathbf{G}$ be the corresponding $n \times M$ matrix. Then, the code $\hat{\mathcal{C}}$, defined by the $(n+1) \times (M+m)$ matrix

$$\hat{\mathbf{G}} = \left(\begin{array}{ccccc|cccc} & & \mathbf{G} & & & 0 & 0 & \cdots & 0 \\ & & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & & 0 & 0 & \cdots & 0 \\ \hline \bullet & \bullet & \bullet & \cdots & \bullet & 1 & 1 & \cdots & 1 \end{array} \right)$$
$$\underbrace{\qquad\qquad\qquad\qquad}_{M}\underbrace{\qquad\qquad\qquad}_{m}$$

is an $[M+m, n+1, m]$ batch code, where $\bullet$ stands for an arbitrary symbol in $\mathbb{F}_q$.

Thank you!

Questions?