ON INJECTIVE HULLS OF S-POSETS

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Abstract. In this paper we describe injectives in the category of S-posets with \( S \)-submultiplicative morphisms and construct injective hulls of S-posets with respect to a specific class \( E \subseteq \) of monomorphisms.

1. Introduction

Injectivity is among properties that were studied in the very first articles which appeared in the area of S-posets ([12] and [5]). It is also a topic of several recent papers on S-posets (see, e.g., [3] and [13]). From those articles it turns out that a regularly injective S-poset has to be complete in such a way that suprema are compatible with S-action. One of the aims of this paper is to show that in a suitable framework injectivity is equivalent to that kind of completeness. The second goal is to show how to construct injective hulls of S-posets with respect to a certain class of monomorphisms.

In this work, S is always a \textit{pomonoid}, that is, a monoid \( S \) equipped with a partial order \( \leq \) such that \( ss' \leq tt' \) whenever \( s \leq t, s' \leq t' \) in \( S \). A poset \(( A, \leq )\) together with a mapping \( A \times S \rightarrow A \) (under which a pair \((a, s)\) maps to an element of \( A \) denoted by \( as \)) is called a \textit{right S-poset}, denoted by \( AS \), if for any \( a, b \in A, s, t \in S \),

\begin{enumerate}
  \item \( a(st) = (as)t \),
  \item \( a1 = a \),
  \item \( a \leq b, s \leq t \) imply that \( as \leq bt \).
\end{enumerate}

A left S-poset can be defined similarly. \textit{Right S-poset homomorphisms} are order-preserving mappings which also preserve the S-action. We denote the category of right S-posets with S-poset homomorphisms as morphisms by \( \text{Pos}_S \). An S\textit{-subposet} of an S-poset \( AS \) is an action-closed subset of \( A \) whose partial order is the restriction of the order of \( A \).

Let \( C \) be a category and let \( M \) be a class of morphisms in \( C \). We recall that an object \( Q \) from \( C \) is \( M \)-\textit{injective} in \( C \) provided that for any morphism \( h : A \rightarrow B \) in \( M \) and any morphism \( f : A \rightarrow Q \) in \( C \) there exists a morphism \( g : B \rightarrow Q \) in \( C \) such that \( gh = f \).

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A morphism \( \eta : A \to B \) in \( \mathcal{M} \) is called \( \mathcal{M} \)-essential if every morphism \( \psi : B \to C \) in \( \mathcal{C} \), for which the composite \( \psi \eta \) is in \( \mathcal{M} \), is itself in \( \mathcal{M} \). An object \( H \in \mathcal{C} \) is called an \( \mathcal{M} \)-injective hull of an object \( A \in \mathcal{C} \) if \( H \) is \( \mathcal{M} \)-injective and there exists an \( \mathcal{M} \)-essential morphism \( A \to H \) (see [1], Def. 9.22).

Let \( A_S \) and \( B_S \) be \( S \)-posets. We say that a mapping \( f : A \to B \) is \( S \)-submultiplicative if \( f(a)s \leq f(as) \) for any \( a \in A \), \( s \in S \). We denote by \( \text{Pos}\_S^\leq \) the category where objects are right \( S \)-posets and morphisms are \( S \)-submultiplicative order-preserving mappings. Clearly every \( S \)-poset homomorphism is an \( S \)-submultiplicative order-preserving mapping, so \( \text{Pos}\_S \) is a subcategory of \( \text{Pos}\_S^\leq \) which is not necessarily full.

**Example 1.** Consider the pomonoid \( S = ((0,1],\cdot,\leq) \) where \( 0 < 1 \). Then the constant mapping

\[
f : S_S \to S_S, 1 \mapsto 1, 0 \mapsto 1
\]

is order-preserving and \( S \)-submultiplicative, but it is not a right \( S \)-act homomorphism, because \( f(1) \cdot 0 = 0 \neq 1 = f(1 \cdot 0) \).

An order embedding from a poset \( (A,\leq_A) \) to a poset \( (B,\leq_B) \) is a mapping \( h : A \to B \) such that \( a \leq_A a' \) iff \( h(a) \leq_B h(a') \), for all \( a, a' \in A \). Every order embedding is necessarily an injective mapping. We will denote by \( \mathcal{E} \) the class of all right \( S \)-poset homomorphisms that are order-embeddings. These are precisely the regular monomorphisms in \( \text{Pos}\_S \) (see [2], Theorem 7).

In this paper we will study injectivity with respect to a specific class of order-embeddings. Let \( \mathcal{E}^\leq \) be the class of morphisms \( e : A_S \to B_S \) in the category \( \text{Pos}\_S^\leq \) which satisfy the following condition: \( e(a)s \leq e(a') \) implies \( a \leq a' \) for all \( a, a' \in A \) and \( s \in S \). Evidently, each morphism in \( \mathcal{E}^\leq \) is an order-embedding. On the other hand, every \( S \)-poset homomorphism that is an order embedding belongs to \( \mathcal{E}^\leq \). In other words, \( \mathcal{E} \subseteq \mathcal{E}^\leq \).

**Lemma 2.** Let \( S \) be a pogroup. Then \( \text{Pos}\_S^\leq = \text{Pos}\_S \) and \( \mathcal{E}^\leq = \mathcal{E} \).

**Proof.** We need to show that every order-preserving \( S \)-submultiplicative mapping \( f : A_S \to B_S \) is a right \( S \)-poset homomorphism. For every \( a \in A \) and \( s \in S \) we have \( f(a)s \leq f(as) \) and \( f(as)s^{-1} \leq f(ass^{-1}) = f(a) \) by \( S \)-submultiplicativity. Multiplying both sides of the last inequality by \( s \) we obtain \( f(as) \leq f(a)s \). We conclude that \( f(as) = f(a)s \), as required.

For the second claim we have to prove the inclusion \( \mathcal{E}^\leq \subseteq \mathcal{E} \). If \( e \in \mathcal{E}^\leq \) then clearly \( e \) is an order-embedding. By the first part of this proof, \( e \) also preserves \( S \)-action.

Inspired by the notion of a quantale (see [8]) we introduce a term “\( S \)-quantale”.

**Definition 3.** We call a right \( S \)-poset \( A_S \) a right \( S \)-quantale if

1. the poset \( A \) is a complete lattice;
2. \( \bigvee M = \bigvee \{ms \mid m \in M \} \) for each subset \( M \) of \( A \) and each \( s \in S \).

In the following, “\( S \)-quantale” is also used to substitute the term “right \( S \)-quantale”.

We note that \( S \)-quantales also appear in [5] under the name of “complete \( S \)-posets”, in [4] under the name of “continuously complete \( S \)-posets” and in [10] under the name of “equivariantly complete \( S \)-posets”.
Injectivity properties of $S$-posets have been studied by several authors. First of all, injectivity with respect to all monomorphisms (i.e., injective homomorphisms) is not an interesting property, because the only $S$-posets with this property are singletons ([3], Theorem 2.5). Therefore it is more natural to study $\mathcal{E}$-injectivity. Since the morphisms in $\mathcal{E}$ are precisely the regular monomorphisms of $\text{Pos}_S$, $\mathcal{E}$-injective $S$-posets have been called also regularly injective (cf. [13]).

The first to study regular injectivity was Skornyakov ([12]) who studied $S$-posets over a discretely ordered monoid $S$ and proved that such $S$-posets are complete as posets if they are regularly injective. Later on, Fakhruddin in [5] found that for a $S$-poset over a cofree $S$-quantale (see [5], Proposition 7.2). Recently, Ebrahimi, Mahmoudi and Rasouli showed in their paper [3] that for a pomonoid $S$, an $S$-poset is regularly injective if and only if it is a retract of a cofree $S$-poset over a complete poset. However, it seems that there exist no descriptions of regularly injective $S$-posets $A_S$ in terms that are internal to $A_S$.

There are some special results. For example, Fakhruddin in [5] found that for a pogroup $S$, a left $S$-poset is regularly injective if and only if it is a left $S$-quantale, thereby generalizing a similar result of Skornyakov.

One approach to obtain necessary and sufficient conditions for injectivity is to allow a larger class of morphisms between $S$-posets. This is inspired by the most recent work [7], and also [14], in which certain injective hulls of posemigroups were constructed in a category where morphisms are submultiplicative order-preserving mappings. In this work, we will first investigate $\mathcal{E}_S$-injectives in the category $\text{Pos}_S^S$ (which has the same objects but possibly more morphisms than $\text{Pos}_S$) and then give an explicit construction of $\mathcal{E}_S$-injective hulls of $S$-posets in $\text{Pos}_S^S$.

2. $\mathcal{E}_S$-Injective $S$-Posets

In this section we show that $\mathcal{E}_S$-injective objects in the category $\text{Pos}_S^S$ are precisely the right $S$-quantales.

**Proposition 4.** Let $Q_S$ be an $S$-quantale. Then $Q_S$ is $\mathcal{E}_S$-injective in the category $\text{Pos}_S^S$.

**Proof.** Let $Q_S$ be an $S$-quantale, $e : A_S \to B_S$ be a morphism in $\mathcal{E}_S$ and let $f : A_S \to Q_S$ be a morphism in $\text{Pos}_S^S$. Define a mapping $g : B_S \to Q_S$ by

$$g(b) = \bigvee \{ f(a)z \mid e(a)z \leq b, \ a \in A, \ z \in S \},$$

for any $b \in B$. Then $g$ is obviously an order-preserving mapping. For any $s \in S$, we have

$$g(bs) = \left( \bigvee \{ f(a)zs \mid e(a)zs \leq bs, \ a \in A, \ z \in S \} \right)s$$

$$\leq \bigvee \{ f(a)t \mid e(a)t \leq bs, \ a \in A, \ t \in S \}$$

$$= g(bs),$$

which means that $g$ is $S$-submultiplicative. (Note that here we used that $e(a)z \leq b$ implies $e(a)zs \leq bs$ if $a \in A, z \in S$.) Finally, for any $a \in A$, we have

$$g(e(a)) = \bigvee \{ f(x)z \mid e(x)z \leq e(a), \ x \in A, \ z \in S \}.$$
If \( x \in A, z \in S \) are such that \( e(x)z \leq e(a) \) then \( xz \leq a \) and hence
\[
f(x)z \leq f(xz) \leq f(a).
\]
Consequently, \((ge)(a) \leq f(a)\). On the other hand, \(f(a)\) is obviously one of the terms in the sup that defines \((ge)(a)\). Therefore, \(ge = f\) as needed.

**Proposition 5.** In the category \( \text{Pos}^\le_S \), every retract of an \( S \)-quantale is an \( S \)-quantale.

**Proof.** Let \( E_S \) be an \( S \)-quantale and let \( A_S \) be a retract of \( E_S \). Then there exist \( S \)-submultiplicative order-preserving mappings \( i : A \to E \) and \( g : E \to A \) such that \( gi = id_A \), where \( id_A \) is the identity mapping on \( A \). It is obvious that \( A \) is complete.

Let \( s \in S, M \subseteq A \). Clearly, \((\vee M)s\) is an upper bound of \( \{ms \mid m \in M\} \).

Suppose that \( u \) is an upper bound of \( \{ms \mid m \in M\} \) in \( A \). Then
\[
u = g(i(u)) \geq g\left(\bigvee_E\{i(ms) \mid m \in M\}\right) \geq g\left(\bigvee_E\{i(m)s \mid m \in M\}\right)
\]
\[
= g\left(\left(\bigvee_E\{i(m) \mid m \in M\}\right)s\right) \geq g\left(\bigvee_E\{i(m) \mid m \in M\}\right)s \geq \left(\bigvee_A\right)s.
\]
This means that \((\vee M)s\) is the least upper bound of \( \{ms \mid m \in M\} \), that is,
\[
\left(\bigvee M\right)s = \bigvee\{ms \mid m \in M\}.
\]
\[\Box\]

A subset \( D \) of a poset \( A \) is said to be a down-set if \( x \leq d \) implies that \( x \in D \) for any \( x \in A, d \in D \). For any \( D \subseteq A \), we denote by \( D \downarrow \) the down-set \( \{x \in A \mid x \leq d \text{ for some } d \in D\} \) and by \( a \downarrow \) the down-set \( \{x \in A \mid x \leq a\} \) for \( a \in A \).

Now we wish to construct an \( \mathcal{E}_\leq \)-injective \( S \)-poset starting from an arbitrary right \( S \)-poset.

Let \( A_S \) be an \( S \)-poset, and let \( \mathcal{P}(A) \) be the set of all down-sets of the poset \( A \). Define a right \( S \)-action \( \cdot \) on \( \mathcal{P}(A) \) by
\[
D \cdot s = (Ds) \downarrow = \{x \in A \mid x \leq ds \text{ for some } d \in D\},
\]
for any \( s \in S, D \in \mathcal{P}(A) \). It is routine to check that \((\mathcal{P}(A), \cdot)\) is an \( S \)-act, and an \( S \)-poset if we consider inclusion as the partial order. Furthermore, \((\mathcal{P}(A), \cdot, \subseteq)\) is a right \( S \)-quantale, that is, \( \mathcal{P}(A) \) is a complete lattice under the inclusion relation with supremum being union, and it satisfies
\[
\left(\bigvee\{M_\alpha \mid \alpha \in \Omega\}\right) \cdot s = \bigvee\{M_\alpha \cdot s \mid \alpha \in \Omega\}
\]
for any \( M_\alpha \in \mathcal{P}(A), \alpha \in \Omega, s \in S \). We denote the right \( S \)-quantale \((\mathcal{P}(A), \cdot, \subseteq)\) shortly by \( \mathcal{P}(A)_S \). By Proposition 4, we have the following result.

**Proposition 6.** Let \( A_S \) be an \( S \)-poset. Then \( \mathcal{P}(A)_S \) is \( \mathcal{E}_\leq \)-injective in the category \( \text{Pos}^\le_S \).

Using the above construction, we obtain a description of \( \mathcal{E}_\leq \)-injectives in the category \( \text{Pos}^\le_S \) in the next theorem.

**Theorem 7.** Let \( A_S \) be an \( S \)-poset. Then \( A_S \) is \( \mathcal{E}_\leq \)-injective in \( \text{Pos}^\le_S \) if and only if \( A_S \) is a right \( S \)-quantale.
Proof. **Necessity.** The mapping \( \eta : A_S \to \mathcal{P}(A)_S \) given by \( \eta(a) = a \downarrow \) for each \( a \in S \) is clearly an order embedding of the poset \( A \) into the poset \( \mathcal{P}(A) \). It is routine to check that \( \eta \) preserves \( S \)-action and hence \( \eta \) is also \( S \)-submultiplicative. Moreover, if \( \eta(a) \cdot s \subseteq \eta(a') \) for \( a, a' \in A, \, s \in S \), then \( (as) \downarrow = a \downarrow \cdot s \subseteq a' \downarrow \). This implies that \( as \leq a' \), which means that \( \eta \in E_S \).

Since \( A_S \) is \( E_S \)-injective by assumption, \( A_S \) is a retract of the \( S \)-quantale \( \mathcal{P}(A)_S \). Consequently, \( A_S \) is an \( S \)-quantale by Proposition 5.

**Sufficiency** follows by Proposition 4. \( \square \)

**Corollary 8** (Cf. [12] and [5]). For a right \( S \)-poset \( A_S \) over a porgroup \( S \) the following assertions are equivalent.

1. \( A_S \) is \( E \)-injective in \( \text{Pos}^S \).
2. \( A_S \) is \( E ^\leq \)-injective in \( \text{Pos}^S \).
3. \( A_S \) is a right \( S \)-quantale.

**Proof.** (1) \( \Leftrightarrow \) (2) by Lemma 2. (2) \( \Leftrightarrow \) (3) by Theorem 7. \( \square \)

We say that a pomonoid \( S \) is right (left) \( E ^\leq \)-self-injective if the \( S \)-poset \( S_S \) \((sS)\) is \( E ^\leq \)-injective in the category \( \text{Pos}^S \). A pomonoid \( S \) is \( E ^\leq \)-self-injective if it is both right and left \( E ^\leq \)-self-injective.

Self-injective (unordered) semigroups have been studied by several authors (see the comments in [6]). In particular, Pääva [9] has given necessary and sufficient conditions for right self-injectivity of a semigroup in terms of certain homomorphisms and right congruences. From Theorem 7 it immediately follows that \( E ^\leq \)-self-injectivity of a pomonoid can be described in a quite simple way.

**Corollary 9.** A pomonoid \( S \) is \( E ^\leq \)-self-injective if and only if it is a quantale.

3. \( E ^\leq \)-injective hulls of \( S \)-posets

In a recent article [7], Lambek et al considered injective hulls in the category of pomonoids and submultiplicative order-preserving mappings. Later on, Zhang and Laan in [14] extended those results to certain posemigroups and submultiplicative order-preserving mappings. Inspired by these results, in this section, we construct \( E ^\leq \)-injective hulls in the category \( \text{Pos}^S \). Similarly to Proposition 2.1 in [7] it can be shown that \( E ^\leq \)-injective hulls are unique up to isomorphism.

Recall that an order-preserving mapping \( j \) on a poset \( P \) is called a closure operator if it satisfies

1. \( a \leq j(a) \),
2. \( j(j(a)) = j(a) \),

for all \( a \in P \). Let us introduce the concept of \( S \)-quantic nucleus similarly to the case of quantales (see [11]).

**Definition 10.** Let \( Q_S \) be an \( S \)-quantale. We say that an \( S \)-submultiplicative closure operator \( j \) on \( Q \) is an \( S \)-quantic nucleus.

**Lemma 11.** Let \( j \) be an \( S \)-quantic nucleus on an \( S \)-quantale \( Q_S \). Then \( j(as) = j(j(a)s) \) for any \( a \in Q, s \in S \).

**Proof.** On one hand, since \( j \) is increasing and order-preserving it follows that

\[
(1) \quad a \leq j(a) \Rightarrow as \leq j(a)s \Rightarrow j(as) \leq j(j(a)s).
\]

Conversely, \( j(as) \leq j(as) \) implies \( j(j(a)s) = j(j(as)) = j(as) \). \( \square \)
Lemma 13. The category \( \mathcal{S} \) is complete. Moreover, \( \mathcal{S} \) is an S-quantic nucleus, then \( \mathcal{S} \) is an S-quantic nucleus.

Proof. Since \( \mathcal{S} \) is complete, it is straightforward to show that

\[
\mathcal{S} \quad \text{and define a right } \mathcal{S}-\text{action } \circ \text{ on } \mathcal{S} \text{ by }
\]

\[
D \circ s := \text{cl}(D \cdot s),
\]

for any \( s \in S \). By Theorem 12, the S-poset \( \mathcal{S} \) is the \( \mathcal{S} \)-injective hull of \( \mathcal{S} \) in the category \( \mathcal{S} \).

Theorem 14. For every S-poset \( \mathcal{S} \), \( \mathcal{S} \) is the \( \mathcal{S} \)-injective hull of \( \mathcal{S} \) in the category \( \mathcal{S} \).
Proof. Since $S$ is a monoid, $a \downarrow \in \mathscr{D}(A)_S$ for any $a \in A$. We will show that the mapping $\eta : A_S \to \mathscr{D}(A)_S$, $a \mapsto a \downarrow$ is an $\mathcal{E}_\infty$-essential morphism in $\text{Pos}_\infty^S$.

To show that $\eta$ is an $S$-poset homomorphism, take $a \in A$, $s \in S$. It is easy to see that $a \downarrow \cdot s = (as) \downarrow$. Hence we have

$$\eta(a) \circ s = \text{cl}(a \downarrow \cdot s) = \text{cl}((as) \downarrow) = (as) \downarrow = \eta(as),$$

i.e. $\eta$ is an $S$-act homomorphism. For every $a, b \in A$, $a \leq b$ if and only if $a \downarrow \subseteq b \downarrow$, which means that $\eta$ is an order embedding. Thus $\eta \in \mathcal{E} \subseteq \mathcal{E}_\infty$.

Finally, let $\psi : \mathscr{D}(A)_S \to B_S$ be a morphism in $\text{Pos}_\infty^S$ such that $\psi \eta \in \mathcal{E}_\infty$. We have to show that $\psi \in \mathcal{E}_\infty$. Suppose that $\psi(D)z \leq \psi(D')$, where $D, D' \in \mathscr{D}(A)$, $z \in S$. First we prove that

$$(\forall a \in A, s \in S)(D's \subseteq a \downarrow \implies Dzs \subseteq a \downarrow).$$

Assume that $D's \subseteq a \downarrow$, $a \in A$, $s \in S$, and take an element $d \in D$. Then $D' \circ s = \text{cl}((D's) \downarrow) \subseteq \text{cl}((a \downarrow) \downarrow) = a \downarrow$ and so

$$(\psi \eta)(d)zs = \psi(d)zs \leq \psi(D)zs \leq \psi(D')s \leq \psi(D' \circ s) \leq \psi(a \downarrow) = (\psi \eta)(a).$$

Since $\psi \eta \in \mathcal{E}_\infty$, we conclude that $dzs \leq a$ in $A$. Consequently, $Dzs \subseteq a \downarrow$.

To complete the proof, we have to show that $D \circ z \subseteq D'$. Take $x \in D \circ z$. Since $D' = \text{cl}(D')$, it suffices to prove that $x \in \text{cl}(D')$, i.e.,

$$(\forall a \in A, s \in S)(D's \subseteq a \downarrow \implies xs \leq a).$$

If $D's \subseteq a \downarrow$ then, by (3.1), we have $Dzs \subseteq a \downarrow$. Since $x \in D \circ z = \text{cl}((Dz) \downarrow)$, we get $xs \leq a$ if we are able to prove that $(Dz) \downarrow \subseteq a \downarrow$. If $d \in D$, $d' \in A$ and $d' \leq dz$ then $d's \leq dzs \leq a$, so $(Dz) \downarrow \subseteq a \downarrow$, as needed. \hfill \square

Example 15. Consider the additive pomonoid $S = (\mathbb{N}, +)$ of nonnegative integers acting on the set $A = \mathbb{N}$ by addition. For the $S$-poset $A_S$ we have

$$\mathscr{P}(A) = \{n \downarrow \mid n \in \mathbb{N}\} \cup \{\emptyset, \mathbb{N}\} = \mathscr{D}(A),$$

where $n \downarrow = \{1, \ldots, n\}$, and the action on $\mathscr{D}(A)$ is defined by

$$\begin{align*}
n \downarrow \circ s &= \text{cl}((n \downarrow + s) \downarrow) = \text{cl}((n + s) \downarrow) = (n + s) \downarrow, \\
\mathbb{N} \circ s &= \text{cl}((\mathbb{N} \downarrow) \downarrow) = \text{cl}(\mathbb{N}) = \mathbb{N}, \\
\emptyset \circ s &= \text{cl}(\emptyset \downarrow) = \emptyset,
\end{align*}$$

$s \in \mathbb{N}$. So $\mathscr{D}(A)_S$ is isomorphic to $(\mathbb{N} \cup \{\emptyset, \infty\})_\mathbb{N}$, where $\infty + s = \infty$, $\emptyset + s = \emptyset$, and $\emptyset \leq a \leq \infty$ for all $s \in \mathbb{N}_0$ and $a \in \mathbb{N}$. In other words, we obtain the injective hull of $A_S$ by adjoining external zero elements, one at the top, the other at the bottom.

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